UCLA MATH 290C Notes

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PRELIMINARY

In this seminar, we follow the first 6 chapters of Riehl and Verity's *Elements of ∞-Category Theory*. A few of the latter talks will be dedicated to exploring examples and applications of infinity-technology.

The notes were compiled live during the seminar, so there is obviously a tradeoff between the quality of notes and the understanding of the material. Therefore, any mistakes or inaccuracies in the notes are faults of my own.

1 INTRODUCTION (APRIL 3, 2023)

Remark 1.1 (What is an ∞-category?). An ∞-category is a place to do homotopy theory. What do we want from this "category"?

- ordinary idea of objects,
- 1-morphisms,
- for each object, there is a "unique" (to some sense) identity,
- and "unique" compositions.

Here "unique" should mean up to a contractible space of choices, since the uniqueness implies a singleton, which is up to contraction. Moreover, our "category" should have 2-morphisms, which is a morphism between 1-morphisms, and continue inductively. One should note that $(\infty, 1)$-category's 2-morphisms and higher are invertible, which is suitable for us to do homotopy theory.

How do we implement our choices? We could look at categories enriched in spaces like **Top**. This category $\mathbf{\Delta}$ has objects of the form $\Delta^n = \{0 < 1 < \cdots < n\}$ and morphisms are non-decreasing maps. (This would look like a poset category.) For instance, $\Delta^0 \to \Delta^1$ has $0 \to 1$ and $0 \to 0$, but $\Delta^1 \to \Delta^0$ only has $0 \to 0$.

We now look at simplicial sets $\mathbf{sSet} = \mathrm{Fun}(\mathbf{\Delta}, \mathbf{Set})$. Now

$$\mathbf{Top} \ni \Delta[n] = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1, x_i \geq 0\}.$$

There is a map $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$, and correspondingly there is a right adjunction $\mathbf{Sing}_\cdot : \mathbf{Top} \to \mathbf{sSet}$ where

$$\mathbf{Sing}_n(X) = \mathbf{Map}_{\mathrm{Top}}(\Delta[n], X).$$

By Quillen, "simplicial sets have the same homotopy theory as spaces" (in some sense). This is where we should look at Kan complexes.

For Δ^n, we define $\Lambda[i]^n$ to be the poset diagram of Δ^n with the edge (morphism) $i \to n$ removed.

Definition 1.2. A Kan complex is a simplicial set X with a diagram

$$\begin{tikzcd}
\Lambda^n_1 \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed, "\exists"] &
\end{tikzcd}$$

This induces

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & \mathbf{Sing}_\cdot(X) \\
\Delta^n &
\end{tikzcd}$$

When looking at the underlying spaces, this corresponds to

$$\begin{tikzcd}
|\Delta^n_i| \arrow[r] \arrow[d, hook] & X \\
|\Delta^n| \arrow[ur, dashed, "?"] &
\end{tikzcd}$$

where we might want to see if the induced morphism exists. Note that the underling map of the inclusion here is just the map that completes the triangle again, so there is an inverse map.

In general, there is a functor

$$\Delta \to \mathbf{Cat}$$
$$\Delta^n \mapsto [n]$$

and corresponds to $h(-) : \mathbf{sSet} \to \mathbf{Cat}$, with $(n\mathcal{C})_n = \mathbf{Fun}([n], \mathcal{C})$. However, this is not a Kan complex in general because there may not be a lifting, as we just saw.

Lemma 1.3. X satisfies unique inner hom lifting if and only if $X \cong N\mathcal{C}$ for some category $\mathcal{C}$.

Definition 1.4. A quasi-category is a simplicial set (sSet) X satisfying inner hom lifting, that is, for $0 < i < n$,

$$\begin{tikzcd}
\Lambda[i]^n \arrow[r] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed, "\exists"] &
\end{tikzcd}$$

We now look at some applications.

Example 1.5. Triangulated categories admit a higher refinement, known as the stable ∞-category. We know that triangulated categories has few limits other than products/coproducts, but the refinement has all limits. Moreover, everything in the refinement is functorial: given any triangle with a cone object, the induced pushout diagram is stable.

This is useful for taking preheaf categories. The presheaves valued in triangulated category may not be triangulated, but the presheaves valued in a stable ∞-category is still stable. Moreover, this helps to glue objects together. (We know gluing more than two things in triangulated categories can be hard.)

Example 1.6. In higher algebra, we look at algebra over spaces, and in particular the spectra. Every stable ∞-category is enriched in the spectra.

Example 1.7. Derived algebraic geometry studies the spectra of "ring objects" in derived categories. This has been used recently in Prismatic cohomology. Brantner and Waldron have studied purely inseparable Galois theory in this setting. Clauser and Scholze works on analytic geometry and on adic spaces.

Example 1.8. Algebraic K-theory admits a number of nice descriptions, like vector bundles over $\mathbb{R}$ (as Cartesian monoidal spaces), whose where we look at the ∞-group completion, more or less related to Fabian K-theory. In some sense, K-theory is the universal additive invariants (if it make sense).

Example 1.9. In mathematical physics like topological quantum field theory, people may work on (∞, n)-categories (see Lurie's book). Some people also work on differential cohomology theories and sheaves of spectra on manifolds.

2 SIMPLICIAL SETS AND QUASI-CATEGORIES (APRIL 7, 2023)

Definition 2.1. The simplex category $\mathbf{\Delta}$ is the full subcategory of $\mathbf{Cat}$ whose objects are $[n] = \{0 \to 1 \to \cdots \to n\}$.

Definition 2.2. Taking the colimit, we obtain a category of simplicial sets $\mathbf{sSet} = \mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$.

The Yoneda embedding $Y : \mathbf{\Delta} \to \mathbf{sSet}$ is defined by sending $[n]$ to $\Delta[n]$, called the standard n-simplex.

Proposition 2.3. Every simplicial set is a colimit of standard simplexes.

To see why this is free, for some cocomplex $\mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathbf{\Delta} \arrow[r, "F"] \arrow[d, "Y"'] & \mathcal{C} \\
\mathbf{sSet} \arrow[ur, "\exists! \tilde{F}"'] &
\end{tikzcd}$$

such that $\tilde{F}$ is cocontinuous.

Example 2.4.

$$\begin{tikzcd}
\mathbf{\Delta}[1] \arrow[r] \arrow[d] & \mathbf{\Delta}[2] \arrow[d] \\
\mathbf{\Delta}[0] = * \arrow[r] & X
\end{tikzcd}$$

where X is the pushout of the diagram.

