Suslin's K-theory of Algebraically Closed Fields

Jiantong Liu

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We discuss the (partial) proof of Lichtenbaum's conjecture for K-theory of algebraically closed fields, especially in the case of fields with positive characteristic, as illustrated in Suslin (1983).

Setup. Fields are assumed to be algebraically closed unless stated otherwise, and are usually denoted by F or k. Let $p > 0$ be the characteristic of the field F, and let prime l be the cardinality of the coefficient ring we are concerned with.

1 Overview

Lichtenbaum's conjecture, according to Quillen (1975) and other sources, says the following:

Conjecture (Lichtenbaum)**.** Let F be an algebraically closed field, then for $i \geqslant 1$, $K_{2i}F$ is uniquely divisible, i.e., a divisible torsion-free abelian group, and $K_{2i-1}F$ is a divisible group whose torsion subgroup is isomorphic to the ith Tate twist $W_{(i)}$ of W, the group of roots of unity in $F^{\times}$.

Remark. Prior to Suslin (1983), several observations have been proven:

- It is well-known that this is true for K_1 and K_2, especially due to Tate.
- Due to Quillen (1972), we know this is true for algebraic closure of finite fields, by passing it to the limit using the theorem he proved. This determines the algebraic K-theory of the algebraic closure of a finite field which has close similarities to the topological K-theory of a point.
- The conjecture is equivalent to the assertion that for primes $l \neq \mathrm{char}(F)$, the cohomology ring $H^*(BGL(F), \mathbb{Z}/l\mathbb{Z})$ is a polynomial ring with generators of degree $2, 4, 6$, etc.

Suslin (1983) makes use of the following result:

Theorem 1.1 (Suslin's First Rigidity Theorem)**.** Let F/F_0 be an extension of algebraically closed fields, and let X be an algebraic variety over F_0, then there is an isomorphism

$$K_*(X; \mathbb{Z}/l\mathbb{Z}) \cong K_*(X \otimes_{F_0} F; \mathbb{Z}/l\mathbb{Z})$$

for all $i \geqslant 0$ between the mod-l K-theory of coherent sheaves on X, respectively its base change to E. This map is the specialization induced by the natural inclusion.

Remark. This is good enough for whatever was discussed in Suslin (1983), but this can be generalized, as mentioned in Suslin (1986). In general, something can be said on the level of any contravariant functor on some category of schemes with values in the category of torsion abelian groups. I will address this in Appendix A.

Corollary 1.2 (Weibel (2013), Theorem VI.1.1)**.** Let C be a smooth curve (i.e., variety) over an algebraically closed field k, with function field $F = k(C)$. If c_0, c_1 are two closed points of C, i.e, given by $c_0, c_1 : \mathrm{Spec}(F) \to C$, then the specializations $K_*(F, \mathbb{Z}/l\mathbb{Z}) \to K_*(k, \mathbb{Z}/l\mathbb{Z})$ coincide.

Corollary 1.3 (Weibel (2013), Theorem VI.1.3)**.** Let F/F_0 be an extension of algebraically closed fields, then we have $K_*(F_0; \mathbb{Z}/l\mathbb{Z}) \cong K_*(F; \mathbb{Z}/l\mathbb{Z})$ for all l.

Theorem 1.4 (Main Result of Suslin (1983))**.** If $i : F_0 \hookrightarrow F$ is an extension of algebraically closed fields, and suppose l is a prime integer, then the induced maps

- $i_* : {}_lK_*(F_0) \to {}_lK_*(F)$,
- $i_* : K_*(F_0)/l \to K_*(F)/l$, and

($\star$) $i_* : K_*(F_0, \mathbb{Z}/l\mathbb{Z}) \to K_*(F, \mathbb{Z}/l\mathbb{Z})$,

are isomorphisms.

Corollary 1.5 (Weibel (2013), Corollary VI.1.3.1)**.** Lichtenbaum's conjecture holds for fields of positive characteristics $p > 0$. That is, for $i \geqslant 1$,

(a) $K_{2i}(F)$ is uniquely divisible,

(b) $K_{2i-1}(F)$ is a direct sum of a uniquely divisible group and the torsion group $\mathbb{Q}/\mathbb{Z}\left[\frac{1}{p}\right]$. In particular, it is divisible with no p-torsion, and the Frobenius automorphism acts on the torsion subgroup as multiplication by p^i,

(c) for $p \nmid l$, the choice of a Bott element $\beta \in K_2(F; \mathbb{Z}/l\mathbb{Z})$ determines a graded ring isomorphism $K_*(F; \mathbb{Z}/l\mathbb{Z}) \cong \mathbb{Z}/l\mathbb{Z}[\beta]$.

Proof Sketch by Weibel (2013), Neukirch et al. (2013). Recall that any divisible abelian group is a direct sum of a uniquely divisible group, i.e, divisible and torsion-free, and a divisible torsion group. Note that a divisible torsion group is just the direct sum of its Sylow subgroups, and any l-primary divisible group is a direct sum of copies of $\mathbb{Z}/l^\infty\mathbb{Z}$, therefore the divisible torsion group portion of $K_{2i-1}(F)$ is of the form $\bigoplus_{l \neq p} \mathbb{Z}/l^\infty\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}\left[\frac{1}{p}\right]$. In particular, recall that if $\gcd(l, p) = 1$, then the ith Tate twist of $\mathbb{Q}/\mathbb{Z}\left[\frac{1}{p}\right]$ is $\varinjlim_m \mathbb{Z}/l^m\mathbb{Z}(i) = \mu_\infty^{\otimes i}$ where μ_∞ is given by the group of all roots of unity in F. ☐

Corollary 1.6. Let F be an algebraically closed field of characteristic $p > 0$, and let l be a prime integer, then

$$H_*(BGL(F); \mathbb{Z}/l\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & l = p \\ \mathbb{Z}/l\mathbb{Z}[c_1, c_2, \ldots], & l \neq p \end{cases}$$

where $c_i \in H_{2i}(BGL(F); \mathbb{Z}/l\mathbb{Z})$.

