Enriched Categories and Applications

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Overview

Motivation

For objects $A, B \in \mathscr{C}$, we denote $\mathscr{C}(A, B)$ as the collection of morphisms from A to B. Two interesting question:

1. Given a category $\mathscr{C}$, what is the structure of such collections, if any?

Example

- For a locally small category $\mathscr{C}$, the collection $\mathscr{C}(A, B)$ is a set for all objects $A, B \in \mathscr{C}$. Therefore, we can denote it $S = \mathscr{C}(A, B) \in$ **Set**.
- In a pre-additive category $\mathscr{C}$, the collection $\mathscr{C}(A, B)$ always has a structure of abelian group such that the composition is bilinear for all objects $A, B \in \mathscr{C}$. Therefore, we can denote it $G = \mathscr{C}(A, B) \in$ **Ab**.

Enrichment: $\mathscr{C}(A, B)$ "looks like" objects from some category.

2. When is a category enriched over itself?

Example

Ab, **Vect**$_{\mathbb{F}}$ and **R-Mod** are categories enriched over themselves.

Definition (Monoidal Category)

A **monoidal category** $\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)$ consists of a category $\mathcal{V}_0$, a functor $\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$, an object I of $\mathcal{V}_0$ and natural isomorphisms $a_{XYZ} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : I \otimes X \to X$, $r_X : X \otimes I \to X$, that satisfies two coherence axioms expressed by the diagram below:

$$\begin{array}{ccccc} ((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{a} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{a} & W \otimes (X \otimes (Y \otimes Z)) \\ {\scriptstyle a \otimes 1} \downarrow & & & & \uparrow {\scriptstyle 1 \otimes a} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow[a]{\hspace{6em}} & & W \otimes ((X \otimes Y) \otimes Z) \end{array}$$

Figure: Pentagon Rule

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow{a} & X \otimes (I \otimes Y) \\ & {\scriptstyle r \otimes 1} \searrow \quad \swarrow {\scriptstyle 1 \otimes l} & \\ & X \otimes Y & \end{array}$$

Figure: Triangle Rule

Definition (Closed)

A monoidal category $\mathcal{V}$ is **closed** if each functor $- \otimes Y : \mathcal{V}_0 \to \mathcal{V}_0$ has a right adjoint $[Y, -]$. In particular, we have the adjunction

$$\pi : \mathcal{V}_0(X \otimes Y, Z) \cong \mathcal{V}_0(X, [Y, Z])$$

or in terms of the unit-counit adjunction with unit $d : X \to [Y, X \otimes Y]$ and counit $e : [Y, Z] \otimes Y \to Z$. Notice that when $X = I$, by the isomorphism $l : I \otimes Y \cong Y$, we get a natural isomorphism

$$\mathcal{V}_0(Y, Z) \cong V[Y, Z]$$

where V is the functor $\mathcal{V}_0(I, -) : \mathcal{V}_0 \to \mathbf{Set}$. Therefore, the object $[Y, Z]$ acts as a lift of the hom set $\mathcal{V}_0(Y, Z)$, and is called the **internal hom**.

Remark

Just like in basic category theory, we focus our attention on locally small monoidal categories.

Definition ($\mathcal{V}$-enriched Category)

Let $\mathcal{V}$ be a monoidal category, then a $\mathcal{V}$-**enriched category** $\mathcal{A}$ consists of a set of objects ob($\mathcal{A}$), a hom-object $\mathcal{A}(A, B) \in \mathcal{V}_0$ for each pair of objects of $\mathcal{A}$, a composition law $M = M_{ABC} : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ for each triple of objects, and an identity element $j_A : I \to \mathcal{A}(A, A)$ for each object; this is subject to the associativity and unit axioms expressed by the commutativity of two diagrams:

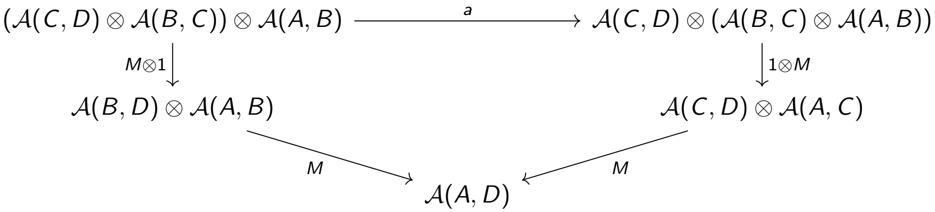


Figure: Associativity on Enriched Category

and

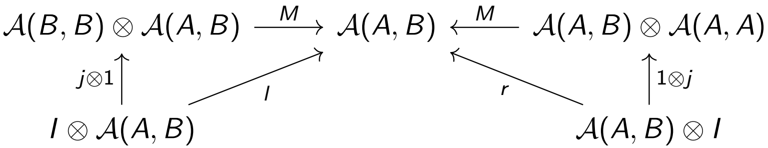


Figure: Unitality on Enriched Category

Examples

- Any pre-additive category can be enriched over **Ab**. In particular, abelian categories are enriched over **Ab**. Moreover, **Ab** is enriched over itself.
- Any (strict) 2-category can be enriched over **Cat**, the category of small categories.
- Any poset category can be enriched over the category of truth values, denoted $\mathbb{2}$.

$$\circlearrowright\ \text{False} \longrightarrow \text{True}\ \circlearrowleft$$

Theorem: Monoidal closed categories are self-enriched

In particular, the structure of $\mathcal{V}(A, B)$ given a monoidal closed category $\mathcal{V}$ and $A, B \in \mathcal{V}$ is $\mathcal{V}(A, B) = [A, B]$, which is the internal hom.

