Model Category Notes

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Throughout the notes, let $\mathcal{M}$ be a model category with classes of maps $\mathcal{W}$, Cof, and Fib, that contains all identity maps. Let us start by recalling the definition of a model category.

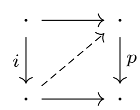
"Recall" 1. A model category $\mathcal{M}$ is a category with classes of maps $\mathcal{W}$ (weak equivalences), Cof (cofibrations), and Fib (fibrations), that contains all identity maps, such that

MC1 $\mathcal{M}$ is complete and cocomplete;

MC2 $\mathcal{W}$ satisfies 2-out-of-3;

MC3 $\mathcal{W}$, Cof, and Fib are closed under retracts;

MC4 consider a commutative diagram



then the lift exists if either

a. i is a cofibration, and p is a weak equivalence and fibration, or
b. i is a weak equivalence and cofibration, and p is a fibration;

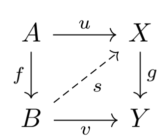
MC5 for any morphism $f : X \to Y$, there exists two factorizations of the form

$$X \xrightarrow{i} Z \xrightarrow{p} Y$$

one such that i is a cofibration and p is a weak equivalence and a fibration, and the other one such that i is a cofibration and a weak equivalence and p is a fibration.

Remark 2. Note that model categories are self-dual. That is, there is a model category $(\mathcal{M}, \mathcal{W}, \mathrm{Cof}, \mathrm{Fib})$ if and only if there is a model category $(\mathcal{M}^{\mathrm{op}}, \mathcal{W}^{\mathrm{op}}, \mathrm{Fib}^{\mathrm{op}}, \mathrm{Cof}^{\mathrm{op}})$.

Definition 3. Given $f; A \to B$ and $g : X \to Y$, we write $f \boxslash g$ if for any commutative square



there exists a lift $s : B \to X$ such that $sf = u$ and $gs = v$, i.e., all relevant diagrams commute. We then say f has the left lifting property with respect to g, or g has the right lifting property with respect to f.

Remark 4. MC4 now says

- $\mathrm{Cof} \boxslash (\mathcal{W} \cap \mathrm{Fib})$, and

- $(W \cap \mathrm{Cof}) \boxslash \mathrm{Fib}$.

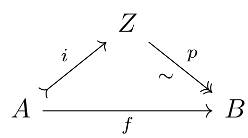
In fact, the lifting property now characterizes the fibrations and cofibrations.

Proposition 5.

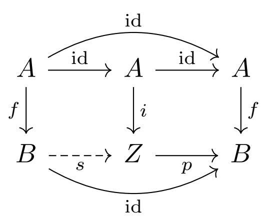
1. *f is a cofibration if and only if $f \boxslash (\mathcal{W} \cap \mathrm{Fib})$;*
2. *$f \in \mathcal{W} \cap \mathrm{Cof}$ if and only if $f \boxslash \mathrm{Fib}$;*
3. *g is a fibration if and only if $(W \cap \mathrm{Cof}) \boxslash g$;*
4. *$g \in \mathcal{W} \cap \mathrm{Fib}$ if and only if $\mathrm{Cof} \boxslash g$.*

Proof. All proofs in ($\Rightarrow$) direction are given by MC4. Moreover, 1. and 3. are formal duals, and so are 2. and 4. Therefore, it suffices to prove 1. and 2.

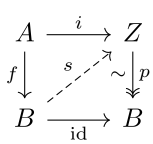
Let us finish the proof of 1. Suppose $f \boxslash (\mathcal{W} \cap \mathrm{Fib})$, and we want to show that $f \in \mathrm{Cof}$. By MC5, we choose a factorization



such that i is a cofibration and p is a fibration and a weak equivalence. It suffices to show that f is a retract of i, thus f is a retract of cofibration, so by MC3 we know $f \in \mathrm{Cof}$ as desired. But that boils down to showing the existence of a map $s : B \to Z$ such that the diagram

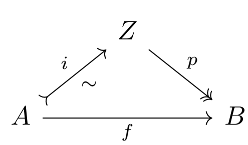


commutes. We construct s as the lifting of the commutative square



which exists because $f \boxslash (W \cap \mathrm{Fib})$.

