Motivic Homotopy Theory Notes

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These notes were taken from a course on Motivic Homotopy Theory taught by Dr. P. Du in Spring 2024 at BIMSA. Any mistakes and inaccuracies would be my own. References for this course include [BH21], [EH23], [Lur18], [Lur09], and others mentioned in the references.

Contents

1 Commutative Monoids and Commutative Semirings as Functors

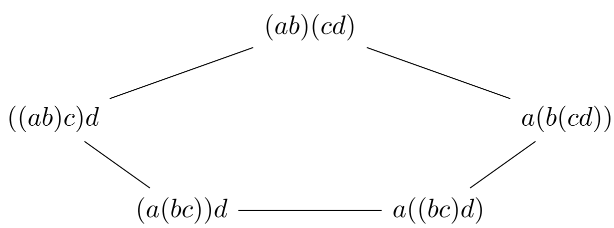
The materials from this section can be found in [EH23], Chapter 1.1-1.2.

1.1 Spans and Monoids

Definition 1.1. A commutative monoid $(M, \times, 1)$ has a multiplication operation

$$\begin{aligned} \times : M \times M &\to M \\ (a, b) &\mapsto a \times b =: ab \end{aligned}$$

that satisfies $ab = ba$, as well as the associativity by the pentagon axiom



Definition 1.2. Denote $\mathbb{F} = \mathbf{FinSet}$ to be the finite category of finite sets, then a commutative monoid M induces a contravariant functor

$$\begin{aligned} \bar{M} : \mathbb{F}^{\text{op}} \to& \mathbf{Set} \\ I \mapsto& M^I \\ (I \xleftarrow{f} S) \mapsto& (M^I \xrightarrow{f^*} M^S) \\ & (a_i)_{i \in I} \mapsto (a_{f(s)})_{s \in S} \end{aligned}$$

and similarly a covariant functor

$$\begin{aligned} \bar{M}' : \mathbb{F} \to& \mathbf{Set} \\ I \mapsto& M^I \\ (s \xrightarrow{g} I) \mapsto& (M^S \xrightarrow{g_\otimes} M^I) \\ & (b_s)_{s \in S} \mapsto \left(\prod_{s \in g^{-1}(j)} b_s \right)_{j \in J} \end{aligned}$$

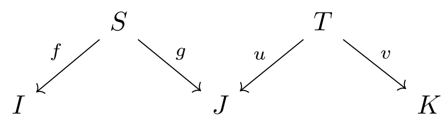
Now given the construction in Definition 1.2 above, suppose we have a zigzag

$$I \xleftarrow{f} S \xrightarrow{g} J \tag{1.3}$$

we can use $\bar{M}$ and $\bar{M}'$ and obtain f^* and $g_\otimes$. One can map Diagram 1.3 to a morphism $g_\otimes f^* : M^I \to M^J$.

Remark 1.4. To define a functor precisely, we need to specify what category Diagram 1.3 lies in. As we will see later, we want a category with the same objects as $\mathbb{F}$, and morphisms are the zigzags of the form Diagram 1.3, which are called spans (or correspondences).

To define the composition of spans as morphisms, we should think of a diagram

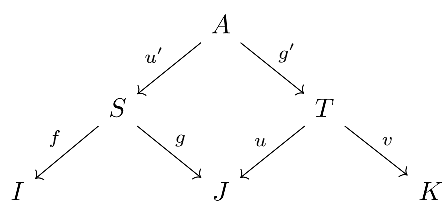


(1.5)

The two zigzags give rise to $g_\otimes f^*$ and $v_\otimes u^*$. For compositions to be well-defined, we should map this diagram to $v_\otimes u^* g_\otimes f^*$. In order to obtain functoriality, we would hope

$$v_\otimes u^* g_\otimes f^* = v_\otimes g_\otimes u^* f^* = (vg)_\otimes (fu)^*$$

using some sort of base-change phenomenon. This is certainly not true. As a remedy, we complete Diagram 1.5 to



(1.6)

as we obtain $u^* g_\otimes : M^S \to M^T$ defined by the composition

$$(b_s)_{s\in S} \mapsto \left(\prod_{s\in g^{-1}(j)} b_s\right)_{j\in J} \mapsto \left(\prod_{s\in g^{-1}(u(t))} b_s\right)_{t\in T}.$$

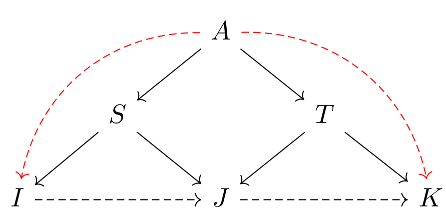
Remark 1.7. If Diagram 1.6 is a commutative diagram, then there is a restriction of u' given by $u' : g'^{-1}(t) \to g^{-1}(u(t))$. In particular, if Diagram 1.6 is a pullback diagram, then this restriction map is a bijection. In this setting, the map $u^* g_\otimes$ sends $(b_s)_{s\in S}$ to

$$\left(\prod_{s\in g^{-1}(u(t))} b_s\right)_{t\in T} = \left(\prod_{a\in g'^{-1}(t)} b_{u'(a)}\right)_{t\in T} = g'_\otimes u'^* (b_s)_{s\in S}.$$

Therefore,

$$v_\otimes u^* g_\otimes f^* = v_\otimes g'_\otimes u'^* f^* = (vg')_\otimes (fu')^*.$$

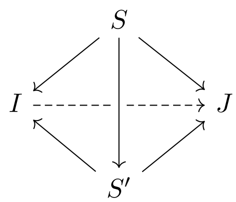
Definition 1.8. We define $\mathbf{Span}(\mathbb{F})$ to be the category of span of $\mathbb{F}$, where objects are finite sets as in $\mathbb{F}$, and morphisms of the form $I \to J$ are the zigzag of the form $I \leftarrow S \to J$. The composition of morphisms $I \to J \to K$ on the zigzag is now defined by $I \leftarrow A \to K$ using the diagram



whenever A is constructed as the pullback, otherwise known as the outer span $S \times_K T$.

Remark 1.9. One issue that persists from this construction is the fact that the pullback A is not unique, thus the composition of morphisms is not unique. (This may be unique up to unique isomorphism.) With this in mind, $\mathbf{Span}(\mathbb{F})$ admits a $(2, 1)$-category structure instead of an ordinary category.

The 2-morphisms of $\mathbf{Span}(\mathbb{F})$ are defined by $S \to S'$ via



Moreover, these 2-morphisms are isomorphisms (of spans) and hence invertible, therefore admitting the $(2,1)$-category structure.

Remark 1.10. The functors we defined in Definition 1.2 can be extended to a functor

$$\tilde{M} : \mathbf{Span}(\mathbb{F}) \to \mathbf{Set}$$

such that $\tilde{M}\big|_{\mathbb{F}^{\text{op}}} \in \mathbf{Fun}^{\times}(\mathbb{F}^{\text{op}}, \mathbf{Set})$. To see this, recall that there is a natural inclusion

$$\begin{aligned} \mathbb{F}^{\text{op}} &\hookrightarrow \mathbf{Span}(\mathbb{F}) \\ A &\mapsto A \\ (I \leftarrow S) &\mapsto (I \leftarrow S \xrightarrow{=} S) \end{aligned}$$

then the extension $\tilde{M}$ is the functor we want, as the product and coproduct of the 2-category $\mathbf{Span}(\mathbb{F})$ are both the coproduct on $\mathbf{FinSet}$, i.e., the disjoint union.

Remark 1.11. In fact, given any category $\mathscr{C}$ with finite products, then there is an identification of commutative monoids on $\mathscr{C}$ with product-preserving functors $\mathbf{Span}(\mathbb{F}) \to \mathscr{C}$. Moreover, this is true homotopically, c.f., [Cra09] and [Cra11].

This is the story of how we induce functors from commutative monoids, where the span exhibits a bivariant phenomenon. We will see below that there is a similar one for commutative semirings, which exhibits distributivity.

1.2 Bispans and Semirings

Definition 1.12. A commutative semiring $(R, +, \times, 0, 1)$ is a set R equipped with operations $+$ and $\times$ as well as additive identity 0 and multiplicative identity 1. However, we do not assume the existence of additive inverse and/or multiplicative inverse. Therefore, R is both an additive monoid and a multiplicative monoid.

Using the same construction in Definition 1.2, we have a functor

$$\begin{aligned} \mathbb{F} &\to \mathbf{Set} \\ I &\mapsto R^I \end{aligned}$$

which induces a functor

$$\begin{aligned} \tilde{R}_{\times} : \text{“}\mathbf{Span}(\mathbb{F})\text{”} &\to \mathbf{Set} \\ (I \xleftarrow{f} S \xrightarrow{g} J) &\mapsto g_{\otimes} f^* \end{aligned}$$

Now note that we still have an additive monoidal structure on R, so we would hope to define a functor of the form

$$\begin{aligned} \tilde{R}_{+} : \text{“}\mathbf{Span}(\mathbb{F})\text{”} &\to \mathbf{Set} \\ ? &\mapsto g_{\oplus} f^* \end{aligned}$$

for some unknown category "$\mathbf{Span}(\mathbb{F})$". These two functors altogether shall define a desired functor $\tilde{R}$: "$\mathbf{Span}(\mathbb{F})$" $\to$ $\mathbf{Set}$. In particular, admitting two different structures here already tells us that the spans are no longer suitable, and a natural adaptation would be bispans.