Remark. It is proven in Suslin (1984) that Lichtenbaum's conjecture holds $\mathbb{C}$, and by Theorem 1.1 in Suslin (1983), K-theory with coefficients of any variety defined over algebraically closed fields does not change over algebraically closed base changes, then these two results together imply that Lichtenbaum's conjecture holds for all algebraically closed fields.

Remark. Lichtenbaum's conjecture has since been generalized to Quillen–Lichtenbaum conjecture, which is a much more étale-flavored statement. Suslin's rigidity theorem was also extended later on, see this paper by Déglise and Cisinski.

2 Chasing Diagrams

Setup. Let $\mathcal{O}$ be a DVR with valuation ν, residue field F and field of fractions E, and let X be a scheme of finite type over $\mathcal{O}$. Let $M(X)$ be the category of coherent sheaves on X, and let $M_0(X) \subseteq M(X)$ be the full subcategory consisting of sheaves with support in the closed fiber of $X \to \operatorname{Spec}(\mathcal{O})$.

In Quillen (1973), the K-groups (respectively, K'-groups) of a (respectively, Noetherian) scheme X is defined over the category of vector bundles (respectively, coherent sheaves) on X, and this gives a canonical map $K_n(X) \to K'_n(X)$ that is an isomorphism when X is regular.

Remark. Since $\mathcal{O}$ is a DVR, then $\operatorname{Spec}(\mathcal{O})$ has two points, often denoted η and s, the generic and the special (or closed) point, corresponding to the ideal (0) and the unique maximal ideal $\mathfrak{m}$, respectively. The names are apparent, as $\{\eta\}$ is dense in $\operatorname{Spec}(\mathcal{O})$, while $\{s\}$ is closed in $\operatorname{Spec}(\mathcal{O})$. Now a scheme over $\operatorname{Spec}(\mathcal{O})$ is a scheme X equipped with a morphism $f : X \to \operatorname{Spec}(\mathcal{O})$. The generic (respectively, special or closed) fibers of X are the fibers over the generic (respectively, closed) point of $\operatorname{Spec}(\mathcal{O})$. As with any morphism of schemes, the fibers are equipped with scheme structures over the residue fields of the corresponding points, that is, X_η, the generic fiber, is a $E := k(\eta) = \operatorname{Frac}(\mathcal{O})$-scheme, while X_s, the special or closed fiber, is a scheme over the residue field $F := k(s) = \mathcal{O}/\mathfrak{m}$.

Remark. The K-groups of a scheme X is defined by $K_*(X) =: K_*\mathbf{P}(X)$, over the category of vector bundles over X, i.e., locally free sheaves of $\mathcal{O}_X$-modules of finite rank. In particular, if X is Noetherian, then we have $K'_*(X) = K_*(M(X))$.

Note that $M(X)$ is an abelian category, and $M_0(X)$, as a full subcategory of $M(X)$, is exact: one can check that it is closed under extension. Therefore, by Quillen's localization theorem, there exists a homotopy fibration

$$BQM_0(X) \longrightarrow BQM(X) \longrightarrow BQM(X_E)$$

where $X_E := X \times_{\mathrm{Spec}(\mathcal{O})} \mathrm{Spec}(E)$ is the base-change scheme over E, given by the fiber product of schemes. By Quillen's dévissage theorem, the embedding $M(X_F) \hookrightarrow M_0(X)$ gives a homotopy equivalence $BQM(X_F) \simeq BQM_0(X)$.

Remark. We need to make sense of the categories over the field change. The category of coherent X_E-modules $M(X_E)$ is the factor category $M(X)/M_0(X)$ where $M_0(X)$ is the full subcategory in $M(X_R)$ consisting sheaves with support in the closed fiber of $\mathrm{Spec}(F)$, which corresponds to the closed fiber.

Therefore, we have a homotopy fibration

$$BQM(X_F) \longrightarrow BQM(X) \longrightarrow BQM(X_E)$$

Let $K'_n(X) := K_n(\mathcal{M}(X))$ be K-groups of the coherent sheaves. Therefore, the homotopy fibration above gives an exact localization sequence

$$\cdots \longrightarrow K'_{i+1}(X_F, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{f_*} K'_{i+1}(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{g^*} K'_{i+1}(X_E, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\partial} K'_i(X_F, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \cdots$$

Theorem (Quillen (1973))**.** If $f : X \to Y$ is a proper map of schemes, and either it is finite, or X has an ample line bundle, then there is a transfer map $f_* : K'_0(X) \to K'_0(Y)$.