Definition 2.5. The set of n-simplices is a simplicial set as

$$X_n = \mathbf{Hom}(\mathbf{\Delta}[n], X) = X([n]).$$

Definition 2.6. A simplex $\mathbf{\Delta}[n] \to X$ is degenerate if it factors by

$$\begin{tikzcd}
\mathbf{\Delta}[n] \arrow[r] \arrow[d] & X \\
\mathbf{\Delta}[m] \arrow[ur] &
\end{tikzcd}$$

for $m < n$.

Example 2.7. $(\mathbf{\Delta}[1])_2 \neq \varnothing$. In particular, $\mathbf{Hom}(\mathbf{\Delta}[2], \mathbf{\Delta}[1]) = \mathbf{Hom}_{\mathbf{\Delta}}([2],[1])$, which collapses the triangle into a line by sending 2 to 1 instead.

To geometrize a standard simplex, we look at a functor from $\mathbf{\Delta}$ to **Top**. In particular, this functor factors via **sSet**, with the map $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$ called the geometric realization. For instance, in Example 2.4, the geometric realization is just the colimit of the pushout.

Remark 2.8. The nerve of a category, denoted $N\mathcal{C}([n]) = \mathrm{Fun}([n], \mathcal{C})$, can be use to interpret the simplices. For example, the 0-simplex in $N\mathcal{C}$ corresponds to the object in $\mathcal{C}$; the 1-simplex in $N\mathcal{C}$ corresponds to the morphism in $\mathcal{C}$; the 2-simplex in $N\mathcal{C}$ corresponds to 2 composable morphisms in $\mathcal{C}$, and so on.

Example 2.9 (Example of simplicial sets).

1. The boundary $\partial\mathbf{\Delta}[n]$ is the boundary of an n-simplex.
2. $\Lambda^i[n] \subseteq \partial\mathbf{\Delta}[n]$ is the simplex with omitting the unique face not including i. For example, $\Lambda^0[2]$ is the simplex with $[0], [1], [2]$, but only with maps $[0] \to [1]$ and $[0] \to [2]$; $\Lambda^1[2]$ is the simplex with $[0], [1], [2]$, but only with maps $[0] \to [1]$ and $[1] \to [2]$.

Definition 2.10 (Horn)**.** An inner horn is something of the form $\Lambda^i[n]$ for $1 \leq i \leq n-1$. An outer horn is $\Lambda^0[n]$ or $\Lambda^n[n]$.

Definition 2.11 (Quasi-category)**.** A quasi-category Q is a simplicial set satisfying the inner horn filling property: given any horn $\Lambda^i[n]$ and a morphism into Q, there exists a diagram

$$\begin{tikzcd}
\Lambda^i[n] \arrow[r] \arrow[d] & Q \\
\mathbf{\Delta}[n] \arrow[ur, dashed] &
\end{tikzcd}$$

Note that the induced map may not be unique.

Remark 2.12.

- In particular, the degenerate 1-simplices are the identity maps.
- A morphism $f \in Q$ is represented by $\mathbf{\Delta}[2] \to \mathbf{\Delta}[1] \to Q$.

Definition 2.13 (Homotopy Category)**.** Given a quasi-category Q, its homotopy category hQ has

1. objects as object sets $\mathrm{ob}(hQ) := Q_0$, the 0-simplices.
2. morphisms $\mathbf{Hom}_{hQ}(a, b)$ as the set of 1-simplices from a to b quotient by some equivalence relation $\sim$, where $f \sim f'$ if and only if there exists a 2-simplex

$$\begin{tikzcd}
 & A \arrow[d, "f"] \\
A \arrow[ur, equal] \arrow[r, "f"] & B
\end{tikzcd}$$

Remark 2.14. The (equivalence) relation is well-defined and therefore this defines a category.

Definition 2.15. A morphism $f : A \to B$ in Q is an isomorphism if there exists $g : B \to A$ such that the diagrams

$$\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[ur, "f"] \arrow[r, equal] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
& A \arrow[d, "f"] \\
B \arrow[ur, "g"] \arrow[r, equal] & B
\end{tikzcd}$$

commute.

Proposition 2.16. A morphism is an isomorphism in Q if and only if it is an isomorphism in hQ.

Proposition 2.17. **sSet** is Cartesian closed and **QCat** is an ideal in **sSet**:

$$\mathbf{Hom}(- \times X, Y) \cong \mathbf{Hom}(\mathbf{\Delta}[n], Y^X)$$

Here we can consider $- \times X$ as a simplicial set Y^X. Moreover, if Y is a quasi-category, then so is Y^X.

Definition 2.18. We say $f : A \to B$ is an equivalence of quasi-categories if and only if there exists $g : B \to A$ such that

$$\begin{tikzcd}
& A \\
A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathrm{II}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\
& A
\end{tikzcd}$$

and

$$\begin{tikzcd}
& B \\
B \arrow[ur, equal] \arrow[r, "\beta"] \arrow[dr, "fg"'] & B^{\mathrm{II}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\
& B
\end{tikzcd}$$

where $\mathrm{II} = N(\cdot \rightleftarrows \cdot)$

Definition 2.19 (Isofibration). A map of quasi-categories $A \to B$ is an isofibration if

$$\begin{tikzcd}
\Lambda[n] \arrow[r] \arrow[d] & A \arrow[d] \\
\mathbf{\Delta}[n] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & A \arrow[d] \\
\mathrm{II} \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

commute.

3 INFINITY COSMOS (APRIL 10, 2023)

Definition 3.1. A category $\mathcal{V}$ is Cartesian closed if it admits finite products and internal hom b^a with the tensor-hom adjunction $\mathbf{Hom}_{\mathcal{V}}(- \times a, b) \cong \mathbf{Hom}_{\mathcal{V}}(-, b^a)$, and that $\mathcal{V}$ is bicomplete.

Definition 3.2. A $\mathcal{V}$-enriched category $\mathcal{C}$ is a set of objects and for each pair x, y, $\mathbf{Hom}_{\mathcal{C}}(x, y) \in \mathcal{V}$, equipped with

1. composition map $\mathbf{Hom}_{\mathcal{C}}(x, y) \times \mathbf{Hom}_{\mathcal{C}}(y, z) \to \mathbf{Hom}_{\mathcal{C}}(x, y)$,
2. identity map $\mathbb{1}_{\mathcal{V}} \to \mathbf{Hom}_{\mathcal{C}}(x, x)$.

Remark 3.3. We can $\mathcal{V}$-enrich $\mathcal{V}$ itself. In particular, $\mathbf{Hom}_{\mathcal{V}}(a, b) = b^a$.