Definition 1.13. A bispan (or a polynomial diagram) from I to J is given by a diagram

$$\begin{array}{ccccc} & X & \xrightarrow{f} & Y & \\ \swarrow^{p} & & & & \searrow^{q} \\ I & & & & J \end{array}$$

The category of bispans, denoted $\mathbf{Bispan}(\mathbb{F})$, has objects (again) the same with objects of $\mathbb{F}$, and morphisms are bispans.

Given a semiring R, we would want to construct a functor

$$\begin{aligned} \mathbf{Bispan}(R) &\to \mathbf{Set} \\ I &\mapsto R^I \\ (I \xleftarrow{p} X \xrightarrow{f} Y \xrightarrow{q} J) &\mapsto q_\oplus f_\otimes p^* \end{aligned}$$

where

$$\begin{aligned} p^* &: R^I \to R^X \\ p^*(\varphi)(x) &= \varphi(px), \end{aligned}$$

$$\begin{aligned} f_\otimes &: R^X \to R^Y \\ f_\otimes(\varphi)(y) &= \prod_{x \in f^{-1}(y)} \varphi(x), \end{aligned}$$

and

$$\begin{aligned} q_\oplus &: R^Y \to R^J \\ q_\oplus(\varphi)(j) &= \sum_{y \in q^{-1}(j)} \varphi(y), \end{aligned}$$

which represent composition (as pullback), fiberwise multiplication (as pushforward), and fiberwise addition (as pushforward), respectively. Altogether, this gives

$$\begin{aligned} q_\oplus f_\otimes p^* &: M^I \to M^J \\ (a_i)_{i \in I} &\mapsto \left(\sum_{y \in q^{-1}(j)} \prod_{x \in f^{-1}(y)} a_{p(x)} \right)_{j \in J}. \end{aligned}$$

Again, to construct such a functor, we need to consider the composition of bispans:

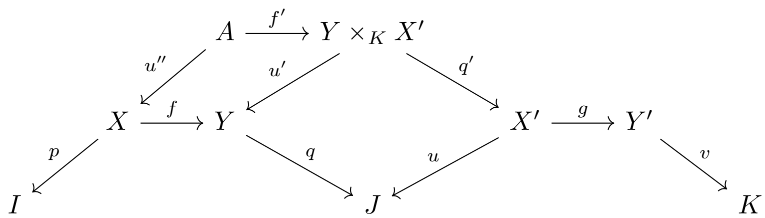
$$\begin{array}{ccccccccc} & X & \xrightarrow{f} & Y & & X' & \xrightarrow{g} & Y' & \\ \swarrow^{p} & & & & \searrow^{q} \;\; \swarrow^{u} & & & & \searrow^{v} \\ I & & & & J & & & & K \end{array}$$

As we have seen previously, we need to study the pullback structure so that we can resolve $v_\oplus g_\otimes u^* q_\oplus f_\otimes p^*$. Using similar construction, we have

$$v_\oplus g_\otimes u^* q_\oplus f_\otimes p^* = v_\oplus g_\otimes q'_\oplus u'^* f_\otimes p^*$$

$$
\begin{aligned}
&= v_{\oplus}q''_{\oplus}g'_{\otimes}u'^{*}f_{\otimes}p^{*} \\
&= v_{\oplus}q''_{\oplus}g'_{\otimes}f'_{\otimes}u''^{*}p^{*} \\
&= (vq'')_{\oplus}(g'f')_{\otimes}(pu'')^{*}
\end{aligned}
$$

assuming we can construct $g'_{\otimes}$ and $q''_{\oplus}$ such that $g_{\otimes}q'_{\oplus} = q''_{\oplus}g'_{\otimes}$. That is, we have constructed two pullback squares



(1.14)

To deal with this, recall that addition distributes over multiplication, therefore given any

$$I \xrightarrow{u} J \xrightarrow{v} K$$

we know $v_{\otimes}u_{\oplus} : R^I \to R^K$ is the mapping defined by

$$(a_i)_{i\in I} \mapsto \left(\prod_{j\in v^{-1}(k)} \sum_{i\in u^{-1}(j)} a_j\right)_{k\in K} = \left(\sum_{(i_j)\in \prod_{j\in v^{-1}(k)} u^{-1}(j)} \prod_{t\in v^{-1}(k)} a_{i_t}\right)_{k\in K}. \tag{1.15}$$

The goal is to identify the said image from Equation (1.15). Recall that the slice categories $\mathbf{FinSet}/\mathbf{K}$ and $\mathbf{FinSet}/\mathbf{J}$ are involved in a pullback/pushforward adjunction

$$
\begin{array}{ccc}
\mathbf{FinSet}/\mathbf{K} & \xrightarrow{\cong} & \mathbf{Fun}(\mathbf{K}, \mathbf{Set}) \\
{\scriptstyle v^*}\downarrow\uparrow{\scriptstyle v_*} & & \\
\mathbf{FinSet}/\mathbf{J} & \xrightarrow{\cong} & \mathbf{Fun}(\mathbf{J}, \mathbf{Set})
\end{array}
\tag{1.16}
$$

where

- $\mathbf{FinSet}/\mathbf{J} \cong \mathbf{Fun}(\mathbf{J}, \mathbf{Set})$ is a Grothendieck correspondence, where given $u : I \to J$, we obtain a functor

$$
\begin{aligned}
J &\to \mathbf{FinSet} \\
j &\mapsto u^{-1}(j)
\end{aligned}
$$

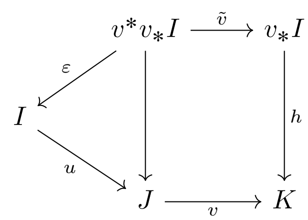
- $\mathbf{FinSet}/\mathbf{K} \cong \mathbf{Fun}(\mathbf{K}, \mathbf{Set})$ is a Grothendieck correspondence, where given $v : J \to K$, we obtain a functor

$$
\begin{aligned}
K &\to \mathbf{FinSet} \\
k &\mapsto v^{-1}(k)
\end{aligned}
$$

- the Grothendieck correspondences give rise to (Co)carteisan fibrations;
- $h = v_* u \in \mathbf{Set}/\mathbf{K}$ is a functor, and by the correspondence we obtain a functor

$$
\begin{aligned}
h' &: K \to \mathbf{Set} \\
k &\mapsto \prod_{j\in v^{-1}(k)} u^{-1}(j) = \prod_{j\in h^{-1}(k)} u^{-1}(j)
\end{aligned}
$$

- v^* is the pullback along $v: J \to K$. In particular, consider the counit $\varepsilon: v^*v_*I \to I$ of the adjunction, then for $X = v_*I$, the pullback $v^*X = v^*v_*I$ gives a counit ε in the diagram



(1.17)

We now make an effort to show that Diagram 1.17 actually commutes.

For any $k \in K$, we pullback $\alpha \in X$ such that $h(\alpha) = k$, but by the correspondence we know α is in the image of k along h'. Similarly, for $k \in K$, we pullback $j \in J$, and using the same argument we would then conclude that the pullback element in v^*X is just a pair (α, j).

Now let $i = \varepsilon(\alpha, j)$, but one can identify i to be the image of α under the projection $h^{-1}(k) \to \prod_{j' \in v^{-1}(k)} u^{-1}(j') \to u^{-1}(j)$. Therefore, $\alpha = (i_j)_{j \in v^{-1}(k)}$. For any fixed α, one can then identify

$$\prod_{t \in v^{-1}(k)} a_{it} = \prod_{j \in v^{-1}(k)} a_{\varepsilon(\alpha, j)}.$$

Therefore, the image of Equation (1.15) is

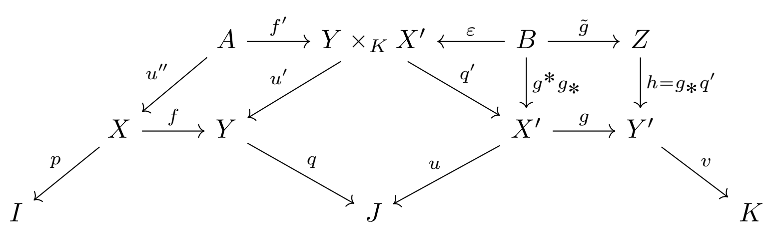
$$\left(\sum_{\alpha \in h^{-1}(k)} \prod_{j \in v^{-1}(k)} a_{\varepsilon(v,j)} \right)_{k \in K} = h_\oplus \tilde{v}_\otimes \varepsilon^*(a_i)_{i \in I}.$$

In particular, we obtain

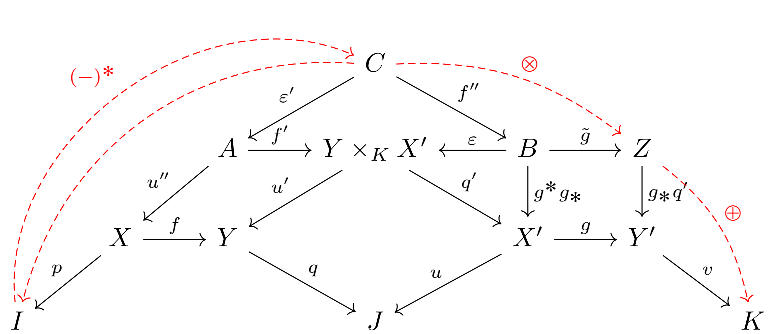
$$v_\otimes u_\oplus = h_\oplus \tilde{v}_\otimes \varepsilon^*,$$

i.e., Diagram 1.17 commutes, which describes the distributivity.