Before proceeding, we examine the maps f_* and g^*. The contravariant map f_* is induced from the projection $f : \mathcal{O} \to \mathcal{O}/\mathfrak{m} = F$ and the finite Tor-dimension closed embedding $X_F \hookrightarrow X$, and so f_* is called the transfer map, c.f., Quillen (1973); the covariaint map g^* is induced from $g : \mathcal{O} \to E$, therefore called the localization, This is to avoid talking about K-theory symmetric spectrum $K(\mathcal{O}_X)$, which acts on the localization sequence

$$K'(X_F)/n \xrightarrow{f_*} K'(X)/n \xrightarrow{g^*} K'(X_E)/n$$

Using this exact sequence, we obtain a homomorphism φ given by the composition

$$K'_n(X_E, \mathbb{Z}/n\mathbb{Z}) \otimes_{\mathcal{O}} E^* = K'_n(X_E, \mathbb{Z}/n\mathbb{Z}) \otimes_{\mathcal{O}} K_1(E) \xrightarrow{\smile} K'_{n+1}(X_E, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\partial} K'_n(X_F, \mathbb{Z}/n\mathbb{Z})$$

Here note that K_1 of a field is just its multiplicative subgroup, so we have an identification $\ell : K_1(E) = E^*$ sending $a \in E^*$ to $\ell(a) \in K_1(E)$, which we will sweep under the rug. We need to explain where the pairing operation $\smile$ came from.

Remark. Note that for any pointed spaces X, Y, and Z, with mapping $f : X \wedge Y \to Z$, we claim that there is an induced bilinear pairing $\pi_n(X) \times \pi_m(Y) \to \pi_{n+m}(Z)$ for $n, m \geqslant 1$. Recall that $\pi_n(X) = \mathrm{hTop}_*(S^n, X)$, and consider the wedge product as $\wedge : \mathrm{hTop}_* \times \mathrm{hTop}_* \to \mathrm{hTop}_*$ to be the functor that induces

$$\mathrm{hTop}^2_*((S^m, S^n), (X, Y)) = \mathrm{hTop}_*(S^m, X) \times \mathrm{hTop}_*(S^n, Y) \to \mathrm{hTop}_*(S^m \wedge S^n, X \wedge Y)$$

Recall that $S^m \wedge S^n \cong S^{m+n}$, then by definition this gives a map $\pi_m(X) \times \pi_n(Y) \to \pi_{m+n}(X \wedge Y)$. This is a bilinear map, and we recover the pairing operation by post-composing with $\pi_{m+n}(X \wedge Y) \to \pi_{m+n}(Z)$. By interpreting the field E and the DVR (as Dedekind domain) as a regular ring, we recover the product structure on K-groups of categories by Quillen's theorem. In particular, the diagrams below consider the K-groups of a field to be the K-groups of the corresponding modules, per Quillen (1973).

With this operation in hand, note that there is an operation on the K-groups given by the localization sequence above, given by the pairing on the K-groups, such that the diagram

$$\begin{tikzcd} K'_p(X_E,\mathbb{Z}/n\mathbb{Z})\otimes_{\mathcal{O}} K_q(E) \arrow[r, "\smile"] \arrow[d, "\partial\otimes\mathrm{id}"'] & K'_{p+q}(X_E,\mathbb{Z}/n\mathbb{Z}) \arrow[d, "\partial"] \\ K'_{p-1}(X_F,\mathbb{Z}/n\mathbb{Z})\otimes_{\mathcal{O}} K_q(E) \arrow[r, "\smile\circ(\mathrm{id}\otimes\partial)"'] & K'_{p+q-1}(X_F,\mathbb{Z}/n\mathbb{Z}) \end{tikzcd}$$

commutes. In particular, we have $\partial(\alpha \smile g^*(\beta)) = \partial(\alpha) \smile f^*(\beta)$ in $K'_{p+q-1}(X_F,\mathbb{Z}/n\mathbb{Z})$. This gives (1.1) on page 2 of Suslin (1983) with the assumption of $q = 1$. Note that the multiplication map defined in φ is technically on $K'_p(X_E,\mathbb{Z}/n\mathbb{Z})\otimes K_q(E)$, which comes from applying $\mathrm{id}\otimes g^*$ in the first place.

We can do something similar to prove (1.2). Fix a uniformizer π from $\mathcal{O}$, as a generator of $\mathfrak{m}$. Taking a commutative diagram

$$\begin{tikzcd} K'_p(X,\mathbb{Z}/n\mathbb{Z})\otimes_{\mathcal{O}} K_q(E) \arrow[r, "g^*\otimes\mathrm{id}"] \arrow[d, "f^*\otimes\mathrm{id}"'] & K'_p(X_E,\mathbb{Z}/n\mathbb{Z})\otimes_{\mathcal{O}} K_q(E) \arrow[d, "\smile"] \\ K'_p(X_F,\mathbb{Z}/n\mathbb{Z})\otimes_{\mathcal{O}} K_q(E) \arrow[d, "\mathrm{id}\otimes\partial"'] & K_{p+q}(X_E,\mathbb{Z}/n\mathbb{Z}) \arrow[d, "\partial"] \\ K'_p(X_F,\mathbb{Z}/n\mathbb{Z})\otimes_{\mathcal{O}} K_{q-1}(F) \arrow[r, "\smile"'] & K_{p+q-1}(X_F,\mathbb{Z}/n\mathbb{Z}) \end{tikzcd}$$

which induces $\varphi(\alpha\otimes\beta) = (-1)^i f^*(\alpha)\nu(\beta)$ for any $\alpha \in K'_i(X,\mathbb{Z}/n\mathbb{Z})$ and $\beta \in E^* = K_1(E)$. This explains (1.2). In particular, we have $\partial(g^*(\alpha) \smile \pi) = (-1)^q f^*(\alpha)$ where $\alpha \in K'_q(\mathcal{O},\mathbb{Z}/n\mathbb{Z})$. The sign comes from the twisting action from the fiber sequence above.

