No Nerves of Steel Notes

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Abstract

In tensor-triangulated geometry, the Nerves of Steel conjecture relates the Balmer spectrum of a tensor-triangulated category with its homological spectrum counterpart. Evidence leaning towards this conjecture dates back to the nilpotence theorem due to Devinatz-Hopkins-Smith. I will talk about the recent work of Barthel-Hyslop-Ramzi that constructed a counterexample of this conjecture using free constructions in higher Zariski geometry.

Throughout, a tt-category (tensor-triangulated category) means an idempotent-complete rigid symmetric monoidal stable ∞-category $\mathscr{C}$. "Rigid" means every compact object is dualizable.

1 Review of tt-geometry

Definition 1. A prime (thick tensor) ideal $\mathcal{P} \subsetneq \mathscr{C}$ is a proper thick $\otimes$-ideal such that $X \otimes Y \in \mathcal{P}$ implies $X \in \mathcal{P}$ or $Y \in \mathcal{P}$.

The Balmer spectrum is

$$\mathrm{Spc}(\mathscr{C}) \ = \ \{\, \text{prime ideals } \mathcal{P} \subseteq \mathscr{C} \,\},$$

topologized so that $\mathrm{supp}(X) = \{\mathcal{P} \mid X \notin \mathcal{P}\}$ is closed for any object $X \in \mathscr{C}$.

Example 2 (Neeman-Thomason). $\mathrm{Spc}(\mathrm{Perf}(X)) \cong X$ for a qcqs scheme X.

The Balmer spectrum parallels the Zariski spectrum not only by construction; it is also equipped with natural maps. Alongside $\mathrm{Spc}(\mathscr{C})$, Balmer constructs the homological spectrum $\mathrm{Spc}^h(\mathscr{C})$ in parallel, c.f., [Bal20b].

Definition 3. Let $\mathscr{C}$ be a (big) tt-category. We define $\mathrm{mod}(\mathscr{C})$ to be the full subcategory of the 1-category of additive presheaves $\mathrm{Mod}(\mathscr{C}) = \mathrm{Fun}^{\oplus}(\mathscr{C}^{\mathrm{op}}, \mathrm{Ab})$ generated by cokernels of $\text{よ}(x) \to \text{よ}(y)$ for $x, y \in \mathscr{C}$. This is a Grothendieck abelian category that inherits a symmetric monoidal structure through Day convolution, such that the restriction to the Yoneda embedding $\text{よ} : \mathscr{C} \to \mathrm{Mod}(\mathscr{C})$ is symmetric monoidal, taking $x \in \mathscr{C}$ to $\text{よ}(x) = \mathrm{Hom}(-, x)$. In this sense, we think of $\mathrm{mod}(\mathscr{C})$ as given by finitely-presented modules.

The homological spectrum of a tt-category $\mathscr{C}$ is the set $\mathrm{Spc}^h(\mathscr{C})$ of maximal Serre $\otimes$-ideals in $\mathrm{mod}(\mathscr{C})$. This is originally defined so that we may define a notion of residue fields on tt-categories, in which case is the quotient, which is an abelian category. To each point $x \in \mathrm{Spc}^h(\mathscr{C})$, we have an associated functor

$$h_x : \mathscr{C} \xrightarrow{\text{よ}} \mathrm{Mod}(\mathscr{C}) \twoheadrightarrow \mathrm{Mod}(\mathscr{C})/\mathrm{Loc}(x)$$

Note that this functor lands in mod.

Again, this can be topologized so that $\mathrm{supp}^h(x) = \{\mathfrak{m} \in \mathrm{Spc}^h(\mathscr{C}) : \text{よ}(x) \notin \mathfrak{m}\}$ is closed for any $x \in \mathscr{C}$.

Theorem 4 ([Bal20b]). There is a natural surjection

$$\phi \colon \mathrm{Spc}^h(\mathscr{C}) \twoheadrightarrow \mathrm{Spc}(\mathscr{C}),$$

that preserves the support, and is given by the preimage of the Yoneda embedding. Moreover, every homological prime in $\mathrm{Spc}^h(\mathscr{C})$ is detected by an abelian residue field.