Remark 3.4. We can change the basis of enrichment with a continuous functor $F : \mathcal{V} \to \mathcal{W}$. Therefore, we have a map $\mathbf{Hom}_{\mathcal{C}}(a, b) \to F(\mathbf{Hom}_{\mathcal{C}}(a, b))$.

Remark 3.5. The underlying category functor is given by $(-)_0 : \mathcal{V} \to \mathbf{Set}$ such that $\mathbf{Hom}_{\mathcal{C}}(a, b) \to \mathbf{Hom}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}}, \mathbf{Hom}_{\mathcal{C}}(a, b))$.

Remark 3.6. Although the Cartesian closed condition may not be required for the enrichment, it is required for the change of basis of enrichment.

Definition 3.7 (Infinity Cosmos). An ∞-cosmos $\mathcal{K}$ is a quasi-enriched category with

- many enriched limits, e.g., cotensors with **sSet**: consider $\mathbf{Hom}_{\mathcal{K}}(A, B^U) \cong \mathbf{Hom}_{\mathcal{K}}(A, B)^U$, where the right-hand side is the exponential objet in the simplicial set. Here B^U is the cotensor, since the quasi-category acts as an ideal in the simplicial sets.

Denote $\mathbf{Fun}(A,B) = \mathbf{Hom}_{\mathcal{K}}(A,B)$. The underlying set would be $\mathbf{Fun}(A,B)_0 = \mathbf{QCat}(*, \mathbf{Fun}(A,B))$. A morphism $A \to B$ is just an element of $\mathbf{Fun}(A,B)_0$. Therefore, we also require

- a class of morphisms called isofibrations (including all isomorphisms and all maps to the terminal object), closed under a few operations. If $f : A \twoheadrightarrow B$ is an isofibration in $\mathcal{K}$, then for all X, the map $\mathbf{Fun}(X,A) \twoheadrightarrow \mathbf{Fun}(X,B)$ is also an isofibration.

Definition 3.8. An ∞-category is an object of an ∞-cosmos.

Definition 3.9. An equivalence $f : A \xrightarrow{\sim} B$ is a morphism such that all induced maps $\mathbf{Fun}(X,A) \to \mathbf{Fun}(X,B)$ is an equivalence.

Definition 3.10. If an isofibration is an equivalence, we say this is a trivial fibration.

Example 3.11.
- $\mathbf{QCat}$, which can be enriched over itself, therefore is a ∞-cosmos.
- $\mathbf{Cat}$ is $\mathbf{Cat}$-enriched, thereby gives a $\mathbf{QCat}$-enriched category (by changing the enrichment after applying nerve), and as a ∞-cosmos. To see this, the isofibration is given by

$$\begin{tikzcd}
\mathbb{1} \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "F"] \\
\mathbb{I} \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

- The category $\mathbf{Kan}$ of Kan complexes.
- $\mathbf{CSS}$, the category of complete Segal spaces.
- Segal categories.
- 1-complicial sets.

Remark 3.12. $f : A \to B$ is an equivalence if and only if it extends to

$$\begin{tikzcd}
 & A \\
A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\
 & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & B \\
B \arrow[ur, equal] \arrow[r, "\beta"] \arrow[dr, "fg"] & B^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"] \\
 & B
\end{tikzcd}$$

Remark 3.13. Equivalences satisfy 2-out-of-3 property.

Remark 3.14. Any morphism $f : A \to B$ factors via $P : PF \to B$ given by the equivalence between A and PF. To see this, note that in **QCat** there is $\mathbb{1} + \mathbb{1} \hookrightarrow \mathbb{I}$ which yields $B^{\mathbb{I}} \twoheadrightarrow B^{\mathbb{1}+\mathbb{1}} \cong B \times B$.

Definition 3.15. A cosmological functor $F : K \to L$ is a simplicial functor preserving the required limits and isobifrations.

Remark 3.16. They automatically preserves equivalences.

Example 3.17. For any $X \in K$, $\mathbf{Fun}_K(X, -)$ is a cosmological functor, so there is $(-)_0 = \mathbf{Fun}_K(\mathbb{1}_K, -)$.

Definition 3.18. $F : K \to L$ is a biequivalence if $\mathbf{Fun}(A, B) \to \mathbf{Fun}(FA, FB)$ is an equivalence in **QCat**.

Definition 3.19. A homotopy 2-category of an ∞-cosmos K is the strict 2-category (**Cat**-enriched category) with the same objects as K, and $\mathbf{hFun}(A, B) = h(\mathbf{Fun}(A, B))$.

Remark 3.20. A simplicial functor $F : K \to L$ induces a 2-functor $hF : hK \to hL$.

Definition 3.21. An equivalence of 2-categories is a pair of functors $f : A \to B$ and $g : B \to A$ that satisfies the equivalence of categories.

Remark 3.22. For $A, B \in K$, $f : A \to B$ is an equivalence if and only if $[f] : A \to B$ is an equivalence.

4 ADJUNCTION (APRIL 14, 2023)

Recall that a ∞-cosmos K enriched over quasi-categories is the category of ∞-categories. There is the notion of a few equivalences upon this, including

- Equivalence 1: $f : A \to B$ in K is an equivalence if for all $X \in K$, $f_* : \mathrm{Fun}(X, A) \to \mathrm{Fun}(X, B)$ is an equivalence of quasi-categories.

  This equivalence concurs with something called equivalence 2:

- Equivalence 2: $f : A \to B$ is an equivalence if and only if there is $g : B \to A$ such that it satisfies Remark 3.12.

Definition 4.1. A homotopy 2-category $\mathrm{hFun}(A, B) = h(\mathrm{Fun}(A, B))$ of ∞-cosmos K is a strict 2-category with objects as ∞-categories, i.e., same as objects of K, 1-cells as ∞-functors, and 2-cells as the so-called ∞-natural transformations, i.e., the homotopy classes of 1-simplices in $\mathrm{Fun}(A, B)$.

This induces a third equivalence:

Remark 4.2 (Equivalence 3)**.** $f : A \to B$ in a 2-category defines an equivalence if there exists $g : B \to A$ and invertible 2-cells α, β such that there is $\alpha : \mathrm{id} \Rightarrow gf$ on A and $\beta : fg \Rightarrow \mathrm{id}$ on B.

Theorem 4.3. Interpreted in some suitable settings, the three equivalences specified above are equivalent.