We now digress a bit and talk about additive property on extensions of $\mathcal{O}$. Suppose we take $\mathcal{O}'$ to be a finite normal algebra over $\mathcal{O}$, then $\mathcal{O}'$ is semi-local. For each maximal ideal $\mathfrak{m}_i$ of $\mathcal{O}'$ we can construct DVRs $\mathcal{O}_i$ of $\mathfrak{m}_i$ and other similar notions. Using commutative diagram of fibrations[1]

$$\begin{tikzcd} BQM_0(X_{\mathcal{O}'}) \arrow[r] \arrow[d] & BQM(X_{\mathcal{O}'}) \arrow[r] \arrow[d] & BQM(X_{E'}) \arrow[d] \\ BQM_0(X) \arrow[r] & BQM(X) \arrow[r] & BQM(X_E) \end{tikzcd}$$

and the localization theorem, we retrieve Lemma 1 immediately: the diagram

$$\begin{tikzcd} K'_{i+1}(X_{E'},\mathbb{Z}/n\mathbb{Z}) \arrow[r, "N_{E'/E}"] \arrow[d, "(\partial_j)"'] & K'_{i+1}(X_E,\mathbb{Z}/n\mathbb{Z}) \arrow[d, "\partial"] \\ \coprod_{j=1}^{k} K'_i(X_{F'_j}) \arrow[r, "\sum N_{F'_j/F}"'] & K'_i(X_F,\mathbb{Z}/n\mathbb{Z}) \end{tikzcd}$$

commutes, where $N_{E'/E}$ and $N_{F'_j/F}$ is the transfer map and/or direct image homomorphism.

3 Rigidity Theorem

We will now restrict our attention to the case where F is algebraically closed. In this case, the specialization map arises naturally. We start with the following observation:

Theorem 3.1 (Fuchs et al. (1960)). A group is isomorphic to $F^\times$ for some algebraically closed field F if and only if it is of the form

$$F^\times \cong \begin{cases} \mathbb{Q}/\mathbb{Z}\oplus D, & \mathrm{char}(F) = 0; \\ \bigoplus_{q\neq p} \mathbb{Z}(q^\infty)\oplus D, & \mathrm{char}(F) \neq 0; \end{cases}$$

[1] Despite the fiber is not connected over components, we will take the total fiber in this case.

Remark. For algebraically closed field F, for every $a \in F$ and prime $p > 0$, the equation $x^p = a$ has p roots in F except when $\mathrm{char}(F) = p > 0$. Therefore, $F^\times$ must be divisible.

Note that the image of $K_i'(X_E, \mathbb{Z}/n\mathbb{Z}) \otimes \mathcal{O}^*$ (as included into $K_i'(X_E, \mathbb{Z}/n\mathbb{Z}) \otimes E^*$) via φ is in the image of the obvious map $K_{i-1}'(X_F, \mathbb{Z}/n\mathbb{Z}) \otimes F^* \to K_i'(X_F, \mathbb{Z}/n\mathbb{Z})$[2] by interpreting the restriction of φ in terms of (1.1) of Suslin. However, since $K_1(F) \cong F^*$ is divisible, then this is the zero map. In particular, the universal property of quotient says that φ must factor through $K_i'(X_E, \mathbb{Z}/n\mathbb{Z}) \otimes (E^*/\mathcal{O}^*) \cong K_i'(X_E, \mathbb{Z}/n\mathbb{Z})$ given by the $\mathfrak{m}$-adic valuation via

$$\begin{tikzcd} K_p'(X_E, \mathbb{Z}/n\mathbb{Z}) \otimes E^* \arrow[r, "\varphi"] \arrow[d, "\pi"'] & K_p'(X_F, \mathbb{Z}/n\mathbb{Z}) \\ K_p'(X_E, \mathbb{Z}/n\mathbb{Z}) \arrow[ur, dashed, "s"'] & \end{tikzcd}$$

denotes by s, called the specialization homomorphism. In particular, we have (1.3) and (1.4) by definition, since π only valuates E^* as a constant.

Remark. In particular, consider the exact sequence by

$$K_1(\mathcal{O}) \xrightarrow{g} K_1(E) \xrightarrow{\nu} \mathbb{Z} \longrightarrow 0$$

where $K_0(F) = \mathbb{Z}$. We now fix a choice of uniformizer π corresponding to ν. In particular, the definition of specialization gives a commutative diagram

$$\begin{tikzcd} K_p'(E, \mathbb{Z}/n\mathbb{Z}) \otimes K_1(E) \arrow[r, "\mathrm{id} \otimes \nu"] \arrow[d, "\smile"'] & K_p'(E, \mathbb{Z}/n\mathbb{Z}) \otimes \mathbb{Z} \arrow[d, "s"] \\ K_{p+1}'(E, \mathbb{Z}/n\mathbb{Z}) \arrow[r, "\partial"'] & K_p'(F, \mathbb{Z}/n\mathbb{Z}). \end{tikzcd}$$

This just means equation (1.3) on page 3 of Suslin (1983) holds, in particular for uniformizer $\pi \in K_1(E)$ this is denoted by $s(t) = s(t) \cdot \nu(\pi) = \partial(\pi \otimes t)$ for $t \in K_i'(F(C), \mathbb{Z}/n\mathbb{Z}$, since the valuation of the uniformizer is 1. (1.4) is obvious by (1.2).

We will now develop Lemma 2, that is, s vanishes on the image of $K_{i-1}'(X, \mathbb{Z}/n\mathbb{Z}) \otimes E^* \to K_i'(X_E, \mathbb{Z}/n\mathbb{Z})$.