Note that it is unclear what a residue field on the Balmer spectrum should look like as of now, so interpreting the result above as relating the homological spectrum with the Balmer spectrum, the abelian residue field in the homological spectrum becomes a crucial substitute that can be traced back to the primes.

Remark. The definition above can be generalized to any (essentially small) symmetric monoidal stable ∞-category. In that case, the natural map still exists, but it is not necessarily a surjection.

Conjecture 5 ([Bal20b], Nerves of Steel). For any (rigid) tt-category $\mathscr{C}$, the comparison map

$$\phi \colon \mathrm{Spc}^h(\mathscr{C}) \twoheadrightarrow \mathrm{Spc}(\mathscr{C})$$

is a bijection, hence a homeomorphism.

The conjecture is known in all computed examples and enjoys strong permanence properties (descent, etc.), e.g., see discussion before [BHR26, Theorem A].

Definition 6. Let $\mathscr{C}$ be a tt-category.

1. We say a map $f : X \to Y$ in $\mathscr{C}$ is $\otimes$-nilpotent if there exists some $n \geq 0$ with $f^{\otimes n} \simeq 0$.

2. We say $\mathscr{C}$ is local if for all objects $X, Y \in \mathscr{C}$, $X \otimes Y \simeq 0$ implies $X \simeq 0$ or $Y \simeq 0$.

   Remark. Equivalently, 0 is a prime ideal in $\mathscr{C}$, or that $\mathrm{Spc}(\mathscr{C})$ has a unique closed point. One should think of the locality condition like Nakayama's lemma.

Lemma 7 ([Bal20b], Theorem 1.1). If $f : x \to y$ is a map in $\mathscr{C}$ with $h_p(f) = 0$ for all $p \in \mathrm{Spc}^h(\mathscr{C})$, then f is $\otimes$-nilpotent.

We want to think of $\mathrm{Spc}^h(\mathscr{C})$ as the universal space for detecting nilpotence in $\mathscr{C}$. In the presence of the surjection above, we hope ϕ would witness a correspondence between nilpotence theorems and thick ideal theorems.

The first ever computation that supports this conjecture would be from Nilpotence theorem due to Devinatz-Hopkins-Smith. This motivates [Bal20b, Theorem 5.6] as a criterion for bijectivity.

Example 8. Let SH be the stable homotopy category and $K = \mathrm{SH}^c$ the stable homotopy category of finite spectra. Then $\phi : \mathrm{Spc}^h(K) \to \mathrm{Spc}(K)$ is a bijection.

The Balmer spectrum consists of points $P(p, n)$ for each prime number p and each chromatic height $1 \leq n \leq \infty$, with the collision $P(0) := P(p, 0) = \mathrm{SH}^{\mathrm{tor}}$ for all p. This prime $P(0)$ is the kernel of rational homology $H\mathbb{Q} \otimes - :$ $\mathrm{SH}^c \to \mathcal{D}^b(\mathbb{Q}) \cong \mathbb{Q}\text{-Mod}_*$. The other primes $P(p, n)$ for $n \geq 1$ are given as the kernels of Morava K-theory $K(p, n)_*$, which are homological functors

$$K(p, n)_* : \mathrm{SH} \to \mathbb{F}_p[v_n^{\pm 1}]\text{-GrMod}$$

for $1 \leq n < \infty$ with v_n of degree $2(p^n - 1)$, and $K(p, \infty)_* : \mathrm{SH} \to \mathbb{F}_p\text{-GrMod}$ is mod-p homology $H\mathbb{Z}/p$.

In this case, the homological primes corresponds to the kernels of the homological functors above, because the family of homological functors above detect nilpotence: if the image of a map f is zero via all functors, then f is $\otimes$-nilpotent, c.f., Lemma 7.