Definition 4.4. An adjunction between ∞-categories is a pair A and B of ∞-categories, a pair $u : A \to B$ and $f : B \to A$ of ∞-functors, and a pair of ∞-natural transformations $\eta : \mathrm{id}_B \Rightarrow uf$ and $\varepsilon : fu \Rightarrow \mathrm{id}_A$ that satisfies the triangle identities:

$$\begin{tikzcd}
u \arrow[r, "\eta u"] \arrow[dr, equal] & ufu \arrow[d, "u\varepsilon"] \\
& u
\end{tikzcd}$$

and

$$\begin{tikzcd}
f \arrow[r, "f\eta"] \arrow[dr, equal] & fuf \arrow[d, "\varepsilon f"] \\
& f
\end{tikzcd}$$

Proposition 4.5 (Calculus of Adjunctions)**.**

- Adjunctions compose: if $f' \dashv u'$ and $f \dashv u$ between C and B, and B and A, respectively, then $ff' \dashv u'u$ is an adjunction between C and A.
- uniqueness: adjoints to a common factor are natural isomorphic, and functors isomorphic to adjoints are adjoints.
- Minimal data: enough to ask that identities in the triangular identities are invertible.
- Can upgrade equivalences to adjoint equivalences.
- Whole situation is equivalence-invariant.

The terminal ∞-category $\mathbb{1}$ in ambient category K for any $A \in K$, an element of A is a map of ∞-categories $\mathbb{1} \to A$.

The initial element is the left adjoint to $! : A \to \mathbb{1}$; the terminal element is the right adjoint to !.

5 LIMITS AND COLIMITS (APRIL 17,2023)

Definition 5.1 (Diagram ∞-category)**.** Let A be an ∞-category. There are two ways to define this:

1. Let $J \in$ sSet, then A^J is the ∞-category of diagrams of shape J in A.
2. Let K be a Cartesian a closed ∞-cosmos, then let J be an ∞-category, then A^J is a functor $J \to A$.

There is now a functor SSet $\times K \to K$ as a map $(J, A) \mapsto A^J$.

Definition 5.2 (Constant Diagram Functor)**.** The terminal object $\mathbb{1}$ is such that $A^{\mathbb{1}} \cong A$ with $! : J \to 1$ and $\Delta : A \to A^J$.

Definition 5.3. 1. A admits all colimits of shape J if Δ has a left adjoint.

A admits all limits of shape J if Δ has a right adjoint.

This corresponds to the colimit-diagonal-limit adjunction in 1-category theory.

Lemma 5.4. Products and coproducts in A also define products and coproducts in the homotopy category $h(A)$.

Definition 5.5 (Absolute Lifting)**.** The absolute left lifting of g through f given by $g : C \to A$ and $f : B \to A$ is a pair (l, λ) given by a lift $l : C \to B$ and $\lambda : g \Rightarrow l$. This satisfies the universal property: given $b : X \to B$ and $c : X \to C$, there exists $\alpha : g \Rightarrow b$.

Similarly, we can define the absolute right lifting.

Lemma 5.6. Given the left lifting

$$\begin{tikzcd} & & B \arrow[d, "f"] \\ X \arrow[r, "c"] & C \arrow[r, "g"] & A \end{tikzcd}$$

there exists the pair $(lr, \lambda c)$ such that gc factors through f with the notation above.

Proposition 5.7. Given an adjunction $f \dashv u$, the functor $\eta : \mathrm{id} \Rightarrow uf$ is a unit if and only if (f, η) acts as the absolute left lifting of u and id_B.

Definition 5.8. The colimit of shape J of $d : D \to A^J$ is the pair $(\text{colim}(d), \eta)$ for $\Delta : A \to A^J$ and $d : D \to A^J$, given by the diagram

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \\ D \arrow[r, "d"] & A^J \arrow[r, equal] & A^J \end{tikzcd}$$

Definition 5.9. Given $F : A \to B$, we say F preserves limits if it sends limits cones in A to limits cones in B; we say F reflects limits if whenever cones in A are sent to limit cones in B, then the cone in A is a limit cone.

Proposition 5.10. If we stack to such lifting diagrams together, and given that the bottom diagram is a lifting diagram, then the whole diagram defines a lifting if and only if the top diagram is a lifting diagram.

Theorem 5.11. Right adjoints preserve limits and left adjoints preserve colimits.

6 Weak Universal Property (April 24, 2023)

Broadly speaking, we need to study the properties of the data of cosmological limits in the homotopy 2-category. Specifically, we look at simplicial cotensors A^2 ("the arrow ∞-category of A") and pullbacks along isofibrations. Viewing these structures in the homotopy 2-category together with their canonical maps we note that they would not have the universal property of a "universal 2-cell with codomain A" and a "2-pullback" (which are notions that can be defined generally in any 2-category. However, they each satisfy a weakened version of the universal property of these notions, namely the weak universal property of the arrow ∞-category and pullback, respectively. Both these weak universal properties consist of three bullet points, which correspond to a certain functor (of 1-categories) in each case being

- surjective on objects,
- full,
- conservative.

Such functors are called smothering functors and allarently they come up a lot in quasi-category theory with regards to these types of limit constructions. Using these weak universal properties, one can prove that

Proposition 6.1. The pullback of an equivalence along an isofibration is an equivalence.

Also, we can use the weak universal property to prove that the arrow ∞-category A^2 of A is unique up to "fibered equivalence" over $A \times A$.

7 COMMA CATEGORY (MAY 1, 2023)

Definition 7.1. For a cospan $C \xrightarrow{g} A \xleftarrow{f} B$ in an ∞-cosmos, the comma ∞-category is the pullback

$$\begin{tikzcd}
\mathbf{Hom}_A(f,g) \arrow[r, "\varphi"] \arrow[d, "{(p_1,p_0)}"] & A^2 \arrow[d] \\
C \times B \arrow[r, "g\times f"] & A \times A
\end{tikzcd}$$

such that the two vertical arrows give isofibrations and

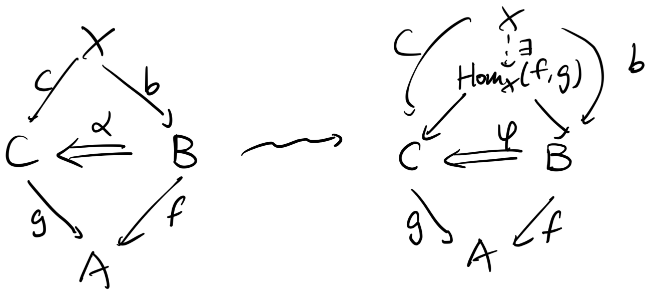
$$\varphi : \mathbf{Hom}_A(f,g) \to A^2 \in \mathrm{Fun}(\mathbf{Hom}_A(f,g), A^2) \cong \mathrm{Fun}(\mathbf{Hom}_A(f,g), A)^2.$$

Therefore, this yields a 2-cell in hK: $\varphi : fp_0 \Rightarrow gp_1$.