The proof of 2. follows a similar idea by choosing a factorization



such that i is a cofibration and p is a fibration and a weak equivalence. $\square$

Definition 6. Given a model category $\mathcal{M}$, a weak factorization system is two classes of maps $\mathcal{L}$ and $\mathcal{R}$, such that

1. $\ell \in \mathcal{L}$ if and only if $\ell \boxslash R$, and $r \in \mathcal{R}$ if and only if $\mathcal{L} \boxslash r$, and

2. for any morphism $f : A \to B$ in $\mathcal{M}$, there exists a factorization

$$\begin{tikzcd} A \arrow[r, "\ell"] \arrow[rr, bend left, "f"] & C \arrow[r, "r"] & B \end{tikzcd}$$

   of f with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$.

Example 7. A model category naturally has two weak factorization systems, namely $(\mathrm{Cof}, \mathcal{W} \cap \mathrm{Fib})$ and $(\mathcal{W} \cap \mathrm{Cof}, \mathrm{Fib})$.

Proposition 8. *Given a weak factorization system $(\mathcal{L}, \mathcal{R})$, then*

1. *if $f, g \in \mathcal{L}$ and gf exists, then $gf \in \mathcal{L}$; similarly, if $f, g \in \mathcal{R}$ and gf exists, then $gf \in \mathcal{R}$;*
2. *$\mathcal{L}$ is closed under cobase-change: if there is a pushout (assuming they exist)*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

   then $f \in \mathcal{L}$ implies $g \in \mathcal{L}$; dually, $\mathcal{R}$ is closed under base-change: if there is a pullback (assuming they exist)

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

   then $g \in \mathcal{R}$ implies $f \in \mathcal{L}$.

Proof.

1. Let us check it for $\mathcal{L}$. Consider

$$A \xrightarrow{f} B \xrightarrow{g} C$$

   for $f, g \in \mathcal{L}$. Let $p : X \to Y$ be any $p \in \mathcal{R}$. Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] \arrow[dd, bend right, "gf"'] & X \arrow[dd, "p"] \\ B \arrow[d, "g"] \arrow[ur, dashed, "s"] & \\ C \arrow[r, "v"'] \arrow[uur, dashed, "t"'] & Y \end{tikzcd}$$

   We claim that lifts $s : B \to X$ and $t : C \to X$ exist, so $gf \boxslash \mathcal{R}$, therefore $gf \in \mathcal{L}$. We construct s as the lifting of the commutative square

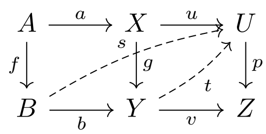
$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r, "vg"'] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

   for $f \in \mathcal{L}$ and $p \in \mathcal{R}$. We then construct t as the lifting of the commutative square

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[d, "g"'] & X \arrow[d, "p"] \\ C \arrow[r, "v"'] \arrow[ur, dashed, "t"] & Y \end{tikzcd}$$

   for $g \in \mathcal{L}$ and $p \in \mathcal{R}$.

2. Let us check it for $\mathcal{L}$. Given a diagram



such that $f \in \mathcal{L}$ and $p \in \mathcal{R}$, and that the left square is a pushout, we want to show that $g \boxslash p$. First, since $f \boxslash p$, we know there exists $s : B \to U$ such that $sf = ua$ and $ps = vb$. Moreover, since g is in the pushout square, then there exists a lifting $t : Y \to U$ such that $tg = u$ and $tb = s$, therefore $tga = ua = sf = tbf$. Finally, since $ptg = pu = vg$ and $ptb = ps = vb$, then $pt = v$ by the pushout property.

