What is a Motive?

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Background. This is a talk given by Professor Jeremiah Heller at the UIUC What is...? Seminar in Fall 2024. *The talk started with Jeremiah playing "Femenine: No. 1, Prime" by Julius Eastman, Wild Up, and Christopher Rountree and explaining how it inspired him to give this talk.*

Suppose we are given a theory of spaces, then inevitably we need to study the corresponding structures via (generalized) cohomology theories. The motives are then the fundamental patterns one can see through the structures, i.e., cohomology theories "factor" through motives.

Example 1. Suppose we study the space of finite CW complexes (or just manifolds), then given a space X, one can study the assocaited numerical invariants, known as Betti numbers $\beta_n(X)$, i.e., the number of n-dimensional "holes" in X, or just $\dim_{\mathbb{Q}}(H^n(X;\mathbb{Q}))$. What kind of information do the Betti numbers see?

- If X is a (compact) surface, then the Betti numbers classify the surfaces (topologically).

In general, they see some, but not all information. In particular, the visible information are captured by $H^*(X;\mathbb{Q})$, which is the information captured by the "motive" of X, or the cochains $C^*(X;\mathbb{Q})$ of X (as CDGA). This is due to the fact that cohomology theories are obtained easily (via Eilenberg-Maclane axioms).

In algebraic geometry, the story is much more complicated. In Grothendieck's vision, one should consider algebraic varieties (over an algebraically closed field).

- For an affine variety $X \subseteq \mathbb{A}^n = k^n$ equipped with Zariski topology, X is the zeros of polynomials.
- If we want thinks to be compact, i.e., as a projective variety, then $X \subseteq \mathbb{P}^n$ is the zeros of homogeneous polynomials.

In this new philosophy, the "spaces" are given by the algebraic varieties.

Remark (Weil's Conjectures)**.** Let $F = \bar{\mathbb{F}}_p$ and X be a variety over F, then the number of $\mathbb{F}_{p^n}$-points $|X(\mathbb{F}_{p^n})|$ is the number of points of X whose coordinates are defined over $\mathbb{F}_{p^n}$. In particular, this is finite. The zeta function Z is defined by the relation

$$\log Z(X,t) = \sum_{m\geqslant 1} |X(\mathbb{F}_{p^m})|\frac{t^m}{m}.$$

Example 2. Since $\mathbb{P}^1 = \mathbb{A}^0 \cup \mathbb{A}^1$, then $|\mathbb{P}^1(\mathbb{F}_{p^n})| = 1+p^n$. Then $\log Z(\mathbb{P}^1,t) = \sum_{m\geqslant 1}(1+p^m)\frac{t^m}{m} = \log\left(\frac{1}{(1-t)(1-pt)}\right)$, so the zeta function of $\mathbb{P}^1$ is $\frac{1}{(1-t)(1-pt)}$.

Weil proved that if X is a curve of genus g, then $Z(x,t)$ is of the form $\frac{P_1(t)}{(1-t)(1-pt)}$ where $P_1(t) \in \mathbb{Z}[t]$ is of the form $(1-a_1t)(1-a_2t)\cdots(1-a_{2g}t)$ where $|a_i| = p^{\frac{1}{2}}$. One may observe that $\deg(P_1(t)) = 2g = b_1(\bar{X})$, which is the first Betti number of $\bar{X}$ defined over $\mathbb{Q}$ with mod p reduction X.

Moreover, Weil conjectured that given a smooth projective variety X of dimension n, then

$$Z(X,t) = \frac{P_1(t)\cdots P_{2n-1}(t)}{(1-t)P_2(t)\cdots P_{2n-2}(t)(1-p^nt)}$$

where each $P_i(t) \in \mathbb{Z}[t]$ is of the form $(1-a_{i1}t)(1-a_{i2}t)\cdots(1-a_{ib_i}t)$ where $|a_{ij}| = p^{\frac{1}{2}}$. In particular, b_i should be the ith Betti number of $\bar{X}$ over $\mathbb{Q}$ with any mod p reduction equal to X.

If X is defined over $\mathbb{F}_p$, then we observe that there is a Frobenius

$$\begin{aligned} \pi : X &\to X \\ [a_0 : \cdots : a_n] &\mapsto [a_0^p : \cdots : a_n^p]. \end{aligned}$$

The points $X(\mathbb{F}_{p^n})$ are the fixed points of the n-fold iteration π^n of the Frobenius. Weil wondered if there is a cohomology theory with a fixed point formula analogous to the Lefschetz fixed point theorem. That is, there should exist an algebraic cohomology $H^*(-;\mathbb{Q})$ with good properties, in particular, satisfying

$$|X(\mathbb{F}_{p^n})| = \sum_i (-1)^i \operatorname{Tr}(\pi^n)$$

where $\operatorname{Tr}(\pi^n)$ is the trace of π^n's action on the cohomology $H^i(X;\mathbb{Q})$. It turns out that such algebraic $\mathbb{Q}$-valued cohomology theory does not exist.