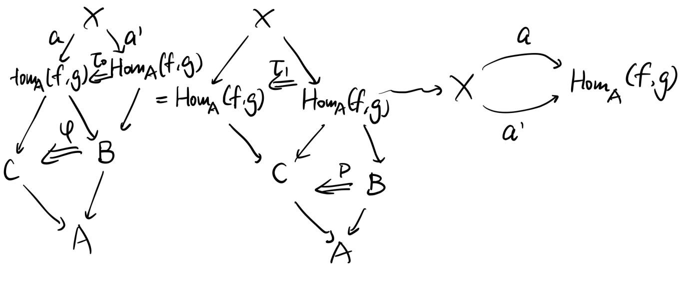
A morphism between cospans induces a morphism between comma categories.

We now look at the weak universal property of comma ∞-categories:

- 1-cell induction:



- 2-cell induction:



- Moreover, given $\tau : a \Rightarrow a'$ between two functors $a, a' : X \to \mathbf{Hom}_A(f,g)$, $p_0\tau, p_1\tau$ being isofibrations implies that τ is an isofibration.

Therefore,

$$\mathrm{hFun}(X, \mathbf{Hom}_A(f, g)) \to \mathbf{Hom}_{hFun(X,A)}(\mathrm{hFun}(X, f), \mathrm{hFun}(X, g))$$

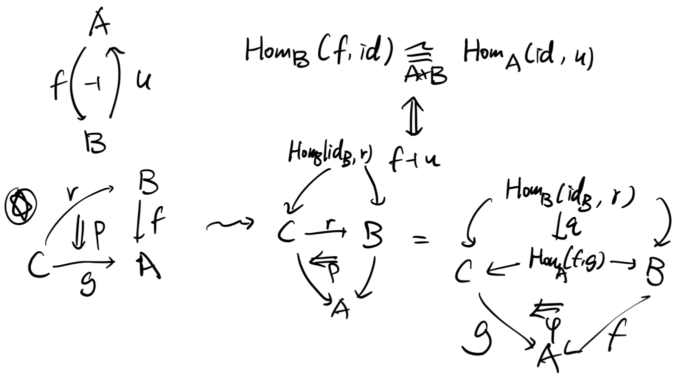
is a smothering. Therefore, applying $\mathrm{Fun}(X, -)$ on the definition (i.e., the pullback diagram) makes perfect sense.

Remark 7.2. Note that the comma ∞-category is unique up to fibered equivalence over $C \times B$:

$$\begin{tikzcd}
\mathbf{Hom}_A(f,g) \arrow[r, "\cong"] \arrow[d] & Q \arrow[dl] \\
C \times B &
\end{tikzcd}$$

However, note that $(hK)_{C\times B} \ncong h(K_{C\times B})$.

Note that we have an absolute right lifting diagram (starred figure)



Proposition 7.3. There is an absolute right lifting (starred above) if and only if q is a fibered equivalence over $C \times B$.

Definition 7.4. A functor $f : A \to B$ is fully faithful if

$$\mathrm{hFun}(X, A) \to \mathrm{hFun}(X, B)$$

if fully faithful for all X.

Under $A \times A$, there is an equivalence given by $A^2 \cong \mathbf{Hom}_B(f, f)$.

Proposition 7.5. $h(K_{/B} \to (hK)_{/B}$ is a smoothering, i.e., surjective on objects and smothering on homomorphisms.

Proposition 7.6. Smothering 2-functors reflect and create equivalences.

Corollary 7.7. There would not be ambiguity about homotopy data of fibered equivalences.

8 Universal Property of Cones (May 3, 2023)

$$\begin{tikzcd}
& \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] & \overset{\varphi}{\Longleftarrow} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}$$

$$\left\{\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr] & \\
C \arrow[dr, "g"'] & \overset{\alpha}{\Longleftarrow} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}\right\} \leftrightsquigarrow \left\{\begin{tikzcd}
& X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, "\alpha"] & \\
C & & B \\
& \mathrm{Hom}_A(f,g) \arrow[ul, "p_1"] \arrow[ur, "p_0"'] &
\end{tikzcd}\right\} \Big/ \underset{C\times B}{\simeq}$$

Special Case:

$$\left\{\begin{tikzcd}
& B \arrow[d, "f"] \\
C \arrow[ur, bend left, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=2ex, "\Downarrow\rho"] & A
\end{tikzcd}\right\} \Longleftrightarrow \begin{tikzcd}
& \mathrm{Hom}_B(\mathrm{id}_B, r) \arrow[dl] \arrow[dr] \arrow[dd, "\gamma"] & \\
C & & B \\
& \mathrm{Hom}_A(f,g) \arrow[ul] \arrow[ur] &
\end{tikzcd} \Big/ \underset{C\times B}{\simeq}$$

$$\begin{tikzcd}
& \mathrm{Hom}_B(\mathrm{id}_B, r) =: X \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[rr, bend left, "r"] \arrow[dr, "g"'] & \overset{\mathrm{id}_r}{\Longleftarrow} \quad \Longleftarrow \quad \overset{\rho}{\Longleftarrow} & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd} \qquad \begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, shift right=2ex, "\Downarrow\exists!"] & B \arrow[d, "f"] \\
C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift left=2ex, "\Downarrow\rho"] & A
\end{tikzcd}$$

Proposition:

$$\begin{tikzcd}
B \arrow[d, bend right, "f"'] \\
A \arrow[u, bend right, "u"']
\end{tikzcd} \qquad f \dashv u \Longleftrightarrow \mathrm{Hom}_A(f, \mathrm{id}_A) \underset{A\times B}{\simeq} \mathrm{Hom}_B(\mathrm{id}_B, u)$$

$$f \downarrow \mathrm{id}_A \underset{A\times B}{\simeq} \mathrm{id}_B \downarrow u$$

slice

Proof.