Our main tools for proving the counterexample will be based on [Hys25]. since our categories are rigid, argument of a morphism $a \to b$ often times boil down to a map of the form $\mathbb{1} \to x$. This motivates the following conditions.

Definition 9. A local tt-category $\mathscr{C}$ satisfies the exact-nilpotence condition if for every fiber sequence

$$Y \xrightarrow{g} \mathbb{1} \xrightarrow{f} X,$$

there exists $0 \neq Z \in \mathscr{C}$ such that either $Z \otimes f$ or $Z \otimes g$ is $\otimes$-nilpotent.

Theorem 10 ([Bal20a], Theorem A.1; [Hys25], Theorem 1.6). The Nerves of Steel Conjecture holds if and only if the exact-nilpotence condition holds for every local tt-category $\mathscr{C}$.

Proof Idea. Consider the pullback diagram

$$
\begin{array}{ccc}
\operatorname{Spc}^h(\mathscr{C}/\mathcal{P}) & \longrightarrow & \operatorname{Spc}^h(\mathscr{C}) \\
\downarrow & & \downarrow \\
\operatorname{Spc}(\mathscr{C}/\mathcal{P}) & \longrightarrow & \operatorname{Spc}(\mathscr{C})
\end{array}
$$

we note that it suffices to prove the local case. Let p be a homological prime over (0). The Nerves of Steel conjecture shows that p is unique in this sense. In the local case, this corresponds to having a unique choice of Z as above by Lemma 7.

The standard proof requires understanding [Bal20a, Theorem A.1], which reduces the conjecture to the following statement: in any local tt-category, for all morphisms $f\colon \mathbb{1} \to X$ and $g\colon \mathbb{1} \to Y$ with $f \otimes g \simeq 0$, there exists $z \neq 0$ such that $z \otimes f$ or $z \otimes g$ is $\otimes$-nilpotent. $\qquad\square$

Remark. In particular, for a local tt-category, the proof above suggests that the Nerves of Steel holds if and only if the exact-nilpotence condition is satisfied.

The exact-nilpotence condition is known for a large family of local tt-categories, c.f., [Hys25, Theorem 3.15]. The class is also closed under filtered colimits with local transition maps, c.f., [Hys25, Theorem 3.14].

2   Free Constructions in 2-Rings

Before the rigid case, it is instructive to see how the exact-nilpotence condition can fail without rigidity. This is studied in [Hys25, Section 2]: since the symmetric monoidal stable ∞-categories we work with are essentially small and idempotent complete (as tt-categories usually are), we have an identification of the ∞-category of symmetric monoidal stable ∞-categories with the category $2\text{-}\mathrm{Ring} = \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$, which is the same as $2\,\mathrm{CAlg}$ in [ABC$^+$25].

Construction. Suppose $\mathcal{O}^\otimes$ is an (essentially small) ∞-operatd and let $\mathscr{C}$ be a symmetric monoidal stable ∞-category $\mathscr{C}$. The goal of a free construction is to get a universal symmetric monoidal stable ∞-category $\mathscr{C}[\mathcal{O}]$ with a symmetric monoidal exact functor from $\mathscr{C}$, such that for any $\mathscr{C}$-algebra $\mathscr{D} \in 2\text{-}\mathrm{Ring}_{\mathscr{C}/}$, there is an equivalence of ∞-categories

$$
\mathrm{Fun}_{2\text{-}\mathrm{Ring}_{\mathscr{C}/}}(\mathscr{C}[\mathcal{O}], \mathscr{D}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathscr{D})
$$

between symmetric monoidal exact $\mathscr{C}$-linear functors $\mathscr{C}[\mathcal{O}] \to \mathscr{D}$ and $\mathcal{O}$-algebra objects in $\mathscr{D}$.

Note that a pointed object $\mathbb{1} \to x$ in $\mathscr{C}$ can be viewed as algebra objects over the ∞-operad $\mathrm{Poi}^\otimes \simeq \mathbb{E}_0^\otimes$. The underlying ∞-category is a wide subcategory of Fin_*. The takeaway, being, its monoidal envelope $\mathrm{Env}(\mathbb{E}_0)$ can be identified with the category $\mathrm{Fin}^{\mathrm{inj}}$ of finite sets and injective maps between them.