□

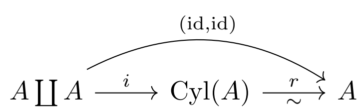
Corollary 9. Cof *and* $\mathcal{W} \cap$ Cof *are closed under cobase-change, while* Fib *and* $\mathcal{W} \cap$ Fib *are closed under base-change.*

We now construct the homotopy category $\mathrm{Ho}(\mathcal{M})$ of a model category $\mathcal{M}$. This requires studying relations on $\mathrm{Hom}_{\mathcal{M}}(X, Y)$, called left homotopy ($\sim_\ell$) and right homotopy ($\sim_r$). There are a few questions we can ask about them:

- Are these equivalence relations?
- Are they compatible with composition?
- Are they the same?
- Are homotopy equivalences just the weak equivalences?

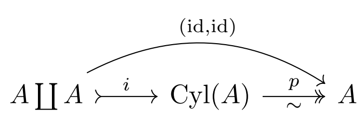
Unfortunately, none of the answers to those questions is a definite "yes", but they are still true under certain circumstances. For instance, if all objects involved are bifibrant, then all the answers are "yes".

Definition 10. A cylinder object $\mathrm{Cyl}(A)$ on $A \in \mathrm{ob}(\mathcal{M})$ is a diagram



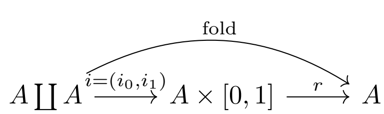
such that $r \in \mathcal{W}$. A good cylinder object is a cylinder object such that $i \in \mathrm{Cof}$. A very good cylinder object is a good cylinder object such that $r \in \mathrm{Fib}$.

Remark 11. The very good cylinder objects always exist according to MC5.

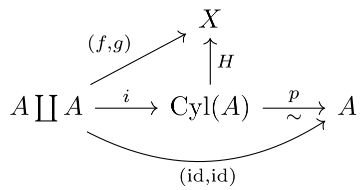


Remark 12. If we write $i = (i_0, i_1)$, then we have $ri_0 = \mathrm{id} = ri_1$, therefore $i_0, i_1 \in \mathcal{W}$.

Remark 13. Definition 10 mimics properties of cylinders. That is, for $A \in \mathsf{Top}$, one can consider

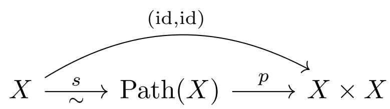


Definition 14. We say $f, g : A \to X$ are left homotopic, i.e., $f \sim_\ell g$, if there exists cylinder objects and a commutative diagram



We then say H is a left homotopy.

Definition 15. A path object on X is a diagram

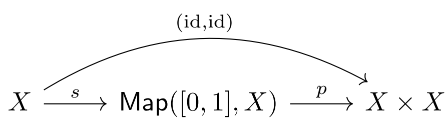


such that $s \in W$. We say a path object on X is good if $p \in \mathrm{Fib}$. We say a good path object is very good if $s \in \mathrm{Cof}$.

Remark 16. The very good path objects always exist according to MC5.

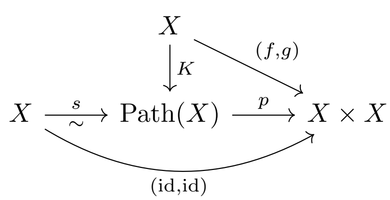
Remark 17. If we write $p = (p_0, p_1)$, then $p_0 s = \mathrm{id} = p_1 s$, therefore $p_0, p_1 \in \mathcal{W}$.

Remark 18. Definition 15 mimics the property of



where p is the evaluation.

Definition 19. We say $f, g : A \to X$ are right homotopic, i.e., $f \sim_r g$ if there exists path objects and a commutative diagram



We then say K is a right homotopy.

Definition 20. We define $[A, X]_\ell = \mathrm{Hom}_\mathcal{M}(A, X)/\sim_\ell$ and $[A, X]_r = \mathrm{Hom}_\mathcal{M}(A, X)/\sim_r$, where $\sim_\ell$ and $\sim_s$ are the equivalence relations generated by left homotopy and right homotopy, respectively.