$f \dashv u \Rightarrow \varepsilon : fu \Rightarrow \mathrm{id}_A$ such that absolute right lifting diagram

$$\begin{tikzcd}
& B \arrow[d, "f"] \\
A \arrow[ur, "u"] \arrow[r, "\mathrm{id}"'] \arrow[ur, phantom, shift right=2ex, "\Downarrow\varepsilon"] & A
\end{tikzcd} \quad \text{ARLD.}$$

Let $a : 1 \to A$, this induces

$$\begin{tikzcd}
1 \arrow[d] \\
\mathrm{Hom}_A(f,g)
\end{tikzcd} \qquad \begin{tikzcd}
1 \arrow[d] \arrow[dd, bend right=60, "\varepsilon_a"'] \\
\mathrm{Hom}_B(\mathrm{id}_B, ua) \arrow[d, phantom, "\wr\Vert"] \\
\mathrm{Hom}_A(f,a)
\end{tikzcd}$$

$$\begin{tikzcd}
& 1 \arrow[dl] \arrow[dr, "ua"] \arrow[d, phantom, "\perp"] & \\
1 \arrow[dr, "ua"'] \arrow[ddr, bend right=60, "a"'] & \mathrm{Hom}_B(\mathrm{id}, ua) \arrow[l] \arrow[r] \arrow[d, phantom, "\overset{\varphi}{\Longleftarrow}"] & B \arrow[dl, "\mathrm{id}"] \\
\overset{\Longleftarrow}{\varepsilon_a} & B \arrow[d, "f"] & \\
& A &
\end{tikzcd}$$

Proposition:

A functor $f: B \to A$ admits a right adjoint iff $\mathrm{Hom}_A(f, A)$ admits a terminal element over A.

Let $d: D \to A^J$

$\Delta: A \to A^J$ (diagonal).

Then the cocones under d is $\mathrm{Hom}_{A^J}(d, \Delta)$

the cones over d is $\mathrm{Hom}_{A^J}(\Delta, d)$.

Proposition:

$\mathrm{Hom}_{A^J}(\mathrm{id}_{A^J}, \Delta) \simeq A^{J \diamond \mathbb{1}}$ fat join.

where

$$\begin{tikzcd}
(J \times 1) \amalg (J \times 1) \arrow[r, "\pi_J \amalg \pi_1"] \arrow[d] & J \amalg 1 \arrow[d] \\
J \times 2 \times 1 \arrow[r] & J \diamond 1 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

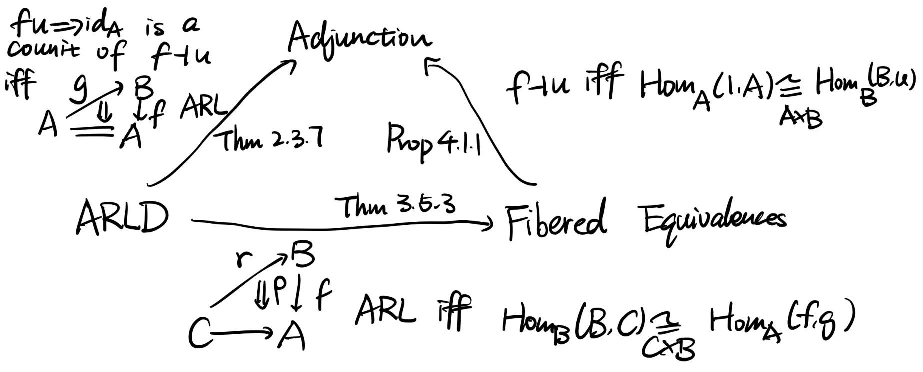
Therefore

$$\begin{tikzcd}
A^J \times A^J & A^J \times A \arrow[l] \\
(A^J)^2 \arrow[u] & A^{J \diamond 1} \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

For quasi-category Q, $Q^{I \diamond J} \simeq Q^{I \star J} \Rightarrow A^{J \diamond 1} \underset{A^J \times A}{\simeq} A^{J \star 1}$

9 Stable ∞-category (May 8, 2023)

Recall



We first look at the universal property of limits and colimits. A family of diagrams $d : D \to A^J$ admits a limit if and only if $\mathbf{Hom}_{A^J}(\Delta, d) \cong_{D \times A} \mathbf{Hom}_A(A, D)$ over d is right representable.

Example 9.1. The cotensors/tensors and pushouts and pullbacks have the usual behaviors: $k \otimes - \dashv (-)^k$ gives an adjunction on A.

Let k be a simplicial set and $a : 1 \to \mathbb{A}$ be an element of some ∞-category. The tensor

$k \otimes a$ is the colimit of the constant diagram index by k at a.

$$\begin{tikzcd}
 & A \arrow[d, "\Delta"] \\
A \arrow[ru, "k\otimes a"] \arrow[r, "\Delta"] & A^k
\end{tikzcd}$$

gives a lifting from Δ to $k \otimes a$. Therefore, $\mathbf{Hom}_A(A, (-)^k) \cong \mathbf{Hom}_A(\Delta, \Delta) \cong \mathbf{Hom}_A(k \otimes (-), A)$ which gives the adjunction.

We will now use $\lrcorner$ and $\ulcorner$ to denote the limit of diagrams indexed by pullbacks and pushouts, respectively.

Lemma 9.2. We have a pullback diagram

$$\begin{tikzcd}
d \arrow[r, "u"] \arrow[d, "v"'] & b \arrow[d, "f"] \\
c \arrow[r, "g"'] & a
\end{tikzcd}$$

if and only if there is the lifting

$$\begin{tikzcd}
 & \mathrm{Hom}_A(A, b) \arrow[d, "f_*"] \\
1 \arrow[ru, "u"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow"] & \mathrm{Hom}_A(A, a)
\end{tikzcd}$$

Definition 9.3. A is pointed if there exists a non-zero element $*$. It admits suspensions if it admits a colimit of loops.

Definition 9.4. $f : 1 \to A^2$ from x to y would admit a fiber if there is an absolute lifting, where we pushout from x and y to the cofiber.

Definition 9.5. *A is stable if*

- *every morphism admits a fiber and cofiber,*
- *and any sequence if fiber if and only if it is cofiber.*

Theorem 9.6. *A pointed ∞-category has the following to be equivalent:*

1. *A is stable,*
2. *A has all pullbacks and pushouts, and they coincide,*
3. *A has all pullbacks and pushouts, pullbacks preserves pushouts and*
4. *A admits cofibers and Σ is equivalent,*
5. *A admits fibers and Ω is equivalent.*

10 ALGEBRA IN HOMOTOPY THEORY, (MAY 10, 2023)

Let X be a connected space and ΩX be a based loop on X. We now want to obtain a group structure on ΩX.

Let A_n be the anima, i.e., ∞-category of spaces.