Proof. Take $t \in K_{i-1}'(X, \mathbb{Z}/n\mathbb{Z})$ and $a \in E^*$, let π be the uniformizer in $\mathcal{O}$, then

$$\begin{aligned} s(t \smile a) &= \partial(t \smile a \smile \pi) \\ &= (-1)^{i-1} f^*(t) \partial(\pi a) \end{aligned}$$

lands in $K_{i-1}'(X_F, \mathbb{Z}/n\mathbb{Z}) K_1(F) = 0$ by divisibility. $\square$

Assuming F is algebraically closed, then in terms of Lemma 1 we note that $F_j' = F$, and $\mathcal{O}_j'$ gives individual specializations $s_j : K_i'(X_{E'}, \mathbb{Z}/n\mathbb{Z}) \to K_i(X_F, \mathbb{Z}/n\mathbb{Z})$. Finally, Lemma 3, $sN_{E'/E} = \sum_j e_j \cdot s_j$, is now an easy consequence of what we have derived so far: fix a uniformizer π, then for any $t \in K_i'(X_{E'}, \mathbb{Z}/n\mathbb{Z})$, we have

$$\begin{aligned} s(N_{E'/E}(t)) &= \partial(N_{E'/E}(t) \cdot \pi) \\ &= \partial N_{E'/E}(t \cdot \pi) \\ &= \sum_j \partial_j(t \cdot \pi) \\ &= \sum_j e_j s_j(t). \end{aligned}$$

The first line is by definition, the second line is by the projection formula, the third line is by Lemma 1, and the last line is by (1.3).

[2] Suslin's writing has the domain from X_E, which seems to be a mistake.

We now arrive at the rigidity theorem. We only care about the case where X is a variety over an algebraically closed field F, and C is a smooth curve over F with function field $E = F(C)$. For any closed point $c \in C$, there is now a corresponding specialization $s_c : K_i'(X_E, \mathbb{Z}/n\mathbb{Z}) \to K_i'(X, \mathbb{Z}/n\mathbb{Z})$. Namely, the point is the unique specialization of itself. By restricting E to C, we have s restricted to $K_i'(X \times C, \mathbb{Z}/n\mathbb{Z})$, then the restriction is, up to a sign change, the same as the homomorphism $t \mapsto t(c)$ induced from the morphism $X \times \{c\} \hookrightarrow X \times C$.

Theorem 3.2 (Rigidity). If $c_0, c_1 \in C$ are closed points and s_0, s_1 are corresponding specializations, then $s_0 = s_1$.

Proof. We first prove the case where $C = \mathbf{P}^1_F$ is a projective variety over F, therefore C is a curve associated to the field $F(t)$, then the DVRs in $F(t)$ are of the form $F[t]_{(t-a)}$ or $F\left[\frac{1}{t-a}\right]_{\left(\frac{1}{t-a}\right)}$, and we define the 0 and ∞ points to be $F[t]_{(t)}$ and $F\left[\frac{1}{t}\right]_{\left(\frac{1}{t}\right)}$, respectively. Here t gives rise to the coordinate function on C. For any point t, we take the localization sequence

$$\cdots \longrightarrow K_q'(F[t], \mathbb{Z}/n\mathbb{Z}) \longrightarrow K_q'(F(t), \mathbb{Z}/n\mathbb{Z}) \longrightarrow \bigoplus_{a \in F} K_{q-1}'(F, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \cdots$$

Recall from last time that there is an isomorphism $K_q'(F, \mathbb{Z}/n\mathbb{Z}) \cong K_q'(F[t], \mathbb{Z}/n\mathbb{Z})$, then one see the isomorphism

$$K_q'(F(t), \mathbb{Z}/n\mathbb{Z}) \cong K_q'(F, \mathbb{Z}/n\mathbb{Z}) \oplus \bigoplus_{a \in F} K_{q-1}'(F, \mathbb{Z}/n\mathbb{Z})$$

by the same result we mentioned. An alternative proof is given in Sherman (1979). In the notation we are used to, this is

$$K_q'(X_E, \mathbb{Z}/n\mathbb{Z}) \cong K_q'(X, \mathbb{Z}/n\mathbb{Z}) \oplus \bigoplus_{a \in F} K_{q-1}'(X, \mathbb{Z}/n\mathbb{Z}) \cdot (t-a).$$

By (1.4), s_0 and s_1 agree on the first summand and is $(-1)^i$, and are zero on the other summands by Lemma 2.

We now tackle the general case. We assume, without loss of generality, that C is complete, because any smooth curve over F has a smooth completion we can use. (Note that these curves are now in 1-1 correspondence with field extensions L/k of transcendental degree 1.)

By taking the specialization map at each point to be

$$s_t : K_p'(X_{E,t}, \mathbb{Z}/n\mathbb{Z}) \to K_p'(X_{F,t}, \mathbb{Z}/n\mathbb{Z}),$$

then they determine a homomorphism (by restricting from $\mathrm{Div}(C)$)

$$\begin{aligned} q : K_p'(X_E, \mathbb{Z}/n\mathbb{Z}) \otimes \mathrm{Div}^0(C) &\to K_p'(X, \mathbb{Z}/n\mathbb{Z}) \\ t \otimes \sum n_i c_i &\mapsto \sum n_i s_i(t), \end{aligned}$$

where $\mathrm{Div}(C)$ is the group of divisors on C, and $\mathrm{Div}^0(C)$[3] is the kernel of the degree homomorphism

$$\deg : \mathrm{Div}(C) = \bigoplus_{c \in C} \mathbb{Z} \to \mathbb{Z}$$

Let $v : X_E^* = F(C)^* \to \mathrm{Div}(C)$ be the valuation map of $F(C)^* = X_E^*$, then v takes values in $\mathrm{Div}^0(C)$.