Let us go back to Diagram 1.14. Using Diagram 1.17, we extend the diagram to



which can be extended by taking one last pullback



and we define the composition to be the outer bispan in this diagram.

Remark 1.18. An explicit construction of this $(2,1)$-category $\mathbf{Bispan}(\mathbb{F})$ can be found in [Cra09], where it is proven that the category has a product structure given by coproducts of $\mathbf{FinSet}$. In this sense, commutative semirings in a category $\mathscr{S}$ correspond to functors $\mathbf{Bispan}(\mathbb{F}) \to \mathscr{S}$ that preserve finite products.

Definition 1.19. As a dual notion to span, a cospan is a zigzag of the form

$$I \xrightarrow{f} S \xleftarrow{g} J$$

Remark 1.20. The duality shows an equivalence of categories $\mathbf{Span}(\mathscr{C}) \cong \mathbf{Cospan}(\mathscr{C}^{\mathrm{op}})$ as $(2,1)$-categories.

2 ∞-CATEGORIES

2.1 CONSTRUCTIONS ON ∞-CATEGORIES

Definition 2.1. Let $r \leqslant n \leqslant \infty$. An (n, r)-category has the usual objects and (1-)morphisms like an ordinary category, but also i-morphisms for $0 \leqslant i < \infty$, such that

- when $i > r$, every i-morphism is invertible, and
- when $i > n$, every i-morphism is trivial.

We mostly consider the $(\infty, 1)$-categories. Let $\mathbf{sSet} = \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set})$ be the category of simplicial sets, then any simplicial set $X \in \mathbf{sSet}$ is a diagram of the form

$$\cdots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} X_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} X_0 \tag{2.2}$$

where $X_n = X([n]) \in \mathbf{Set}$ is the set of n-simplices of X. In Diagram 2.2, the blue arrows $X_n \to X_{n-1}$ are called the face maps, as they assign each n-simplex to the face not containing the ith vertex for $0 \leqslant i \leqslant n$; the red arrows $X_n \to X_{n+1}$ are called the degeneracy maps, as they assign each n-simplex to the degenerate $(n+1)$-simplex by duplicating the ith vertex. These maps satisfy the simplicial identity, c.f., [GJ09], that is,

- if $i < j$, then $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$;
- if $i > j$, then $s_i \circ s_j = s_j \circ s_{i-1}$;
- face maps and degeneracy maps are compatible, as

$$\partial_i \circ s_j = \begin{cases} s_{j-1} \circ \partial_i, & i < j \\ \mathrm{id}_n, & i \in \{j, j+1\} \\ s_i \circ \partial_{i-1}, & i > j+1 \end{cases}$$

Remark 2.3. The degeneracy maps usually play a role when the algebraic structure involves a unital object, e.g., existence of monoidal structure. They are often times omitted when, for example, we study non-unital monoidal objects, in which case we only draw the face maps.

Definition 2.4. A morphism $p : X \to Y$ is called an inner fibration if for every $0 < i < n$, any commutative diagram

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y \end{array} \tag{2.5}$$

in $\mathbf{sSet}$ admits a solution.

Definition 2.6. We say a simplicial set $X \in \mathbf{sSet}$ is an $(\infty, 1)$-category[1] if $X \to * \cong \Delta^0$ is an inner fibration.

Remark 2.7. The following conditions are equivalent:

- $X \in \mathbf{sSet}$ is an $(\infty, 1)$-category;
- the induced map $i^* : X^{\Delta^n} \to X^{\Lambda_i^n}$ of i from Diagram 2.5 is a trivial Kan fibration for all $0 < i < n$;

[1] Alternatively, we also call it an ∞-category or a quasi-category, depending on sources.

- the induced map $i^* : X^{\Delta^2} \to X^{\Lambda^2_1}$ of i from Diagram 2.5 is a trivial (acyclic) Kan fibration;[2]

  Note that a vertex $v \in X^{\Lambda^2_1}$ is just a map $\Lambda^2_1 \to X$ from the inner horn. Let us draw the inner horn as

$$\begin{array}{ccccc} & & 1 & & \\ & \nearrow & & \searrow & \\ 0 & & & & 2 \end{array}$$

  then its image in X is the pair of composable morphisms in X. Now a vertex in X^{Δ^2} is a 2-simplex in X, therefore for each pair of composable morphisms

$$\begin{array}{ccccc} & & \bullet & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ \bullet & & & & \bullet \end{array}$$

  it can be completed to a 2-simplex σ

$$\begin{array}{ccccc} & & 1 & & \\ & \overset{f=d_2\sigma}{\nearrow} & \sigma & \overset{g=d_0\sigma}{\searrow} & \\ 0 & & \underset{d_1\sigma}{\longrightarrow} & & 2 \end{array}$$

  We then define the composite $g \circ f = d_1\sigma$. The trivial fibration condition then means that once we pick composable arrows in X, then the fiber over vertex $v \in X^{\Lambda^2_1}$ will be a trivial Kan complex. Therefore, the conditions above are equivalent to

- the composition problem has a unique solution (up to contractible space of choices).

The $(\infty, 1)$-categories in **sSet** forms a full subcategory, denoted **qCat**.

Remark 2.8. For $K \in \mathbf{sSet}$, and $\mathscr{C} \in \mathbf{qCat}$, we have $\mathscr{C}^K = \mathbf{Fun}(\mathbf{K}, \mathscr{C}) \in \mathbf{sSet}$.

We call an $(\infty, 1)$-category a weak Kan complex, which describes the property that it is a simplicial set for which all inner horns have a filler. It is weaker than a Kan complex, which exhibits the property that every horn has a filler.

Definition 2.9. Let $\mathscr{C} \in \mathbf{qCat}$, we say a simplicial subset $\mathscr{C}' \subseteq \mathscr{C}$ is an $(\infty, 1)$-subcategory if $i : \mathscr{C}' \hookrightarrow \mathscr{C}$ is an inner fibration.

Recall that for any $(\infty, 1)$-category $\mathscr{C}$, its homotopy category $h\mathscr{C}$ is given by quotienting homotopy relations, which identifies 1-morphisms that are connected by some 2-morphism.[3] With this, $h\mathscr{C} \in \mathbf{Cat}$ is an ordinary 1-category. Conversely, given any ordinary category, one can show that its nerve has a simplicial set structure, and in particular becomes an $(\infty, 1)$-category. Now note that the two functors $h(-)$ and $N(-)$ give an adjunction

$$\begin{array}{c} \mathbf{qCat} \\ h\big\downarrow \big\uparrow N \\ \mathbf{Cat} \end{array} \tag{2.10}$$

of 1-categories. The unit of this adjunction

$$\begin{aligned} F : \mathbf{qCat} &\to \mathbf{qCat} \\ \mathscr{C} &\mapsto N(h\mathscr{C}) \end{aligned}$$

[2] This equivalence is given by the Joyal model structure on **sSet**, c.f., [Lur09].
[3] In particular, this is the restriction of the natural functor $\mathbf{sSet} \to \mathbf{Cat}$ to **qCat**.

is an inner fibration. For any subcategory $A \subseteq h\mathscr{C}$, we consider the mapping

$$A \mapsto F^{-1}(NA) = NA \times_{N(h\mathscr{C})} \mathscr{C}.$$

Here it is notable that $F^{-1}(NA)$ acts as the pullback diagram

$$\begin{array}{ccc} F^{-1}(NA) & \hookrightarrow & \mathscr{C} \\ \downarrow & & \downarrow F \\ NA & \hookrightarrow & Nh\mathscr{C} \end{array} \tag{2.11}$$

Now the defined mapping above gives a bijection from subcategories of the ordinary category $h\mathscr{C}$ to the subcategories of the $(\infty, 1)$-category $\mathscr{C}$.