Lemma 11 ([Hys25], Lemma 2.1)**.** The free symmetric monoidal category on an object X is $\mathrm{Fin}^{\simeq}$ of finite sets with bijective maps, and the free symmetric monoidal category on an object X equipped with a map $\mathbb{1} \to X$ is $\mathrm{Fin}^{\mathrm{inj}}$ of finite sets with injective maps, and the induced map sending X to X is the canonical inclusion $\mathrm{Fin}^{\simeq} \to \mathrm{Fin}^{\mathrm{inj}}$.

We need to first check out the free construction in tt-geometry, dropping the rigidiy assumptions for now. We will see that the construction is analogous, but far more complicated in that case. For now, note that there is an adjunction $\mathrm{Ind}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \rightleftarrows \mathrm{Pr}_{\mathrm{st}}^L \colon (-)^\omega$, which gives an equivalence $\mathrm{Cat}_\infty^{\mathrm{perf}} \cong \mathrm{Pr}_{\mathrm{st}}^{L,\omega}$, therefore we get an equivalence between $2\text{-}\mathrm{Ring}$ and $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{L,\omega})$. We see that the following is the natural free construction.

Construction ([Hys25], Lemma 2.2)**.** The free symmetric monoidal stable ∞-category over $\mathscr{C}$ with an $\mathcal{O}$-algebra is the presheaf category $\mathrm{Fun}(\mathrm{Env}(\mathcal{O})^{\mathrm{op}}, \mathrm{Ind}(\mathscr{C}))^\omega$ with Day convolution.

Definition 12. The free idempotent-complete symmetric monoidal stable ∞-category over such an (idempotent-complete symmetric monoidal stable) ∞-category $\mathcal{T}$ is the category

$$
\mathrm{Fun}((\mathrm{Fin}^{\mathrm{inj}})^{\mathrm{op}}, \mathrm{Ind}(\mathcal{T}))^\omega
$$

of compact objects in $\mathrm{Fun}((\mathrm{Fin}^{\mathrm{inj}})^{\mathrm{op}}, \mathrm{Ind}(\mathcal{T}))$, equipped with the Day convolution tensor product.

With this in mind, given a small symmetric monoidal ∞-category $\mathscr{C}$, we define

- the non-rigid affine line $\mathscr{C}[X] := \operatorname{Fun}((\operatorname{Fin}^{\simeq})^{\operatorname{op}}, \operatorname{Ind}(\mathscr{C}))^{\omega}$ is the free symmetric monoidal stable ∞-category on an object X.

- the free pointed object $\mathscr{C}[\operatorname{Poi}] := \operatorname{Fun}((\operatorname{Fin}^{\operatorname{inj}})^{\operatorname{op}}, \operatorname{Ind}(\mathscr{C}))^{\omega}$ is the free symmetric monoidal stable ∞-category on a pointed object $\mathbb{1} \to X$ for $X \in \mathscr{C}$. Equivalently, this is the free (non-rigid) commutative 2-ring on a pointed object.

Theorem 13 ([Hys25], Theorem 2.5). *If $\mathscr{C}$ is local, then $\mathscr{C}[\operatorname{Poi}]$ is local but the exact-nilpotence condition fails.*

Proof sketch. The natural map gives a restriction functor $\operatorname{Res}: \mathscr{C}[\operatorname{Poi}] \to \mathscr{C}[X]$. This is the right adjoint of a symmetric monoidal functor which classifies the free object of $\mathscr{C}[\operatorname{Poi}]$, so it is lax symmetric monoidal, and in this case upgrades to a monoidal functor.

By [Hys25, Proposition 2.6], $\mathscr{C}[X]$ is local: this requires understanding

$$\mathscr{C}[X] \cong \coprod_{n \geqslant 0} \operatorname{Fun}(B\Sigma_n, \operatorname{Ind}(\mathscr{C}))^{\omega},$$

as we can check levelwise and unpack the Day convolution. Since Res is faithful on objects, $\mathscr{C}[\operatorname{Poi}]$ is local as well.