Remark. This is true for $\alpha \in F(C)^* = E^*$ by choosing homomorphism

$$\begin{aligned} F(t) &\to F(C) \\ t &\mapsto \alpha \end{aligned}$$

and this is true in general by comparison of localization sequences

$$\begin{tikzcd} F(t)^* \arrow[r, "\nu"] \arrow[d] & \bigoplus_{c \in \mathbb{P}^1} \mathbb{Z} \arrow[r] \arrow[d] & K_0(\mathbb{P}^1) \arrow[r] \arrow[d] & K_0(F(t)) \arrow[d] \\ F(C)^* \arrow[r, "\nu"] & \bigoplus_{c \in C} \mathbb{Z} \arrow[r] & K_0(C) \arrow[r] & K_0(F(C)) \end{tikzcd}$$

for any morphism $C \to \mathbf{P}^1$.

[3]Note that the difference of any two divisors, i.e., rational points, is in $\mathrm{Div}^0(Y)$.

Remark. In general, look at

$$\begin{tikzcd}
F(X)^* \arrow[r, "\nu"] \arrow[d, "\varphi^*"'] & \bigoplus_{c\in X}\mathbb{Z} \arrow[r] \arrow[d] & K_0(X) \arrow[r] \arrow[d] & K_0(F(X)) \arrow[d] \\
F(Y)^* \arrow[r, "\nu"] & \bigoplus_{c\in Y}\mathbb{Z} \arrow[r] & K_0(Y) \arrow[r] & K_0(F(Y))
\end{tikzcd}$$

We will study $\varphi' : \bigoplus_{c\in X}\mathbb{Z} \to \bigoplus_{c\in Y}\mathbb{Z}$. Say y is sent to x through $Y \to X$, then φ', studying componentwise, gives $\varphi'(x) = \nu_y\varphi^*(\pi_x) = e_y$, the ramification index of x at y, i.e., $\varphi^*(\pi_x) = \pi_y^{e_y}$. Therefore, for any $x \in X$ in $\mathrm{Div}(Y)$, $\varphi'(x) = \sum_{y\in\varphi^{-1}(x)} n_y y$.

Let $\mathrm{Pic}^0(C)$ be the Picard group of divisors of degree 0[4], then this agrees with the Jacobian variety $J(C)$ which is the cokernel of $v : X_E^* \to \mathrm{Div}^0(C)$, which is the group of points of an abelian variety over k, and is uniquely n-divisible. To prove the rigidity theorem, we just need to show that the composition

$$F(C)^* \otimes K_p'(X_E, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{v\otimes\mathrm{id}} \mathrm{Div}^0(C) \otimes K_p'(X_E, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{q} K_p'(X, \mathbb{Z}/n\mathbb{Z})$$

is 0, then note that q factors through $\mathrm{Pic}^0(C) \otimes K_p'(X_E, \mathbb{Z}/n\mathbb{Z}) \cong 0$ since $\mathrm{Pic}^0(C)$ is now n-divisible, and since $x - y \in \mathrm{Div}^0(C)$ for any points on C. In particular, q is 0 and we are done.

Therefore, to prove our claim, we need to reduce the problem on arbitrary curve C to a problem on projective curve $\mathbf{P}^1$. Let $f : C \to \mathbf{P}_F^1$ be a rational function of smooth complete curves such that $t \mapsto \alpha$ under the induced map $F(t) \to F(C) = X_E$. Fix $\alpha \in F(C)^*$, let $x \in \mathbf{P}_F^1$ and $\gamma \in K_p'(X_E, \mathbb{Z}/n\mathbb{Z})$ be arbitrary. We have a commutative diagram

$$\begin{tikzcd}
K_{p+1}'(X_E = F(C), \mathbb{Z}/n\mathbb{Z}) \arrow[r, "(\partial_c)"] \arrow[d, "f_*"'] & \bigoplus_{c\in f^{-1}(x)} K_p'(F, \mathbb{Z}/n\mathbb{Z}) \arrow[d, "f_*"] \\
K_{p+1}'(F(t), \mathbb{Z}/n\mathbb{Z}) \arrow[r, "\partial x"'] & K_p'(F, \mathbb{Z}/n\mathbb{Z})
\end{tikzcd}$$

where f_* on the right is the transfer map. To interpret this, we consider $\bigoplus_{c\in f^{-1}(x)} K_p'(F, \mathbb{Z}/n\mathbb{Z})$ to be the reduced scheme of fiber $\mathrm{Spec}(F) \times_X Y$ over $x : \mathrm{Spec}(F) \to X$, which is finite. (This is for $Y = C$ and $X = \mathbf{P}_F^1$.) Therefore, using projective formula, we have

$$\begin{aligned}
s_x(f_*(\gamma)) &= \partial_x(\pi_x \smile f_*(\gamma)) \\
&= \sum_{c\in\varphi^{-1}(x)} \partial_c(f^*(\pi_x) \smile \gamma) \\
&= \sum_{c\in\varphi^{-1}(x)} n_c\partial_c(\pi_c \smile \gamma) \\
&= \sum_{c\in\varphi^{-1}(x)} n_c s_c(\gamma)
\end{aligned}$$