Definition 2.12. If, in addition, that A is a full subcategory of $h\mathscr{C}$, then we say $\mathscr{C} \times_{Nh\mathscr{C}} NA$ is *the* full $(\infty, 1)$-subcategory of $\mathscr{C}$.

Remark 2.13. Given a functor $F : \mathscr{C} \to \mathcal{E}$ of $(\infty, 1)$-categories, and suppose we have an ∞-subcategory $\mathscr{C}' \subseteq \mathscr{C}$, then the idea is that the restricted functor $F' : \mathscr{C}' \to \mathcal{E}$ can be realized if we just look at the effect of F on vertices and edges in $\mathscr{C}'$.

Definition 2.14. Given an $(\infty, 1)$-category $\mathscr{C}$, the core of $\mathscr{C}$ is $\mathsf{core}(\mathscr{C}) = \mathscr{C}^{\simeq}$, the underlying ∞-groupoid (an $(\infty, 0)$-category) obtained by discarding non-invertible morphisms. Alternatively, we can it the largest Kan complex contained in $\mathscr{C}$.

Remark 2.15. For an $(\infty, 1)$-category $\mathscr{C}$, let $A = (h\mathscr{C})^{\simeq}$ be the core of the homotopy category, then the core $\mathscr{C}^{\simeq}$ of $\mathscr{C}$ is just $F^{-1}(NA)$, i.e., fits into Diagram 2.11.

Remark 2.16. For a 1-category $\mathscr{C}$, there is a canonical isomorphism

$$N(\mathscr{C})^{\simeq} \cong N(\mathscr{C}^{\simeq}).$$

Remark 2.17. Given a functor $F : \mathscr{C} \to \mathscr{D}$ of $(\infty, 1)$-categories, F sends $\mathscr{C}^{\simeq}$ to $\mathscr{D}^{\simeq}$, and therefore there is a morphism $F^{\simeq} : \mathscr{C}^{\simeq} \to \mathscr{D}^{\simeq}$ of $\mathbf{sSet}$.

Now suppose $X \in \mathbf{sSet}$, then X has a dual object $X^{\mathrm{op}} \in \mathbf{sSet}$.

Definition 2.18. The dual object X^{op} is a simplicial set, with $X_n^{\mathrm{op}} = X_n$, where $d_i : X_n^{\mathrm{op}} \to X_{n-1}^{\mathrm{op}}$ is defined by $d_{n-i} : X_n \to X_{n-1}$, and $s_i : X_n^{\mathrm{op}} \to X_{n+1}^{\mathrm{op}}$ is defined by $s_{n-i} : X_n \to X_{n+1}$.

Definition 2.19. Suppose $C, D \in \mathbf{qCat}$, and fix a vertex $d \in \mathscr{D}$, which is viewed as a map $d : \Delta^0 \to \mathscr{D}$. We denote $\mathscr{D}_{/d}$ to be the slice category of $\mathscr{D}$ over d, and $\mathscr{D}_{d/}$ to be the coslice category of $\mathscr{D}$ under d. Now let $p : \mathscr{C} \to \mathscr{D}$ be a morphism in $\mathbf{sSet}$, then the pullback of the (co)slice category gives rise to another (co)slice category. To be precise,

- $\mathscr{C}_{/d} = p^*\mathscr{D}_{/d}$ is the slice $(\infty, 1)$-category of $\mathscr{C}$ over $d \in \mathscr{D}$, fitting into the Cartesian square

  $$\begin{array}{ccc} \mathscr{C}_{/d} & \longrightarrow & \mathscr{C} \\ \downarrow & & \downarrow p \\ \mathscr{D}_{/d} & \longrightarrow & \mathscr{D} \end{array} \tag{2.20}$$

  in $\mathbf{sSet}$;

- $\mathscr{C}_{d/} = p^*\mathscr{D}_{d/}$ is the coslice $(\infty, 1)$-category of $\mathscr{C}$ under $d \in \mathscr{D}$, fitting into the Cartesian square

$$\begin{array}{ccc} \mathscr{C}_{d/} & \longrightarrow & \mathscr{C} \\ \downarrow & & \downarrow p \\ \mathscr{D}_{d/} & \longrightarrow & \mathscr{D} \end{array} \tag{2.21}$$

  in $\mathbf{sSet}$;

Here there exists natural functors $\mathscr{D}_{/d} \to \mathscr{D}$ and $\mathscr{D}_{d/} \to \mathscr{D}$ by forgetting the vertex d.

More generally, there exists a version of (co)slice categories over morphisms $f : K \to \mathscr{D}$ for $K \in \mathbf{sSet}$.
To define limits and colimits on $(\infty, 1)$-categories, we require the notion of join.

2.2 (Co)limits

Definition 2.22. For 1-categories $\mathscr{A}, \mathscr{B}$, the join $\mathscr{A} \star \mathscr{B}$ is a 1-category with objects $\mathrm{Ob}(\mathscr{A}) \coprod \mathrm{Ob}(\mathscr{B})$ and morphisms $\mathrm{Mor}(\mathscr{A}) \coprod (\mathrm{Ob}(\mathscr{A}) \times \mathrm{Ob}(\mathscr{B})) \coprod \mathrm{Mor}(\mathscr{B})$. That is,

$$\mathrm{Hom}_{\mathscr{A}\star\mathscr{B}}(x,y) = \begin{cases} \mathrm{Hom}_{\mathscr{A}}(x,y), & x,y \in \mathrm{Ob}(\mathscr{A}) \\ \mathrm{Hom}_{\mathscr{B}}(x,y), & x,y \in \mathrm{Ob}(\mathscr{B}) \\ \{*\}, & x \in \mathrm{Ob}(\mathscr{A}), y \in \mathrm{Ob}(\mathscr{B}) \\ \varnothing, & x \in \mathrm{Ob}(\mathscr{B}), y \in \mathrm{Ob}(\mathscr{A}) \end{cases}$$

Definition 2.23. For a category $\mathscr{A}$, the left cone is $\mathscr{A}^{\lhd} = [0] \star \mathscr{A}$, and the right cone is $\mathscr{A}^{\rhd} = \mathscr{A} \star [0]$.

Therefore, the cone adjoins an extra vertex onto the simplicial set.

Definition 2.24. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor. A limit of F is a functor $\hat{F}' : \mathscr{A}^{\lhd} \to \mathscr{B}$ which is terminal among all functors that extend F. Similarly, a colimit of F is a functor $\hat{F} : \mathscr{A}^{\rhd} \to \mathscr{B}$ which is initial among functors which extend F.

Here we need to explain what initial and terminal means in terms of universal properties of $(\infty, 1)$-categories.

Definition 2.25. Let $\mathscr{C}$ be an $(\infty, 1)$-category. We say $x \in \mathscr{C}$ is terminal if the canonical map $\mathscr{C}_{/x} \to \mathscr{C}$ is an acyclic Kan fibration of simplicial. That is, the mapping spaces $\mathrm{Map}_{\mathscr{C}}(c, x)$ are acyclic Kan complexes for all objects $c \in \mathscr{C}$. We say $x \in \mathscr{C}$ is initial if it is a terminal object in $\mathscr{C}^{\mathrm{op}}$.

Remark 2.26. Let $\mathscr{C}$ be an ordinary category. Note that the mapping space $\mathrm{Map} = \mathrm{Map}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathcal{S}$ is a functor into $\mathcal{S}$, the $(\infty, 1)$-category of (small) spaces (anima/∞-groupoids). Therefore, the mapping space is a simplicial set. With this in mind, a general (co)limit in $\mathscr{C}$ should preserve limits in each variable. That is,

$$\mathrm{Map}_{\mathscr{C}}(\operatorname*{colim}_{i \in I}(a_i), b) \cong \lim_{i \in I^{\mathrm{op}}} \mathrm{Map}_{\mathscr{C}}(a_i, b)$$

and

$$\mathrm{Map}_{\mathscr{C}}(b, \operatorname*{colim}_{i \in I}(a_i)) \cong \lim_{i \in I^{\mathrm{op}}} \mathrm{Map}_{\mathscr{C}}(b, a_i)$$

for $a_i, b \in \mathscr{C}$. These are restatements of the universal property of (co)limits, as we view Map as hom sets.