Definition 10.1. Let $\mathcal{C}$ be an ∞-category with finite limits, a Cartesian monoid in $\mathcal{C}$ is a functor $X : \Delta^{\text{op}} \to \mathcal{C}$ such that

1. X is reduced, i.e., $X_0 \simeq *$ the terminal object, and
2. the Segal condition, i.e., if $\rho_i : [1] \to [n]$ sends $0 \mapsto i$ and $1 \mapsto i+1$, then the product $\prod_{i=1}^{n} \rho_i : X_n \to \prod_{i=1}^{n} X_i$ is an isomorphism. Therefore, this encodes multiplication.

Now $\text{Mon}(\mathcal{C}) \subseteq s\mathcal{C} = \mathcal{C}^{\Delta^{\text{op}}}$ for many categories $\mathcal{C}$ like A_n, the monoids in the anima, etc.

Definition 10.2. We say a Carteisan monoid $X : \Delta^{\text{op}} \to \mathcal{C}$ is group-like if $(p_1, \circ) : X \times X \to X \times X$ defined by $(f, g) \mapsto (f, f \circ g)$ is an isomorphism.

In particular, in an anima, it suffices to check on π_0.

Remark 10.3. Given two operations $\circ, \times$ on S, which are both unital, and satisfy $(a \circ b) \times (c \circ d) = (a \times c) \circ (b \times d)$, then $\circ = \times$, and the operation is commutative and associative.

A Carteisan group in A_n will be called an $\mathbb{E}_1$-group. This becomes a full subcategory $\mathrm{Grp}(\mathcal{C}) \subseteq \mathrm{Mon}(\mathcal{C})$.

There is a correspondence between $\mathbb{E}_1$-monoids and ∞-categories with one object. Also, there is a correspondence between $\mathbb{E}_1$-groups and connected anima.

There is an adjunction $B : \mathrm{Grp}(A_n) \rightleftarrows (*/A_n)_{\geq 1} : \Omega$, where $B = \mathrm{colim}_{\Delta^n}(X)$.

Theorem 10.4 (Stasheff's Recognition Principle)**.** $\mathbb{E}_1$-groups are equivalent to loop spaces.

Proposition 10.5. The inclusion $\mathrm{Grp}(A_n) \hookrightarrow \mathrm{Mon}(A_n)$ has a left adjoint $(-)^{\infty\text{-grp}} : \mathrm{Mon}(A_n) \to \mathrm{Grp}(A_n)$. Through this we get $X^{\infty\text{-grp}} = \Omega BX$.

Proof. There is a one-to-one correspondence between $(*/A_n)_{\geq 1}$ and $(*/\infty\text{-Cat})_{\geq 1}$, which is isomorphic to the two categories above. ☐

Remark 10.6. The structure of the ∞-category is already determined by the Cartesian structure on the monoid, along with the group homomorphisms.

Proposition 10.7. The map $\mathrm{ev}_1 : \mathrm{Mon}(A_n) \to A_n$ has a left adjoint $\mathrm{Free}^{\mathrm{Mon}}(-)$, defined by $X \mapsto \bigsqcup_{n \geq 0} X^n$.

Proof. A morphism in Δ^{op} is inert if the inclusion of an interval, and is active if it preserves minimum and maximum.

We look at $\mathrm{Fun}(\mathrm{Seg}/\Delta^{\mathrm{op}}_{\mathrm{inert}}(A_n) \cong A_n$ under the evaluation map. This induces an adjoint to $\mathrm{Fun}^{\mathrm{Seg}}(\Delta^{\mathrm{op}}, A_n)$ with left adjoint as a left Kan extension Lan. This functor acts as sending X to the colimit over X, i.e., $\bigcup_{n \geq 0} X^n$. ☐

Corollary 10.8. $\mathrm{Free}^{\mathrm{Mon}}(*) \cong \mathbb{N}$.

The functor $\mathrm{Free}^{\mathrm{Grp}} : A_n \to \mathrm{Grp}(A_n)$ is defined by $X \mapsto \Omega\Sigma(X)_+$. Therefore, $\mathrm{Free}^{\mathrm{Grp}}(*) = \Omega\Sigma(S^0) \cong \Omega(S^1) \cong \mathbb{Z}$.

Example 10.9. We have a pushout diagram

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{N} \arrow[d] \\
* \arrow[r] & \cdot
\end{tikzcd}$$

Then this pushout gives $\mathbb{Z}/2\mathbb{Z}$. Over the functor on ∞-groups, we know the pushout gives $\mathbb{R}P^2$ as in

$$\begin{tikzcd}
SS^1 \arrow[r, "\cdot 2"] \arrow[d] & S^1 \arrow[d] \\
D^2 \arrow[r] & \mathbb{R}P^2
\end{tikzcd}$$

Over the Ω, this is just

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[d] \\
* \arrow[r] & \Omega\mathbb{R}P^1
\end{tikzcd}$$

Example 10.10. $(\mathrm{proj}(R)^{\cong}, \oplus) \in \mathrm{Mon}(\mathrm{Grp}^R) \cong \mathrm{Mon}(A_n)$. The algebraic K-theory space is given by $K(R) = (\mathrm{proj}(R)^{\cong})^{\infty\text{-grp}}$.

Let Γ^{op} be the category of finite set sand partially defined maps, with $\langle n \rangle = \{1, \dots, n\}$.

Definition 10.11. A commutative monoid in $\mathcal{C}$ is a functor $X : \Gamma^{\mathrm{op}} \to \mathcal{C}$ such that

1. it is reduced, i.e., $X_0 \simeq *$,
2. and satisfies the Segal condition, i.e., $\rho_i \langle n \rangle \to \langle 1 \rangle$ is uniquely defined only at i.

Again, this gives $\prod_{i=1}^{n} \rho_i : X_n \to \prod_{i=1}^{n} X_i$ as an isomorphism.

Now, $\mathrm{Free}^{\mathrm{CMon}}(X) = \bigsqcup_{n \geq 0} X^n_{h\Sigma_n}$, $\mathrm{Free}^{\mathrm{CMon}}(*) = \bigsqcup_{n \geq 0} B\Sigma n = \mathrm{Finset}^{\cong}$, and $\mathrm{Free}^{\mathrm{Ab}}(*) = (\mathrm{FinSet}^{\cong})^{\infty\text{-grp}} \cong \mathbb{S}$.