Remark

The converse does not hold in general: observe that **Top** *is a monoidal category that is not closed (the product functor* $- \otimes Y$ *does not commute with colimits, therefore does not have a right adjoint), but it can be self-enriched regardless: consider the structure* $\mathbf{Top}(A, B)$ *as equipped with the trivial topology where every subset is open.*

Example: $([0,1], \leq)$

- Consider $([0,1], \leq)$ as a monoidal (poset) category, with $\leq$ as the usual comparison relation. Again, this is enriched over the category of truth values $\mathbb{2}$.
- The categorical product $\times$ is given by the minimum function $\min(X, Y)$. The terminal object of the poset category is 1. By taking $\otimes = \times$ and $I = 1$, we obtain a (cartesian) monoidal category.
- Is this category closed? i.e. for $X, Y, Z \in [0,1]$, is there a hom object structure $[Y, Z]$ such that there is a natural isomorphism $\mathbf{Hom}(X \otimes Y, Z) \cong \mathbf{Hom}(X, [Y, Z])$?
  Yes, denote $[Y, Z] = \begin{cases} 1, & \text{if } Y \leq Z \\ Z, & \text{if } Y > Z \end{cases}$. This is the Gödel conditional, an operator of Gödel logics.

Example: still $([0,1], \leq)$, but less canonical

- This time, denote $\otimes$ as the usual multiplication operation $\times : [0,1] \times [0,1] \to [0,1]$ on numbers. This is not a categorical product.
- Still take $I = 1$.
- The category is still closed. For $X, Y, Z \in [0,1]$, there is a hom object structure $[Y, Z]$ such that $\mathbf{Hom}(X \otimes Y, Z) \cong \mathbf{Hom}(X, [Y, Z])$. This time, $[Y, Z] = \begin{cases} 1, & \text{if } Y \leq Z \\ \frac{Z}{Y}, & \text{if } Y > Z \end{cases}$. This is "the product structure".

Application: Fuzzy Logic

Suppose a group of students are taking Putnam. Suppose A, B and C scores $10, 50$ and 100 respectively. We may consider a fuzzy preorder $\preccurlyeq$ as "at most as good as".

What can we say about the relative performance between the three individuals using fuzzy logic? Intuition: $C \preccurlyeq A$ should be very close to 0, $B \preccurlyeq A$ should be less closer to 0, $C \preccurlyeq B$ should be much closer to 1.

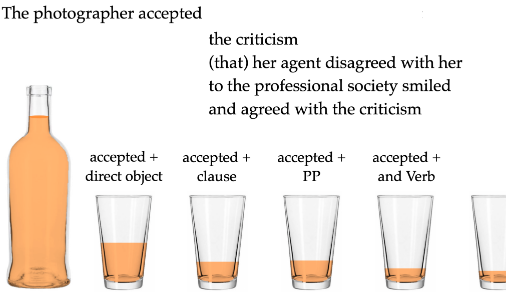
Let $\mathscr{C} = (A, B, C)$ form a preorder category with respect to $\preccurlyeq$, then the category can be considered as "enriched" over $([0,1], \leq)$ with $\mathscr{C}(X, Y) = \begin{cases} 1, & \text{if } X \leq Y \\ \frac{\text{score}(Y)}{\text{score}(X)}, & \text{if } X > Y \end{cases}$. In particular, we have $\mathscr{C}(C, A) = \frac{10}{100} = 0.1$, $\mathscr{C}(B, A) = \frac{10}{50} = 0.2$ and $\mathscr{C}(C, B) = \frac{50}{100} = 0.5$,

This can be quantified ("enriched") over the category $([0,1], \leq)$.

Application: Language Model

From the perspective of psycholinguistics,

- language prediction starts almost immediately after the speech starts! i.e. **we start to predict what others are going to say as soon as they start talking.**
- this prediction is **ranked parallel** instead of all-or-none, i.e. our brain evaluates the most possible follow-up dynamically instead of just picking the suggestion with the highest probability.



Levy (2008), *Cognition*

Application: Language Model

Bradley and Vlassopoulos (2021) described "a way of assigning to any word or expression s in a language a particular linear operator ρ_s." From the perspective of the Yoneda lemma, the "meaning" of the word s is described by ρ_s, as the operator captures the environment and hence the meanings of word/expression.

Finally, they showed that the assignment $s \mapsto \rho_s$ is a functor between categories enriched over probabilities, i.e. $([0,1], \leq)$. More precisely, this is an enriched functor from language (as a preorder category enriched over probabilities) to the category of those associated operators enriched over probabilities.

References

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