Let $K \in \mathbf{sSet}$, then the slice category $\mathscr{D}_{/d}$ has the universal property: the hom set of 1-category $\mathbf{sSet}$ satisfies

$$\mathbf{sSet}(K, \mathscr{D}_{/d}) \cong \mathrm{Hom}_d(K^{\rhd}, \mathscr{D}).$$

In particular, if $K = \Delta^n$, then we get all n-simplices in the simplicial set $\mathscr{D}_{/d}$, which gives a description of this category. The hom set $\mathrm{Hom}_d(K^{\triangleright}, \mathscr{D})$ is a subset of the simplicial sets $\mathbf{sSet}(K^{\triangleright}, \mathscr{D})$. Dually, there is an isomorphism

$$\mathbf{sSet}(K, \mathscr{D}_{d/}) \cong \mathrm{Hom}_d(K^{\triangleleft}, \mathscr{D}).$$

Now recall that $(\infty, 1)$-categories are weak Kan complexes, therefore $\mathbf{Kan}$, the $(\infty, 1)$-category of Kan complexes[4], becomes a subcategory of $\mathbf{qCat}$. Both categories are simplicially-enriched, i.e., as $\mathbf{sSet}$-categories. By applying the homotopy coherent nerve functor N^{hc} on the inclusion, we obtain another inclusion $\mathcal{S} \subseteq \mathbf{Cat}_\infty$ of $(\infty, 1)$-categories. This inclusion functor gives an adjunction triple

$$\begin{array}{c} \mathcal{S} \\ {\scriptstyle |\cdot|}\uparrow\ \downarrow\ \uparrow{\scriptstyle \mathrm{Core}(-)} \\ \mathbf{Cat}_\infty \end{array} \tag{2.27}$$

where $|\cdot|$ is the ∞-groupoid completion and $\mathrm{Core}(-)$ is the core functor.

Finally, we study cofinality in simplicial sets.

Definition 2.28. A map $v : K' \to K$ in $\mathbf{sSet}$ is right cofinal if it satisfies all of the following (equivalent) conditions:

- v respects all colimits, i.e., for every $(\infty, 1)$-category $\mathscr{C}$ and any colimit cocone $K^{\triangleright} \to \mathscr{C}$, the composition $K'^{\triangleright} \xrightarrow{v} K^{\triangleright} \to \mathscr{C}$ is also a colimit cocone.
- v respects colimits in $\mathcal{S}^{\mathrm{op}}$.

  If, in addition, K is an $(\infty, 1)$-category, then they are equivalent to
- for every object $k \in K$, the simplicial set $K'_{x/}$ is weakly contractible.[5] That is, $K'_{x/} \sim \Delta^0$.

A map $v : K' \to K$ in $\mathbf{sSet}$ is left cofinal if $v^{\mathrm{op}} : K'^{\mathrm{op}} \to K^{\mathrm{op}}$ is right cofinal.

Remark 2.29.

- Left (respectively, right) cofinal maps are stable under products, i.e., if $v : K' \to K$ is left (respectively, right) cofinal, then so is $K' \times L \to K \times L$ for any $L \in \mathbf{sSet}$.
- A left (respectively, right) adjoint is left (respectively, right) cofinal.
- Left (respectively, right) cofinal maps are stable under pushforwards (respectively, pullbacks) along Cartesian (respectively, Cocartesian) fibrations.

Definition 2.30. A morphism $p : A \to B$ in $\mathbf{sSet}$ is proper if, for any pullback pairs in $\mathbf{sSet}$ of the form

$$\begin{array}{ccccc} A'' & \xrightarrow{u} & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ B'' & \xrightarrow[v]{} & B' & \longrightarrow & B \end{array} \tag{2.31}$$

$u : A'' \to A'$ is right cofinal whenever $v : B'' \to B'$ is.

Definition 2.32. A morphism $p : A \to B$ in $\mathbf{sSet}$ is smooth if its opposite morphism $p^{\mathrm{op}} : A^{\mathrm{op}} \to B^{\mathrm{op}}$ in $\mathbf{sSet}$ is proper. That is, given Diagram 2.31, $u : A'' \to A'$ is left cofinal whenever $v : B'' \to B'$ is.

Remark 2.33.

[4] This is sometimes referred to as "the ∞-category of spces".

[5] Unless stated otherwise, we always mean this is in terms of Kan-Quillen model structure of $\mathbf{sSet}$.

- The class of proper morphisms in **sSet** and the class of smooth morphisms in **sSet** are both stable under composition and under base-change.
- Cocartesian fibrations are proper; Cartesian fibrations are smooth.

Definition 2.34. Let κ be a regular cardinal.

- A simplicial set K is κ-small if the number of its non-degenerate simplices is less than κ. In particular, if $\kappa = \omega = \aleph_0$ is the countable regular cardinal, then it is finite.
- A simplicial set L is κ-filtered if, for any κ-small simplical set K and any map $v : K \to L$ in **sSet**, the simplicial set $L_{v/}$ is non-empty. That is, v can be extended to a cocone $\bar{v} : K^{\triangleright} \to L$. Moreover, we say L is filtered if it is ω-filtered, that is, every finite diagram in L extends to a cocone.

  Remark 2.35. An $(\infty, 1)$-category $\mathscr{C}$ is filtered if and only if for any integer $n \geqslant 0$, every morphism $\partial\Delta^n \to \mathscr{C}$ of simplicial sets can be extended to a morphism of the form $(\partial\Delta^n)^{\triangleright} \to \mathscr{C}$.

  Dually, a simplicial set L is (κ-)cofiltered if its dual L^{op} is (κ-)filtered.

  Remark 2.36.

  - Any (co)filtered $(\infty, 1)$-category is weakly contractible.
  - A Kan complex is (co)filtered if and only if it is weakly contractible.

- A simplicial set K is sifted if $K \neq \varnothing$ and its diagonal $\Delta : K \to K \times K$ is cofinal. Equivalently, the diagonal $\Delta : K \to K^I$ is right cofinal for any finite set I. A simplicial set K is cosifted if K^{op} is sifted.

  Remark 2.37.

  - A (co)sifted simplicial set is weakly contractible.
  - Let $\mathscr{C}$ be an $(\infty, 1)$-category. $\mathscr{C}$ is sifted if and only if, for every pair of objects $a, b \in \mathscr{C}$, the underlying simplicial set $\mathscr{C}_{a/} \times_{\mathscr{C}} \mathscr{C}_{b/}$ is weakly contractible. In particular, any $(\infty, 1)$-category with finite coproducts is sifted.

    To prove this, consider an $(\infty, 1)$-category K and any map $K' \to K$, then the sifted property says that $K'_{x/} \sim \Delta^0$ is weakly contractible for any $x \in K$. By definition, any arbitrary pair of vertices $a, b \in K$ gives a commutative square

    $$\begin{array}{ccc} K_{(a,b)/} & \longrightarrow & K \\ \downarrow & & \downarrow \Delta \\ K_{a/} \times K_{b/} & \longrightarrow & K \times K \end{array} \tag{2.38}$$

    One can check $K_{(a,b)/} \cong K_{a/} \times_K K_{b/}$ pointwise. Therefore, it is equivalent to saying that $K_{a/} \times_K K_{b/}$ is weakly contractible.
  - Let $\mathcal{A}$ be an 1-category. $\mathcal{A}$ is sifted if and only if $A_{a/} \times_A A_{b/}$ is connected for any $a, b \in A$. (That is, the diagonal functor respects all limits in 1-categories.)
  - If $v : K' \to K$ is a right cofinal map of simplicial sets and K is sifted, then K' is also sifted.
  - Let $F : \mathscr{C} \to \mathscr{D}$ be a functor of $(\infty, 1)$-categories where $\mathscr{C}$ is cocomplete, then F preserves sifted colimits if and only if it preserves filtered colimits and geometric realizations. Here the geometric realization means any colimit indexed by Δ^{op}.
  - Any colimit can be written as a sifted colimit of finite coproducts.

Example 2.39.

- $N\Delta^{\mathrm{op}}$ is sifted.
- Any non-empty filtered simplicial set is sifted.

Proposition 2.40. Let $v : \mathscr{C} \to \mathscr{D}$ be a right cofinal functor of $(\infty, 1)$-categories, then $\mathscr{D}$ is filtered if $\mathscr{C}$ is.

Theorem 2.41. Let $\mathscr{C}$ be an $(\infty, 1)$-category, then the following are equivalent:

- $\mathscr{C}$ is filtered;
- there exists a right cofinal functor $NA \to \mathscr{C}$ for some directed poset A;
- the diagonal map $\Delta : \mathscr{C} \to \mathscr{C}^K$ is right cofinal for every finite simplicial set K.

Definition 2.42. Let $p : \mathscr{C} \to \mathscr{D}$ be a functor of $(\infty, 1)$-categories. We say p is fully faithful if

$$p_* : \mathrm{Map}_{\mathscr{C}}(c, c') \to \mathrm{Map}_{\mathscr{D}}(pc, pc')$$

is an equivalence for any vertices $c, c' \in \mathscr{C}$.

Remark 2.43. Note that we did not give a precise definition of the mapping space. However, We should think of the hom set $\mathrm{Map}_{\mathscr{C}}(c, c')$ to be the pullback of

$$\begin{array}{ccc} \mathrm{Map}_{\mathscr{C}}(c, c') & \longrightarrow & \mathscr{C}^{\Delta^1} \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow[(c,c')]{} & \mathscr{C} \times \mathscr{C} \end{array} \tag{2.44}$$

Here

- $\mathscr{C}^{\Delta^1}$ gives the edges in $\mathscr{C}$, then the map $\mathscr{C}^{\Delta^1} \to \mathscr{C} \times \mathscr{C}$ is the map landing at the source and the target;
- $(c, c') : \Delta^0 \to \mathscr{C} \times \mathscr{C}$ is the map landing at the pair (c, c').