Given a family of compact generators $\{c_i\}_{i \in I}$ of $\mathscr{C}$, we get a family of compact generators $\{ょ[n] \otimes c_i\}_{i \in I, n \in \mathbb{N}}$. In particular, properties of $\mathscr{C}[\operatorname{Poi}]$ can just be checked on these test objects.

Note that $ょ[0] \to ょ[1]$ in $\mathscr{C}[\operatorname{Poi}]$ restricts to a split map in $\mathscr{C}[X]$ via this restriction as $\operatorname{Res}(ょ[0]) = ょ[0] = \mathbb{1}$ and $\operatorname{Res}(ょ[1]) = ょ[0] \oplus ょ[1]$, so the free map $\mathbb{1} \to ょ([1])$ cannot be $\otimes$-nilpotent after nonzero tensor. Taking the involution $\mathscr{C}^{\operatorname{op}}[\operatorname{Poi}] \simeq \mathscr{C}[\operatorname{Poi}]^{\operatorname{op}}$, the fiber of $\mathbb{1} \to ょ([1])$ also cannot be $\otimes$-nilpotent after nonzero tensor. $\square$

This does not disprove the conjecture, since $\mathscr{C}[\operatorname{Poi}]$ is not rigid. In the rigid case, the analogous free construction is far more complicated.

Before doing anything, we claim that understanding the rigid counterpart of these free constructions should solve our problem. Indeed, if the conjecture holds for such a category $\mathscr{C}$, then for any local tt-category $\mathscr{D}$ with a testing fiber sequence $Y_D \to \mathbb{1}_D \to X_D$, there is an exact $\otimes$-functor $F : \mathscr{C} \to \mathscr{D}$ sending the universal pointed object $\mathbb{1}_{\mathscr{C}} \to X$ to $\mathbb{1}_{\mathscr{D}} \to X$. Because $\mathscr{D}$ is local, then $\ker(F)$ is a prime ideal, and thus $\mathbb{1}_D \to X_D$ factors over $\mathscr{C}/\ker(F)$. If the Nerves of Steel holds for $\mathscr{C}$, then the exact-nilpotence condition holds for $\mathscr{C}/\ker(F)$, thus there is some $0 \neq Z \in \mathscr{C}/\ker(F)$ as assumed, so that $F(Z) \in \mathscr{D}$ has the exact-nilpotence condition.

The first step, therefore has to be describing what the universal category is. The toy rigid analogue is the following:

Definition 14 (Cobordism Hypothesis, Lurie-Harpaz). The free rigid symmetric monoidal ∞-category on an object is the 1-dimensional oriented cobordism category $\operatorname{Cob} = \operatorname{Cob}^{1d,\operatorname{or}}$, with

- objects: oriented closed 0-manifolds (finite sets of signed points $\pm$);

- morphisms from M to N: compact oriented 1-manifolds W with $\partial W \cong \overline{M} \sqcup N$.

This is equipped with a Day convolution structure. Denote by $X^{i,j} := X^{\otimes i} \otimes X^{\vee, \otimes j}$ the generators, where $X = +$ is the universal dualizable object.

Note that this is not the right universal model, as we would need to include more bordisms to show its pointed nature.

Proposition 15 (Key structural facts about Cob). For natural numbers i, j, r, s:

1. $\operatorname{Hom}_{\operatorname{Cob}}(X^{i,j}, X^{r,s}) = \varnothing$ unless $i - j = r - s$ (orientation conservation);

2. $\operatorname{Hom}_{\operatorname{Cob}}(X^{i,j}, X^{r,s})$ is free as a module over $\operatorname{Hom}_{\operatorname{Cob}}(\mathbb{1}, \mathbb{1}) \simeq \operatorname{Free}_{E_{\infty}}(BS^1)$ on the set of simply-connected cobordisms.