However, Grothendieck realized that such cohomology theories exist over other fields of chracteristic 0. For instance,

- étale cohomology $H^*_{\text{ét}}(X;\mathbb{Q}_\ell)$ as ℓ-adic $\mathbb{Q}_\ell$-vector spaces;
- de Rham cohomology $H^*_{\text{dR}}(X)$ as k-vector spaces for base field k;
- crystalline cohomology $H^*_{\text{crys}}(X)$ as vector space/field of fractions of Witt vectors.

Eventually, $H^*_{\text{ét}}(X;\mathbb{Q}_\ell)$ was used to prove Weil's conjecture.

Remark. These cohomology theories are not equivalent, but they behave similarly. For instance, given an endomorphism $\alpha : X \to X$, there is an induced endomorphism α^* on any one of those cohomology theories. It is not obvious, but the corresponding traces $\operatorname{Tr}(\alpha^*) \in \mathbb{Q}$ and is independent of the chosen cohomology theory. Therefore, there is some sort of motif that lies under the cohomology theories that governs the arithmetic/geometric behavior.

More precisely, given the category of smooth projective spaces $\mathbf{SmProj}_k$ over k, a good enough cohomology theory H is now a functor H on $\mathbf{SmProj}_k$ into graded k-vector spaces that factors via $h : \mathbf{SmProj}_k \to M_{\mathbb{Q}}(k)$ with the following properties:

- the maps are $\mathbb{Q}$-vector spaces;
- it is an abelian category;
- it is semi-simple;
- it is a Tannakian category, so it is equivalent to the category of representations of a profinite group, called the motivic Galois group.

A good cohomology theory under these restrictions should satisfy the following:

- there is a Poincaré duality $H^i(X)^\vee \cong H^{2n-i}(X)$ where $n = \dim(X)$;
- there is a Künneth formula $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$;
- there is a cycle class map
$$\mathrm{cl}_X : Z^i(X) \to H^{2i}(X)$$
where $Z^i(X)$ is the set of finite formal sums $\sum\limits_{Z \subseteq X} n_Z[Z]$ where Z is a codimension-i subvariety of X. (Every algebraic cycle gives a cohomology class.)

Without keeping track of the gradings, we have

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y) \cong H^*(X)^\vee \otimes H^*(Y) \cong \mathrm{Hom}(H^*(X), H^*(Y)).$$

Therefore, the elements of the cohomology $H^*(X \times Y)$ gives operators from $H^*(X)$ to $H^*(Y)$. Moreover, the image of $Z^*(X \times Y) \to H^*(X \times Y)$ gives algebraic operators in $\mathrm{Hom}(H^*(X), H^*(Y))$. Therefore, this gives an idea of constructing $M_{\mathbb{Q}}(k)$. For instance, one may construct the category $\mathbf{CV}_k$ (of correspondence of varieties) whose objects are smooth projective varieties, with morphisms as correspondences $\mathrm{Hom}_{\mathbf{CV}_k}(X, Y) = Z^{\dim(X)}(X \times Y)_{\mathbb{Q}}$. A map of varieties $f : Y \to X$ gives a graph $\Gamma_f \subseteq Y \times X$ as an element in $Z^{\dim(X)}(X \times Y)$, under some equivalences such that the intersection product is defined. Therefore, as we have

$$\begin{aligned} V_k^{\mathrm{op}} := \mathbf{SmProj}_k^{\mathrm{op}} &\to \mathbf{CV}_k \\ X &\mapsto X \\ (f : Y \to X) &\mapsto (\Gamma_f : X \to Y) \end{aligned}$$

there is a choice of representatives. Each choice leads to a category of motives. The finest choices are the rational equivalences (which leads to Chow motives), and the coarsest choices are the numerical equivalences (which leads to Grothendieck motives). Some choices in between are homological equivalences and algebraic equivalences. (Finally, we need to add idempotent elements into $\mathbf{CV}_k$, and some other step that we omit here.)

The talk ended here while Jeremiah resumed playing a segment of the song.

Remark. The construction for *the* category of motives is unclear, i.e., as a universal choice of category of motives (arising from equivalences). One obvious choice to make would be the rational equivalence, but the corresponding cohomology theory turns out not to be good. If Grothendieck's standard conjectures of type C and D are true, we do get such a universal category.

One way of attacking this issue is to follow the motivic theory of Suslin-Voevodsky, where one hopes to construct derived category of motives directly, then the motivic category is just a non-abelian version of the derived category.

In particular, $M_{\mathbb{Q}}(k)$ sits inside the category of mixed motives as an ambient category. On the other hand, every motive is a direct sum of pure motives.