Theorem 2.45. Let

$$\begin{array}{c} \mathscr{C} \\ p \downarrow \uparrow q \\ \mathscr{D} \end{array}$$

be an adjunction of $(\infty, 1)$-categories with unit η and counit ε, then η is a natural equivalence if and only if p is fully faithful. In addition, the essential image of p consists of objects $d \in \mathscr{D}$ such that ε_d is an equivalence. That is, the essential image gives the full subcategory of $\mathscr{D}$ to which the restriction of q is conservative.

Lemma 2.46. Given an adjunction triple $F \dashv U \dashv G$ of $(\infty, 1)$-categories, there is an adjunction pair $UF \dashv UG$ of $(\infty, 1)$-categories.

Theorem 2.47. Let

$$\begin{array}{c} \mathscr{C} \\ p \downarrow \uparrow q \\ \mathscr{D} \end{array}$$

be an adjunction of $(\infty, 1)$-categories. For any $K \in \mathbf{sSet}$ and $\mathscr{E} \in \mathbf{qCat}$, we obtain adjunctions

$$\begin{array}{c} \mathrm{Fun}(K, \mathscr{C}) \\ p_* \downarrow \uparrow q_* \\ \mathrm{Fun}(K, \mathscr{D}) \end{array}$$

and

$$\begin{array}{c} \mathrm{Fun}(\mathscr{C},\mathscr{E}) \\ q^* \Big\downarrow \Big\uparrow p^* \\ \mathrm{Fun}(\mathscr{D},\mathscr{E}) \end{array}$$

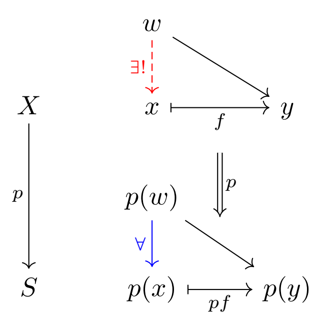
of $(\infty, 1)$-categories.

3 (Co)cartesian Fibration and Grothendieck-Lurie Correspondences

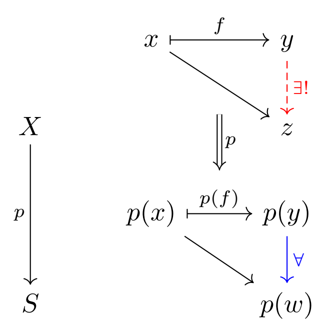
Remark 3.1.

- Vaguely speaking, let $f : X \to S$ be a morphism in $\mathbf{sSet}$. If f is a (Co)cartesian fibration over S, then f is a categorical fibration (over a model $(\infty, 1)$-category) over S, which is an inner fibration over S. ([Lur09], Remark 2.0.0.5.)
- For a morphism a Cocartesian fibration $f : X \to S$ over S is determined exactly by a functor $S \to \mathbf{Cat}_\infty$, according to the $(\infty, 1)$-Grothendieck construction.

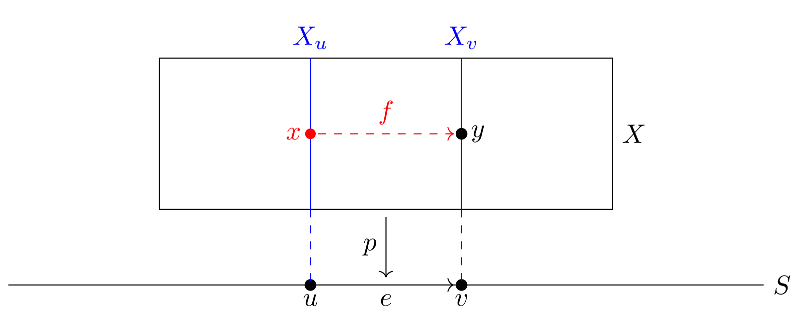
Definition 3.2. Let $p : X \to S$ be an inner fibration of simplicial sets with an edge $f : x \to y$ in X_1. We say f is p-Cartesian (or a Cartesian morphism) if $X/f \to X/y \times_{S/p(y)} S/p(f)$ is a trivial Kan fibration. Equivalently, f is terminal among all morphisms to y that lifts $p(f) \in S_1$.



Dually, we say f is p-Cocartesian (or a Cocartesian morphism) if $f^{\mathrm{op}} : y \to x$ is p^{op}-Cartesian. That is, f is initial among all morphisms from x that lifts $p(f) \in S_1$.



Definition 3.3. We say a morphism $p : X \to S$ is a Cartesian fibration if it is an inner fibration, and for any edge $e : u \to v$ in S_1 and any $y \in p^{-1}(v) =: X_v$, there exists a p-Cartesian edge $f : x \to y$ in X_1 over e such that $p(f) = e$.



Dually, we say f is a Cocartesian fibration if $p^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is a Cartesian fibration. Moreover, if f is both Cartesian and Cocartesian, we then say f is a Bicartesian fibration.

For an $(\infty,1)$-category S, note that there is a category $\mathbf{Cat}_{\infty/S}$, the slice $(\infty,1)$-category of $(\infty,1)$-categories over S. With Cartesian and Cocartesian fibrations, we can construct a $(\infty,1)$-subcategory $\mathbf{Cat}_{\infty/S}^{\mathrm{Cart}} \subseteq \mathbf{Cat}_{\infty/S}$, the $(\infty,1)$-category of Cartesian fibrations $X \to S$ for some fixed $S \in \mathbf{qCat}$. Any morphism of $\mathbf{Cat}_{\infty/S}^{\mathrm{Cart}}$ between $p : X \to S$ and $p' : X' \to S$ is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & S & \end{array}$$

of simplicial sets, where $X \to X'$ preserves Cartesian edges, i.e., sending p-Cartesian edges to p'-Cartesian edges. Therefore, the set of morphisms between p and p' is $\mathrm{Map}^{\mathrm{Cart}}_{\mathbf{Cat}_{\infty/S}}(p,p') \subseteq \mathrm{Map}_{\mathbf{Cat}_{\infty/S}}(p,p')$.

Recall that we have a functor

$$\begin{aligned} F : \mathbf{qCat} &\to \mathbf{qCat} \\ \mathscr{C} &\mapsto N(h\mathscr{C}) \end{aligned}$$

that is the unit of the adjunction in Diagram 2.10. For any subcategory $A \subseteq h\mathscr{C}$, we know $NA \times_{N(h\mathscr{C})} \mathscr{C}$ is an $(\infty,1)$-subcategory of $\mathscr{C}$, and we established that there is a correspondence between $(\infty,1)$-subcategories of $\mathscr{C}$ and subcategories of $h\mathscr{C}$. Therefore, showing $\mathbf{Cat}_{\infty/S}^{\mathrm{Cart}} \subseteq \mathbf{Cat}_{\infty/S}$ is an $(\infty,1)$-subcategory of $\mathbf{Cat}_{\infty/S}$ boils down to showing the diagrams are closed under composition, which is obvious: composition of morphisms preserving Cartesian edges should still preserve Cartesian edges.

Remark 3.4. By definition, being a (Co)cartesian fibration is invariant under base-change. Therefore, if $\pi : X \to S$ is a (Co)carteisan fibration and $T \to S$ is any map, then $X \times_S T \to T$ is a (Co)cartesian fibration as well. In particular, $\pi^{-1}(s)$ is an $(\infty,1)$-category for every $s \in S$.

Definition 3.5. Let $p : X \to S$ be a Cartesian fibration, then a Cartesian section of p is a section $s : S \to X$ that sends all 1-morphisms of S to Cartesian morphisms in X. The collection of all Cartesian sections is denoted

$$\Gamma^{\mathrm{Cart}}(p) = \{s : S \to X : ps = \mathrm{id}_S, s(S_1) \text{ are Cartesian edges in } X_1\} \subseteq \mathbf{Fun}_{/S}(S,X).$$

This becomes the $(\infty,1)$-category of Cartesian sections over p.

Proposition 3.6. Let X and S be $(\infty,1)$-categories, $p : X \to S$ be an inner fibration, and let $f : x \to y$ be an edge in X_1.

1. f is p-Cartesian if and only if for all $w \in X_0$, the diagram

$$\begin{array}{ccc} \mathrm{Map}_X(w,x) & \xrightarrow{f_*} & \mathrm{Map}_X(w,y) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Map}_S(pw,px) & \xrightarrow[p(f)_*]{} & \mathrm{Map}_S(pw,py) \end{array}$$

is a pullback in S.