Working rationally (over $\mathcal{D}(\mathbb{Q})$), we simplify to the following.

Notation. The (rational) affine line is

$$\mathbb{A}^1 := \operatorname{Fun}(\operatorname{Cob}^{\operatorname{op}}, \mathcal{D}(\mathbb{Q}))^{\omega},$$

with Day convolution monoidal structure. This is the free rational rigid 2-ring on an object. Generators are denoted $X^{i,j}$ via the Yoneda embedding $ょ : \operatorname{Cob} \to \mathbb{A}^1$, then it is generated under finite colimits, desuspensions and retracts by images of $ょ$.

Remark. This is also the free rigid stably symmetric monoidal ∞-category over $\mathcal{D}^b(\mathbb{Q})$ on an object. To get the description above, one just needs to follow the free 2-ring functor constructed in [ABC$^+$25, Construction 2.7, Definition 2.11]. In that setting, this is exactly the affine line of higher zariski geometry.

Lemma 16. The graded endomorphism ring of the unit satisfies:
$$\pi_*(\mathbb{1}_{\mathbb{A}^1}) \;\cong\; \mathbb{Q}[t_0, t_1, t_2, \ldots], \quad |t_i| = 2i.$$

In particular, $\mathrm{Spc}(\mathbb{A}^1)$ is large: it surjects onto $\mathrm{Spec}^h(\mathbb{Q}[t, t_1, t_2, \ldots])$.

Notation. Let $\mathbb{A}^1_\eta$ be the localization/stalk of $\mathbb{A}^1$ at "the generic point": the quotient of $\mathbb{A}^1$ by the thick $\otimes$-ideal generated by $\mathrm{cofib}(a : \Sigma^i \mathbb{1} \to \mathbb{1})$ for all nonzero homogeneous $a \in \pi_i(\mathbb{1}_{\mathbb{A}^1})$.

Lemma 17 ([BHR26], Lemma 4.1). There is an isomorphism of graded rings:
$$\pi_* \mathrm{End}_{\mathbb{A}^1}(X^{i,j}) \;\cong\; \pi_*(\mathbb{1}_{\mathbb{A}^1}) \otimes_{\pi_0(\mathbb{1}_{\mathbb{A}^1})} \mathrm{End}_{\mathrm{Rep}(GL_t)}(X^{i,j}),$$

where $\mathrm{Rep}(GL_t)$ is Deligne's interpolation category. At the generic point:
$$\pi_0 \mathrm{End}_{\mathbb{A}^1_\eta}(X^{i,j}) \;\cong\; B_{i,j}(t),$$

the walled Brauer algebra with parameter t over the field $\mathbb{Q}(t)$.

Theorem 18 ([BHR26], Theorem 4.6). $\mathbb{A}^1_\eta$ is semisimple with simple unit. In particular, $\mathbb{A}^1_\eta$ is a tt-field[1], and $\mathrm{Spc}(\mathbb{A}^1_\eta) = *$ is a single point.

Proof sketch. $\mathbb{A}^1_\eta$ is generated by $X^{i,j}$. By semisimplicity of $B_{i,j}(t)$, c.f., [BHR26, Theorem A.2], and periodicity, every object of $\mathrm{ho}(\mathbb{A}^1_\eta)$ has a semisimple endomorphism ring. Any map between objects with semisimple endomorphism rings is isomorphic to $\mathrm{id} \oplus 0$, so the subcategory of such objects is closed under (co)fibers. Hence $\mathbb{A}^1_\eta$ itself is semisimple. $\square$

3   The Disproof

Notation. Let Cob^+ denote the free rigid symmetric monoidal ∞-category on a pointed dualizable object $\mathbb{1} \to X$. More rigorously, this is allowing half-open intervals $\varnothing \to +$ with positive orientation on the endpoint in the cobordism category. The category Cob^+ is more intricate than Cob: objects of Cob^+ include cobordisms that "remember" the basepoint, and the pointing introduces extra morphisms ("arcs from/to the basepoint"). In particular, it is the pushout of

$$
\begin{array}{ccc}
\mathrm{Fin}^{\simeq} & \longrightarrow & \mathrm{Cob} \\
\downarrow & & \downarrow \\
\mathrm{Fin}^{\mathrm{inj}} & \longrightarrow & \mathrm{Cob}^+
\end{array}
$$