Therefore the diagram

$$\begin{tikzcd}
\mathrm{Div}^0(\mathbf{P}^1) \otimes K_p'(F(C), \mathbb{Z}/n\mathbb{Z}) \arrow[r, "\mathrm{id}\otimes f_*"] \arrow[d, "f'\otimes\mathrm{id}"'] & \mathrm{Div}^0(\mathbf{P}^1) \otimes K_p'(F(t), \mathbb{Z}/n\mathbb{Z}) \arrow[d, "s_{\mathbb{P}^1}"] \\
\mathrm{Div}^0(C) \otimes K_p'(F(C), \mathbb{Z}/n\mathbb{Z}) \arrow[r, "s_C"'] & K_p'(F, \mathbb{Z}/n\mathbb{Z})
\end{tikzcd}$$

[4] The notion of degree is well-defined and invariant when we are working over irreducible complete curves.

commutes. Now we know

$$\begin{aligned} s_C(v \otimes \mathrm{id})(\alpha \otimes \gamma) &= s_C(v_C \otimes \mathrm{id})(f^* \otimes \mathrm{id})(t \otimes \gamma) \\ &= s_C(f' \otimes \mathrm{id})(v_{\mathbf{P}^1} \otimes \mathrm{id})(t \otimes \gamma) \\ &= s_{\mathbf{P}^1}(\mathrm{id} \otimes f_*)(v_{\mathbf{P}^1} \otimes \mathrm{id})(t \otimes \gamma) \\ &= s_0(f_*(\gamma)) - s_\infty(f_*(\gamma)). \end{aligned}$$

This gives $q(t \otimes \alpha) = s_0(N(t)) - s_\infty(N(t))$, where N is taken with respect to the embedding $F(\mathbf{P}^1) \hookrightarrow F(C) = E$. (The degree of t is just $0 - \infty$ in the divisor of $\mathbb{P}^1$.) From the calculation above, we note that the composition map is zero whenever $s_0 = s_\infty$, where we interpret $0 = k[t]_{(t)}$ and $\infty = k\left[\frac{1}{t}\right]_{\left(\frac{1}{t}\right)}$. But as we have shown this is true on the image of $K_q'(F, \mathbb{Z}/n\mathbb{Z}) \cong K_q'(F[t], \mathbb{Z}/n\mathbb{Z}) \to K_q'(F(t), \mathbb{Z}/n\mathbb{Z})$ for $\mathbf{P}^1$. □

4 Consequences

Corollary 4.1. Suppose A is a connected smooth affine F-algebra and $h_0, h_1 : A \to F$ are any F-algebra homomorphisms. The induced homomorphisms $K_i'(X_A, \mathbb{Z}/n\mathbb{Z}) \to K_i'(X, \mathbb{Z}/n\mathbb{Z})$ coincide.

Proof. Let $Z = \mathrm{Spec}(A)$, and let z_0, z_1 be closed points on Z corresponding to h_0 and h_1. By algebraic geometry, there exists an irreducible curve $C \subseteq Z$ joining z_0 and z_1. Without loss of generality, suppose C is normal (otherwise take the normalization), then apply the theorem. □

Corollary 4.2. Let F/F_0 be an extension of algebraically closed fields and let X_0 be a variety over F_0. Denote the fiber product $(X_0)_F$ by X. If A is any smooth affine F_0-algebra with no zero divisors, and $h_0, h_1 : A \to F$ are any F_0-algebra homomorphisms, then the induced homomorphisms $K_i'((X_0)_A, \mathbb{Z}/n\mathbb{Z}) \to K_i'(X, \mathbb{Z}/n\mathbb{Z})$ coincide.

Theorem 4.3. Using the notation above, the homomorphisms ${}_nK_*'(X_0) \to {}_nK_*'(X)$, $K_*'(X_0)/n \to K_*'(X)/n$, and $K_*'(X_0, \mathbb{Z}/n\mathbb{Z}) \to K_*'(X, \mathbb{Z}/n\mathbb{Z})$ are isomorphisms.

Proof. Denote R to be ${}_nK_*'(-)$, $K_*'(-)/n$, or $K_*'(-, \mathbb{Z}/n\mathbb{Z})$. We can write F in the form of direct limit $F = \varinjlim A$ for all smooth affine F_0-subalgebras A of F. Since R respects direct limits we have $R(X) = \varinjlim R((X_0)_A)$, and the every element of $R(X_0)$ killed in $R(X)$ should also be killed in some $R((X_0)_A)$. Since F_0 is algebraically closed, then there exists some F_0-algebra homomorphism $h : A \to F_0$, and by definition it induces a splitting $R((X_0)_A) \to R(X_0)$ for the map $R(X_0) \to R((X_0)_A)$. This shows that $R(X_0) \to R((X_0)_A)$ is split injective and therefore $R(X_0) \to R(X)$ is injective.

We know $A \hookrightarrow F$ induces a homomorphism $K_i'(X_0, \mathbb{Z}/n\mathbb{Z}) \to K_i'(X, \mathbb{Z}/n\mathbb{Z})$ coincides with the homomorphism induced by $A \xrightarrow{h} F_0 \hookrightarrow F$, and hence its image is contained in the image $K_i'(X_0, \mathbb{Z}/n\mathbb{Z}) \to K_i'(X, \mathbb{Z}/n\mathbb{Z})$. This proves surjectivity for K-groups with coefficients. By the snake lemma, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i'(X_0)/n & \longrightarrow & K_i'(X_0, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & {}_nK_{i-1}'(X_0) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_i'(X)/n & \longrightarrow & K_i'(X, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & {}_nK_{i-1}'(X) & \longrightarrow & 0 \end{array}$$

commutes and shows surjectivity for the other two. □

Corollary 4.4. Using the same notation, the embedding $F_0 \hookrightarrow F$ induces isomorphism

$$H_i(\mathrm{GL}_n(F_0), \mathbb{Z}/l\mathbb{Z}) \cong H_i(\mathrm{GL}_n(F), \mathbb{Z}/l\mathbb{Z})$$

when $i \leqslant n$.