2. f is p-Cocartesian if and only if for all $z \in X_0$, the diagram

$$\begin{array}{ccc} \mathrm{Map}_X(y,z) & \xrightarrow{f_*} & \mathrm{Map}_X(x,z) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Map}_S(py,pz) & \xrightarrow[p(f)^*]{} & \mathrm{Map}_S(px,pz) \end{array}$$

is a pullback in S.

Example 3.7. Let $\mathbf{CMon}(\mathbf{Ab})$ be the category of commutative rings, and define $\mathbf{Mod}(\mathbf{Ab}) = \{(A, M) : A \in \mathbf{CMon}(\mathbf{Ab}), M \in \mathbf{Mod}(\mathbf{A})\}$, then $\mathbf{Mod}(\mathbf{Ab})$ is a category with morphisms $(A, M) \to (B, N)$ of the form (u, f), given by $u : A \to B$ and $f : M \to u_* N$. To construct f, recall we have $u : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ which gives rise to an adjunction

$$\begin{array}{c}\mathbf{Mod}(\mathbf{A}) \\ u^* \big\downarrow\big\uparrow u_* \\ \mathbf{Mod}(\mathbf{B})\end{array} \tag{3.8}$$

and therefore $f : M \to u_* N$ is dual to $f^\# : u^* M \cong M \otimes_A B \to N$. Now there is a forgetful functor

$$\begin{aligned} U : \mathbf{Mod}(\mathbf{Ab}) &\to \mathbf{CMon}(\mathbf{Ab}) \\ (A, M) &\mapsto A \\ (u, f) &\mapsto u \end{aligned}$$

Then one can check that

- $(u, f) : (A, M) \to (B, N)$ is U-Cartesian if and only if $f : M \to u_* N$ is an isomorphism of A-modules, and
- $(u, f) : (A, M) \to (B, N)$ is U-Cocartesian if and only if $f^\#$ is an isomorphism of B-modules.

This notion can be generalized to the ∞-groupoid $\mathbf{Ab}^{\mathrm{ani}} = \mathbf{Fun}^\times(\mathbf{Free}_{\mathbb{Z}}, \mathbf{Ab})$.

Example 3.9. Consider the target map

$$\begin{aligned} d_0 : \mathscr{C}^{\Delta^1} &\to \mathscr{C} \\ (\alpha : x \to y) &\mapsto y \end{aligned}$$

and which sends $f = (f_0, f_1)$ as a diagram

$$\begin{array}{ccc} x & \xrightarrow{\alpha} & y \\ {\scriptstyle f_0}\big\downarrow & & \big\downarrow{\scriptstyle f_1} \\ x' & \xrightarrow[\alpha']{} & y' \end{array}$$

to $f_1 : y \to y'$. In this case,

- f is d_0-Cocarteisan if and only if $f_1 = d_0 f : y \xrightarrow{\sim} y'$ in $\mathscr{C}$, and
- suppose $\mathscr{C}$ has pullbacks, then f is d_0-Cartesian if and only if f is a pullback square.

Theorem 3.10 (Straightening-unstraightening Equivalence/Grothendieck-Lurie Correspondence)**.** For any $\mathscr{C} \in \mathbf{qCat}$, we have an adjunction given by striaghtening functor and unstraightening functor

$$\begin{array}{c}\mathbf{Cat}_{\infty/\mathscr{C}}^{\mathrm{Cocart}} \\ \mathrm{St} \big\downarrow\big\uparrow \mathrm{un} = \textstyle\int_{\mathscr{C}} \\ \mathbf{Fun}(\mathscr{C}, \mathbf{Cat}_\infty)\end{array} \tag{3.11}$$

where the straightening functor is defined over any Cocartesian fibration $\mathcal{E} \to \mathscr{C}$ via

$$\begin{aligned} \mathrm{St} : \mathbf{Fun}(\mathscr{C}, \mathbf{Cat}_\infty) &\to \mathbf{Cat}_{\infty/\mathscr{C}}^{\mathrm{Cocart}} \\ (\mathcal{E} \to \mathscr{C}) &\mapsto (c \mapsto \mathcal{E}_c) \end{aligned}$$

and the unstraightening functor is an end. There is also a dual version of this equivalence given by

$$\begin{array}{c} \mathbf{Cat}_{\infty/\mathscr{C}}^{\mathrm{Cart}} \\ {\scriptstyle \mathrm{St}}\big\downarrow\big\uparrow{\scriptstyle \mathrm{un}=\int^{\mathscr{C}}} \\ \mathbf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}_\infty) \end{array} \tag{3.12}$$

Definition 3.13. Let $f : X \to S$ be a morphism in $\mathbf{sSet}$, then we say f is a left fibration if it has the right lifting property with respect to all horn inclusions $\Lambda_k^n \to \Delta^n$ except possibly the right outer ones, i.e., for all $0 \leqslant k < n$. Similarly, it is a right fibration if it extends against all horns except possibly the left outer ones, i.e., for all $0 < k \leqslant n$.

Remark 3.14. $f : X \to S$ is a left fibration if and only if there exists a lift in the commutative diagram

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow \\ \Delta^n & \longrightarrow & S \end{array} \tag{3.15}$$

for any $n \in \mathbb{N}$ and $0 \leqslant k < n$.

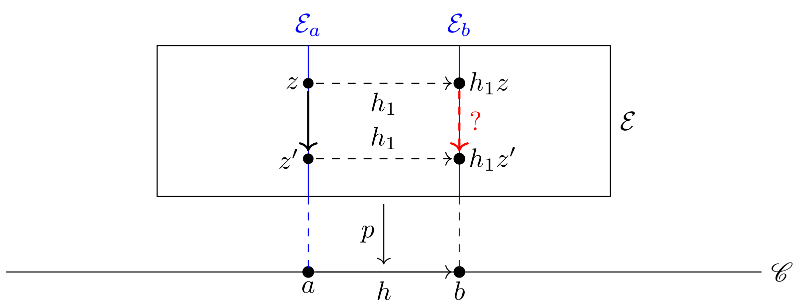
Remark 3.16. Let $\mathbf{LFib}(\mathscr{C})$ be the category of left fibrations of $\mathscr{C}$, then there is a straightening-unstraightening adjunction given by

$$\begin{array}{c} \mathbf{LFib}(\mathscr{C}) \\ {\scriptstyle \mathrm{St}}\big\downarrow\big\uparrow{\scriptstyle \mathrm{un}=\int_{\mathscr{C}}} \\ \mathbf{Fun}(\mathscr{C}, \mathcal{S}) \end{array} \tag{3.17}$$

Remark 3.18. Say $p : \mathcal{E} \to \mathscr{C}$ is a Cocartesian fibration, then we have

$$\mathrm{St}(p) : \mathscr{C} \to \mathbf{Cat}_\infty$$
$$(h : a \to b) \mapsto (\mathscr{E}_a \to \mathcal{E}_b)$$

where $\mathcal{E}_a = p^{-1}(a)$. Given any $z \to z'$, we know it descend to the identity map on u via p. However, we need to study what $h_1 z \to h_1 z'$ looks like, so that the involved square commutes.



Since $z \mapsto h_1 z$ is Cocartesian, then there exists a lift giving $h_1 f : h_z \to h_1 z'$, as desired.

Example 3.19. In light of Example 3.7, if (U, f) is U-Cartesian, then by Theorem 3.10 we obtain

$$\mathbf{CMon}(\mathbf{Ab})^{\mathrm{op}} \longrightarrow \mathbf{Cat}_\infty$$

$$\begin{array}{ccc} A & \longmapsto & \mathbf{Mod}(A) \\ {\scriptstyle u}\big\downarrow & & \big\uparrow{\scriptstyle f=u_*} \\ B & \longmapsto & \mathbf{Mod}(B) \end{array}$$

Similarly, if (U, f) is U-Cocartesian, then we obtain

$$\begin{array}{ccc} \mathbf{CMon}(\mathbf{Ab}) & \longrightarrow & \mathbf{Cat}_\infty \\ A & \longmapsto & \mathbf{Mod}(A) \\ {\scriptstyle u}\downarrow & & \downarrow {\scriptstyle f^\# = - \otimes_A B} \\ B & \longmapsto & \mathbf{Mod}(B) \end{array}$$

by Theorem 3.10.

Example 3.20. In light of Example 3.9, consider $p : \mathscr{C}^{\Delta^1} \to \mathscr{C}$ where $\mathscr{C}$ has fiber products and pullbacks, then

- as a Cocartesian fibration, we obtain

$$\begin{array}{ccc} \mathscr{C} & \longrightarrow & \mathbf{Cat}_\infty \\ c & \longmapsto & \mathscr{C}_{/c} \\ {\scriptstyle \alpha}\downarrow & & \downarrow \\ c' & \longmapsto & \mathscr{C}_{/c'} \end{array}$$

  via straightening;

- as a Cartesian fibration, we obtain

$$\begin{array}{ccc} \mathscr{C}^{\mathrm{op}} & \longrightarrow & \mathbf{Cat}_\infty \\ c & \longmapsto & \mathscr{C}_{/c} \\ {\scriptstyle \alpha}\downarrow & & \uparrow \\ c' & \longmapsto & \mathscr{C}_{/c'} \end{array}$$

  via straightening.