The free rational rigid 2-ring on a pointed object is:
$$\mathbb{A}^{1,+} \;:=\; \mathrm{Fun}((\mathrm{Cob}^+)^{\mathrm{op}}, \mathcal{D}(\mathbb{Q}))^\omega.$$

There is a canonical functor $\mathrm{can} : \mathbb{A}^{1,+} \to \mathbb{A}^1$ taking the universal pointing $\mathbb{1} \xrightarrow{f} X$ to the zero map $\mathbb{1} \xrightarrow{0} X$ in $\mathbb{A}^1$.

Remark. Note that this category is not local; nevertheless, if we can understand its localization, we would be able to produce a counterexample of the conjecture.

Let $\mathbb{A}^{1,+}_\eta$ be the localization of $\mathbb{A}^{1,+}$ at the generic point (same construction as for $\mathbb{A}^1_\eta$), then the canonical functor above induces an equivalence on endomorphism rings of the unit, and so we may invert all endomorphisms of the unit and get an induced functor of 2-rings, denoted $\mathrm{can}_\eta : \mathbb{A}^{1,+}_\eta \to \mathbb{A}^1_\eta$.

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[1]That is, every object is a coproduct of compact objects and such that tensoring with any object is faithful. A tt-field F must have a minimal spectrum: $\mathrm{Spc}(F^c) = *$. In other words, every non-zero object generates the whole category. This matches the intuition that a field should be very small.

Remark. The Balmer spectra of $\mathbb{A}^1$ and $\mathbb{A}^{1,+}$ are both very large (not local). To study the exact-nilpotence condition, we must pass to localizations: the universal candidates are $\mathbb{A}^{1,+}_\eta/\mathcal{P}$ for $\mathcal{P} \in \mathrm{Spc}(\mathbb{A}^{1,+})$.

Lemma 19 ([BHR26], Lemma 5.6). In $\mathbb{A}^{1,+}_\eta$, the hom spectrum $\mathrm{Hom}_{\mathbb{A}^{1,+}_\eta}(X^{i,j}, X^{r,s})$ is nonzero only when $r - s \geqslant i - j$.
Moreover, if $r - s = i - j$, the map induced by can_η,

$$\mathrm{Hom}_{\mathbb{A}^{1,+}_\eta}(X^{i,j}, X^{r,s}) \;\xrightarrow{\sim}\; \mathrm{Hom}_{\mathbb{A}^1_\eta}(X^{i,j}, X^{r,s}),$$

is an equivalence.

Proof. The hom spectra are base-changes of $\mathbb{Q}$-homologies of mapping spaces in Cob^+. In Cob^+, a cobordism from $X^{i,j}$ to $X^{r,s}$ must send the marked point to something, and an orientation count forces $r - s \geqslant i - j$. When $r - s = i - j$, can_η has a section (the canonical section $\mathbb{A}^1 \to \mathbb{A}^{1,+}$ sending $X \mapsto X$ forgetting the pointing), and one checks this section is an equivalence on these hom pieces. $\qquad\square$

This lemma provides a canonical grading on $\mathbb{A}^{1,+}_\eta$ by the "degree" $i - j$, with $X^{i,j}$ in degree $i - j \geqslant 0$. The map can_η is an equivalence on the degree-0 piece (the "flat" part).

Theorem 20 ([BHR26], Proposition 5.1). The functor $\mathrm{can}_\eta \colon \mathbb{A}^{1,+}_\eta \to \mathbb{A}^1_\eta$ is conservative.