Proof. Consider the morphism $BSL(F_0)^+ \to BSL(F)^+$ of $+$-construction. By the theorem, it induces isomorphism of homotopy groups with coefficients $\mathbb{Z}/l\mathbb{Z}$, therefore this is a homotopy equivalence of simply connected spaces and so has the same homology groups with coefficients in $\mathbb{Z}/l\mathbb{Z}$. We retrieve the isomorphism $H_*(\mathrm{SL}(F_0), \mathbb{Z}/l\mathbb{Z}) \cong H_*(\mathrm{SL}(F), \mathbb{Z}/l\mathbb{Z})$ without plus construction. By comparing Hochschild-Serre spectral sequences of group extensions

$$1 \longrightarrow \mathrm{SL}(F_0) \longrightarrow \mathrm{GL}(F_0) \longrightarrow F_0^* \longrightarrow 1$$

and

$$1 \longrightarrow \mathrm{SL}(F) \longrightarrow \mathrm{GL}(F) \longrightarrow F^* \longrightarrow 1$$

we note that $H_*(\mathrm{GL}(F_0), \mathbb{Z}/l\mathbb{Z}) \cong H_*(\mathrm{GL}(F), \mathbb{Z}/l\mathbb{Z})$. To prove this for $\mathrm{GL}_n(F_0)$ and $\mathrm{GL}_n(F)$ for all n, we use homology stability theorem for infinite fields. □

Corollary 4.5. Let F be an algebraically closed field of characteristic $p > 0$, then $H_*(\mathrm{GL}(F), \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$, and for $l \neq p$, we have $H_*(\mathrm{GL}(F), \mathbb{Z}/l\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ as a polynomial ring over $\mathbb{Z}/l\mathbb{Z}$ over variables $c_i \in H_{2i}(\mathrm{GL}(F), \mathbb{Z}/l\mathbb{Z})$.

Proof. Let F be the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$, then by Quillen (1972) (c.f., Theorem 8.9 in Mitchell's account of this paper) and the previous corollary we are done. □

A Discussion on Rigidity Theorem

There are multiple versions of rigidity theorem, other than the one Suslin proved. For instance, Suslin (1986) gives a more categorical version:

Theorem. Let V be a contravariant functor on some category of schemes with values in the category of torsion abelian groups. Suppose that

- any finite flat morphism $X \to Y$ gives rise to a transfer homomorphism $N_{X/Y} : V(X) \to V(Y)$ satisfying the usual properties,
- V is homotopy invariant, i.e., $V(X \times \mathbb{A}^1) = V(X)$ for any X, then

let X/F be a connected variety over an algebraically closed field, then for any two points $x, y : \mathrm{Spec}(F) \to X$, the induced maps $V(X) \rightrightarrows V(\mathrm{Spec}(F)) = V(F)$ coincide.

Corollary. Let F/F_0 be an extension of algebraically closed field, and let X_0/F_0 be a connected variety. If $x, y : \mathrm{Spec}(F) \to X_0$ are any two F_0-points, then the induced maps $V(X_0) \rightrightarrows V(F)$ coincide. Moreover,

- if in addition, for any F_0-point $x : \mathrm{Spec}(F) \to X_0$, the image of the corresponding homomorphism $V(X_0) \to V(F)$ is contained in the image of $V(F_0)$, and
- suppose, in addition, that V commutes with limits, i.e., $V(\mathrm{Spec}(\varinjlim A_i)) = \varinjlim V(\mathrm{Spec}(A_i))$, then $V(F) = V(F_0)$ for any extension F/F_0 of algebraically closed fields.

More generalizations can be found in Suslin (1986), now known as Gabber rigidity:

Theorem (Gabber Rigidity Theorem)**.** Let A be a Henselian ring and $\mathfrak{m}$ be a maximal ideal, then for any $n \geqslant 1$ that is invertible in A, we have $K_*(A, \mathbb{Z}/n\mathbb{Z}) = K_*(A/\mathfrak{m}, \mathbb{Z}/n\mathbb{Z})$.

Along the lines, we obtain

Theorem. Let A be a Henselian ring with field of fractions F and residue field k, and let $X/\mathrm{Spec}(A)$ be a smooth affine curve. Suppose $x, y : \mathrm{Spec}(A) \to X$ are sections that coincide in the closed point of $\mathrm{Spec}(A)$. Suppose, in addition, that

- $nV(X) = 0$ for $\gcd(n, \mathrm{char}(k)) = 1$, and
- $V(A) \hookrightarrow V(F)$,

then the induced maps $x^*, y^* : V(X) \to V(A)$ coincide.

Theorem. Let V/F be a smooth variety and let $v \in V$ be a rational point. Denote A_v^h the hensalization of a local ring A_v. For any m such that $\gcd(m, \mathrm{char}(F)) = 1$, the natural homomorphism $K_*(A_v^h, \mathbb{Z}/m\mathbb{Z}) \to K_*(F, \mathbb{Z}/m\mathbb{Z})$ is bijective.

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