Example 3.21. For $C \in \mathbf{sSet} = \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set})$. Recall that any set can be described as a discrete category, then from C we obtain a functor

$$\Delta^{\mathrm{op}} \mapsto \mathcal{S}$$
$$[n] \mapsto C_n$$

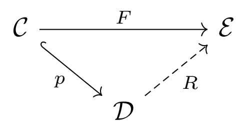
Therefore, we have a category $\mathbf{LFib}_{/\mathbf{\Delta}^{\mathrm{op}}}$ of functors $\Delta^{\mathrm{op}}_{/C} \to \Delta^{\mathrm{op}}$.

Proposition 3.22. Let $B \in \mathcal{S} \subseteq \mathbf{Cat}_\infty$, then the unstraightening functor is involved in a commutative diagram

$$\begin{array}{ccccc} \mathbf{Fun}(B, \mathcal{S}) & & \xrightarrow[\sim]{\mathrm{Un}} & & \mathcal{S}_{/B} \\ & {\scriptstyle \mathrm{Un}}\searrow & & \nearrow{\scriptstyle \sim} & \\ & & \mathbf{LFib}(B) & & \end{array}$$

4 Kan Extensions

Consider a functor $F : \mathcal{C} \to \mathcal{E}$ and an inclusion of subcategory $p : \mathcal{C} \hookrightarrow \mathcal{D}$, then we want to find a functor $G : \mathcal{D} \to \mathcal{E}$ such that the diagram

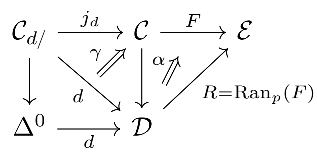


commutes. Unfortunately, this is not possible in general, but it is not possible to find an approximation as a substitute, by constructing a Kan extension.

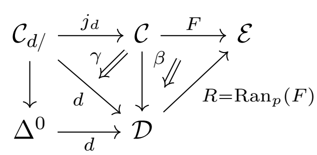
Definition 4.1. A right Kan extension of F along p is a functor $R : \mathcal{D} \to \mathcal{E}$ with a natural transformation $\alpha : R \circ p \Rightarrow F$, such that fro any $d : \mathcal{D}_0$,

$$R(d) \longrightarrow R \circ p \circ j_d \xrightarrow{\alpha \circ j_d} F \circ j_d$$

is a limit diagram, where $j_d : \mathcal{C}_{d/} \to \mathcal{C}$ is defined by a pullback diagram

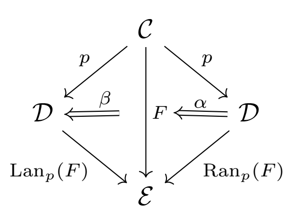


Dually, a left Kan extension $L = \mathrm{Lan}_p(F)$ is determined by the diagram



Remark 4.2. In general, $\alpha : R \circ p \Rightarrow F$ and $\beta : F \Rightarrow L \circ p$ are not equivalences. However, if $p : \mathcal{C} \hookrightarrow \mathcal{D}$ is a full embedding, then it is actually an equivalence.

With left and right Kan extensions, we determine a diagram



Remark 4.3. $(L, \beta : F \Rightarrow L \circ p)$ is a left Kan extension if and only if $(L^{\mathrm{op}}, \beta^{\mathrm{op}} : L^{\mathrm{op}} \circ p^{\mathrm{op}} \Rightarrow F^{\mathrm{op}} \Rightarrow F^{\mathrm{op}})$ is a right Kan extension.

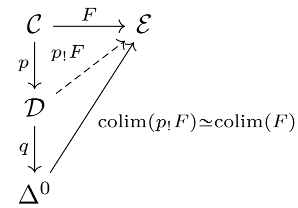
Remark 4.4. If $p : \mathcal{C} \to \Delta^0 = \mathcal{D}$, then

$$\mathrm{Ran}_p(F) \simeq \lim(F) : \Delta^0 \to \mathcal{E}$$

is just a limit, and similarly $\mathrm{Lan}_p(F) \simeq \mathrm{colim}(F)$ is a colimit.

Remark 4.5. Denote $L = \mathrm{Lan}_p(F) = p_! F$, then $(q \circ p)_! \simeq q_! \circ p_!$.

Therefore, we have



Remark 4.6. Let $(R, R \circ p \Rightarrow F)$ be a right Kan extension of F along p, then it is invertible upon replacing $\mathcal{C}, \mathcal{D}, \mathcal{E}$ by (categorical) equivalent categories, p and F by isomorphic functors, and α by homotopic natural isomorphisms.

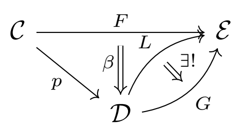
Remark 4.7. We have an adjunction triple

$$\begin{array}{c} \mathbf{Fun}(\mathcal{D},\mathcal{E}) \\ {\scriptstyle p_!=\operatorname{Lan}_p(-)}\uparrow \quad \downarrow{\scriptstyle p^*} \uparrow{\scriptstyle p_*=\operatorname{Ran}_p(-)} \\ \mathbf{Fun}(\mathcal{C},\mathcal{E}) \end{array} \tag{4.8}$$

Proposition 4.9 (Universal Property of Left Kan Extensions)**.** $(L, \beta : F \Rightarrow L \circ p)$ is a left Kan extension of F along p if and only if for any $G : \mathcal{D} \to \mathcal{E}$, we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbf{Fun}(\mathcal{D},\mathcal{E})}(L,G) & \xrightarrow{\quad p^* \quad} & \operatorname{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{E})}(L\circ p,) \\ & {\scriptstyle\simeq}\searrow \qquad \swarrow{\scriptstyle -\circ[\beta]} & \\ & \operatorname{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{E})}(F, G\circ p) & \end{array}$$

Therefore, the universal property originates as in



Proposition 4.10. Suppose we have an adjunction

$$\begin{array}{c} \mathcal{D} \\ {\scriptstyle p}\uparrow\downarrow{\scriptstyle q} \\ \mathcal{C} \end{array} \tag{4.11}$$

then there are induced mappings

$$\operatorname{Lan}_p(-) \simeq q^* : \operatorname{Fun}(\mathcal{C}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{D}, \mathcal{E})$$

and

$$\operatorname{Ran}_q(-) \simeq p^* : \operatorname{Fun}(\mathcal{D}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{E})$$

A Notations of Categories

- $\mathbf{Cat}$: 1-category of 1-categories.
- $\mathbb{F} = \mathbf{FinSet}$: category of finite sets.
- $\mathbf{FinSet}/K$: slice category of $\mathbf{FinSet}$ over $K \in \mathbb{F}$.
- $\mathbf{Span}(\mathbb{F})$: category of span of $\mathbb{F}$.
- $\mathbf{Bispan}(\mathbb{F})$: category of bispan of $\mathbb{F}$.
- $\mathbf{Cospan}(\mathbb{F})$: category of cospan of $\mathbb{F}$.
- $\mathbf{sSet}$: 1-category of simplicial sets.
- $\mathbf{qCat}$: 1-category of $(\infty, 1)$-categories.
- $\mathbf{CAT}$: 2-category of categories.
- $\mathscr{C}_{/c}$: slice category of $(\infty, 1)$-category $\mathscr{C}$ over $c \in \mathscr{C}$.
- $\mathscr{C}_{c/}$: coslice category of $(\infty, 1)$-category $\mathscr{C}$ under $c \in \mathscr{C}$.
- $\mathscr{C} \star \mathscr{D}$: (1-category) join of 1-categories $\mathscr{C}$ and $\mathscr{D}$.
- $\mathcal{S}$: $(\infty, 1)$-category of (small) spaces (anima/∞-groupoids).
- $\mathbf{Kan}$: $(\infty, 1)$-category of Kan complexes.
- $\mathbf{Cat}_\infty$: $(\infty, 1)$-category of $(\infty, 1)$-categories.
- $\mathbf{Cat}_{\infty/S}$: $(\infty, 1)$-category of $(\infty, 1)$-categories over $S \in \mathbf{qCat}$.
- $\mathbf{Cat}_{\infty/S}^{\text{Cart}}$: $(\infty, 1)$-category of Cartesian fibrations over $S \in \mathbf{qCat}$.
- $\mathbf{Cat}_{\infty/S}^{\text{Cocart}}$: $(\infty, 1)$-category of Cocartesian fibrations over $S \in \mathbf{qCat}$.
- $\mathbf{LFib}(\mathscr{C})$: category of left fibrations of $\mathscr{C}$.
- $\mathbf{RFib}(\mathscr{C})$: category of right fibrations of $\mathscr{C}$.

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