Proof Idea. By the Schwede–Shipley theorem, it suffices to show that base-change along

$$\mathrm{End}_{\mathbb{A}^{1,+}_\eta}\Big(\bigoplus_{i+j \leqslant n} X^{i,j} \Big) \;\longrightarrow\; \mathrm{End}_{\mathbb{A}^1_\eta}\Big(\bigoplus_{i+j \leqslant n} X^{i,j} \Big)$$

is conservative for each n, and then pass to the colimit. This is equivalent to checking the increasing union of the thick subcategories generated $\mathbb{X}^{i,j}$ for $i + j \leqslant n$ and taking $n \to \infty$, getting the colimit $\mathbb{A}^{1,+}_\eta$. $\qquad\square$

Corollary 21 ([BHR26], Corollary 5.2). The universal test 2-ring $\mathbb{A}^{1,+}_\eta$ is local.

Proof. By Theorem 18, $\mathbb{A}^1_\eta$ is local. Any category admitting a conservative symmetric monoidal functor to a local category is itself local. $\qquad\square$

Construction. There is an 2-ring involution and autoequivalence $\iota \colon \mathbb{A}^{1,+} \xrightarrow{\sim} \mathbb{A}^{1,+}$ induced at the level of cobordism categories by the symmetry that reverses orientation and swaps evaluation/coevaluation. Concretely, ι sends the universal pointing $f \colon \mathbb{1} \to X$ to (a shift of) $g^\vee \colon \mathbb{1} \to Y^\vee$, where $g = \mathrm{fib}(f)$. This is unique up to a contractible space of choices.

This involution descends to $\mathbb{A}^{1,+}_\eta$ and will allow us to assume that g is the relevant map in the proof below.

Theorem 22 ([BHR26], Theorem 5.7). The category $\mathbb{A}^{1,+}_\eta$ is local, and the exact-nilpotence condition fails for $\mathbb{A}^{1,+}_\eta$, witnessed by the universal fiber sequence

$$Y \xrightarrow{g} \mathbb{1} \xrightarrow{f} X.$$

In particular, $\phi \colon \mathrm{Spc}^h(\mathbb{A}^{1,+}_\eta) \twoheadrightarrow \mathrm{Spc}(\mathbb{A}^{1,+}_\eta)$ is not injective, and the Nerves of Steel Conjecture is false.

Proof. Locality was established above. Suppose for contradiction that there exists $0 \neq z \in \mathbb{A}^{1,+}_\eta$ such that either $z \otimes f$ or $z \otimes g$ is $\otimes$-nilpotent. Applying the involution ι if necessary, we may assume $z \otimes g$ is $\otimes$-nilpotent (since ι swaps f and g up to duality, and $\otimes$-nilpotence is preserved by symmetric monoidal functors and duality). Since can_η is symmetric monoidal, we know $\mathrm{can}_\eta(z) \otimes \mathrm{can}_\eta(g)$ is $\otimes$-nilpotent in $\mathbb{A}^1_\eta$. As can sends the universal pointing $f \colon \mathbb{1} \to X$ to the zero map $\mathbb{1} \xrightarrow{0} X$ in $\mathbb{A}^1$, then $\mathrm{can}_\eta(g) = \mathrm{fib}(\mathbb{1} \xrightarrow{0} X)$, which is $\mathrm{fib}(\mathbb{1} \xrightarrow{0} X) \simeq \Omega X \oplus \mathbb{1}$ in $\mathbb{A}^1_\eta$. The map $g = (\mathrm{fib}(\mathbb{1} \to X) \to \mathbb{1})$ becomes the projection $\Omega X \oplus \mathbb{1} \to \mathbb{1}$, which is split surjective. Since $\mathrm{can}_\eta(z) \neq 0$ (conservativity of can_η shows that it reflects zero objects, but $z \neq 0$), and $\mathrm{can}_\eta(g)$ is split surjective, $\mathrm{can}_\eta(z) \otimes \mathrm{can}_\eta(g)$ is also split surjective to target $\mathrm{can}_\eta(z) \neq 0$. But like in Theorem 13, split surjection can only be $\otimes$-nilpotent if the target is zero, we have a contradiction. $\qquad\square$

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