The spectra $\mathbb{S}_p$ of the ∞-category is given by the limits at infinity of $B^{(n)} : \mathrm{Ab}(A_n) \to \mathrm{Ab}(A_n)$, as composing Ω infinitely many times. For Ω, we know there is an adjunction B defined. For the spectra at infinity, there is a corresponding limit at infinity given by $B^{\infty} : \mathrm{Ab}(A_n) \to \mathbb{S}_p$.

Given a ring R, then $HR \in S_p$ and $\mathrm{Mod}_{HR} = D(R)$ is the derived category of R.

Lemma 10.12 (Tate Orbit Lemma)**.** Let X be a bounded below spectrum with C_{p^2}-action, then this gives $(X_{hC_p})^{tC_{p^2}/C_p} \simeq 0$. This is given by $X_{hG} \to X^{hG} \xrightarrow{\text{cofiber}} X^{tG}$ for any finite group G.

11 YONEDA LEMMA (MAY 15, 2023)

Let K be an ∞-cosmos and B be an ∞-category. Consider $\mathfrak{b} : \mathbb{1} \to B$, then this gives $(X : \mathbb{1} \to B) \mapsto \mathbf{Hom}_B(X, \mathfrak{b})$.

In the 1-category setting, for a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, there is the category of elements $\mathbf{El}(F)$ with objects (A, a) for $A \in \mathcal{C}$ and $a \in FA$, and morphisms $(A, a) \to (B, b)$ is a morphism $f : A \to B$ in $\mathcal{C}$ with $(Ff)(b) = a$. We now denote the corresponding functor to be $\mathcal{T} : \mathbf{El}(F) \to \mathcal{C}$. In fact, for any $f : X \to \mathcal{T}(Y, y)$, there exists a unique $\varphi : (X, x) \to (Y, y)$ such that $\mathcal{T}(\varphi) = f$.

Definition 11.1. An arrow $\psi : X \to E$ is a $\mathcal{T}$-Cartesian arrow if it is a trivial fibration as a composition of two pullbacks:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\xi"'] & E^3 \arrow[d, two heads, "\delta"] \\
\cdot \arrow[r] \arrow[d] & B^3 \times_B E \arrow[d, two heads, "\mathcal{T}_{12}"] \\
X \arrow[r, "\psi"] & E^2
\end{tikzcd}$$

Theorem 11.2. The following are equivalent:

1. $\psi : X \to E^2$ is $\mathcal{T}$-Cartesian.
2. The commutative triangle

$$\begin{tikzcd}
& E^2 \arrow[d, "\mathcal{T}"] \\
X \arrow[ur, "\psi"] \arrow[r, "\mathcal{T}\psi"] & \mathbf{Hom}(\mathrm{id}_B, \mathcal{T})
\end{tikzcd}$$

   is an absolute right lifting diagram.

Definition 11.3. A $\mathcal{T}$-Cartesian lift of $\beta : X \to \mathbf{Hom}_B(\mathrm{id}_B, \mathcal{T})$ is a $\mathcal{T}$-Cartesian map $\psi : X \to E^2$ such that

$$\begin{tikzcd}
& E^2 \arrow[d, "\mathcal{T}"] \\
X \arrow[ur, dashed, "\psi"] \arrow[r, "\beta"] & \mathbf{Hom}(\mathrm{id}_B, \mathcal{T})
\end{tikzcd}$$

Definition 11.4. An isofibration $\mathcal{T} : E \to B$ is a carteisan fiber if every $\beta : X \to \mathbf{Hom}_B(\mathrm{id}_B, \mathcal{T})$ has a $\mathcal{T}$-Cartesian lift.

Lemma 11.5. $\mathcal{T} : E \twoheadrightarrow B$ is Cartesian if and only if $\mathrm{id}_{\mathbf{Hom}_B(\mathrm{id}_B, \mathcal{T})}$ has a $\mathcal{T}$-Carteisn lift χ.

Theorem 11.6. The following are equivalent for an isofibration $\mathcal{T} : E \to B$:

1. $\mathcal{T}$ is a Cartesian fibration.
2. $E^2 \to \mathbf{Hom}_B(\mathrm{id}_B, \mathcal{T})$ has a right adjoint.

Proposition 11.7. For any ∞-category B, $\mathcal{T}_0 : B^2 \to B$ is a Cartesian fibration.

12 ∞-Yoneda Lemma and Homotopy Theory

Given $F : X \to Y$ to be a natural transformation between $X, Y : \mathbb{1} \to B$, then we have a lifting

$$\begin{tikzcd}
\mathbf{Hom}(Y,b) \arrow[d, dashed, "F^*"] \arrow[dr] & \\
\mathbf{Hom}(X,b) \arrow[r] \arrow[d] & \mathbf{Hom}(B,b) \arrow[d, two heads, "p"] \\
\mathbb{1} \arrow[r, bend left, "y"] \arrow[r, bend right, "x"'] & B
\end{tikzcd}$$

If $p : \mathbf{Hom}(B,b) \to B$ is Cartesian filtration, these maps exist.

In **Top**, $p : E \twoheadrightarrow B$ is a fibration if we have

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & E \arrow[d, two heads, "p"] \\
X \times [0,1] \arrow[r] \arrow[ur, dashed, "h"] & B
\end{tikzcd}$$

where there exists a lifting h. We then denote the fiber $E_X = p^{-1}(X)$. Observe that given a path $l : [0,1] \to B$ by connecting x to y, then this lifts to a path $l' : [0,1] \to E$ such that $l(0)$ is fixed in E_X and $l(1)$ is fixed in E_Y, then we have

$$\begin{tikzcd}
E_X \arrow[r] \arrow[d] & E \arrow[d, two heads] \\
E_X \times [0,1] \arrow[r, "p"] \arrow[ur, dashed] & B
\end{tikzcd}$$

Apply singular complex everywhere, this makes E_X into a pullback on

$$\begin{tikzcd}
E_X \arrow[r] \arrow[d] & \mathrm{Sing}(E) \arrow[d, two heads] \\
\mathbb{1} \arrow[r, bend left, "y"] \arrow[r, bend right, "x"'] & \mathrm{Sing}(B)
\end{tikzcd}$$

Recall that ∞-category $E \in K$ is discrete if all natural transformations with codomain E are invertible, i.e., for all X, $\mathrm{Fun}(X, E)$ is a Kan complex.

An isofibration $p : E \twoheadrightarrow B$ is a discrete fibration if it is discrete.

Lemma 12.1. $p : E \twoheadrightarrow B$ is discrete if and only if p is conservative in $h(K/B)$. Therefore, if pX is invertible, then so is X.

Lemma 12.2. Discrete fibrations have discrete fibers.