We state the following lemma for cylinder objects.

Lemma 21.

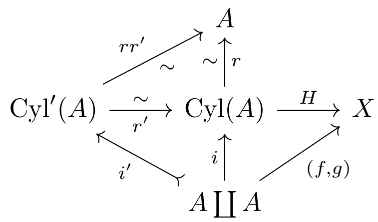
1. *If $f \sim_\ell g : A \to X$, then there exists a good left homotopy, i.e., there exists a good cylinder object and a left homotopy defined from it.*

2. *Moreover, if X is fibrant, then there exists a very good left homotopy, i.e., there exists a very good cylinder object and a left homotopy defined on it.*

There are also dual statements about right homotopy and good path object.

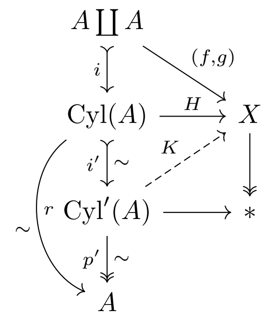
Proof.

1. By MC5, we factor $i = r'i'$, then we have a commutative diagram



   where rr' is a weak equivalence by MC2. Then $H' = Hr'$ is the good left homotopy we want.

2. Suppose we are given a left homotopy H, then we have a diagram



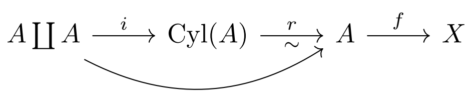
   where we factorize $r = p'i'$ via some object $\mathrm{Cyl}'(A)$. By MC2, i' is also a weak equivalence, therefore $\mathrm{Cyl}'(A)$ is a very good cylinder object since $i'i$ is a cofibration. Finally, the lift K exists by MC4.

□

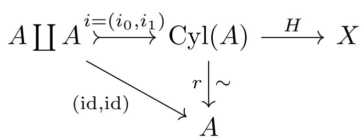
Lemma 22.

1. *If A is cofibrant, then $\sim_\ell$ is an equivalence relation on* $\mathrm{Hom}_{\mathcal{M}}(A, X)$.
2. *If X is fibrant, then $\sim_r$ is an equivalence relation on* $\mathrm{Hom}_{\mathcal{M}}(A, X)$.

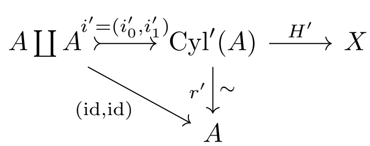
Proof. We will prove the first statement. Symmetry is obvious. To prove reflexivity, consider



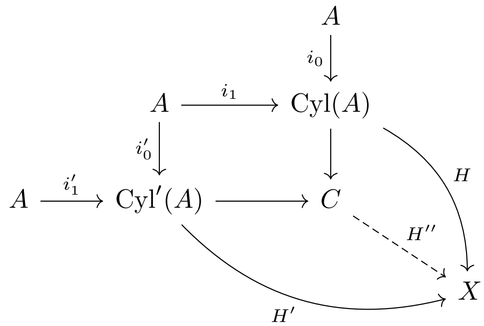
then $fr = H$ is a left homotopy which shows that $f \sim_\ell f$. Moreover, we need to show transitivity, which mimics the proof for gluing on cylinders. Take $f, g, h : A \to X$, and $H : f \sim_\ell g$ and $H' : g \sim_\ell h$. Consider cylinders



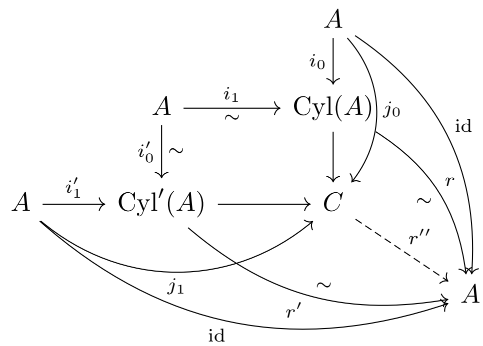
and



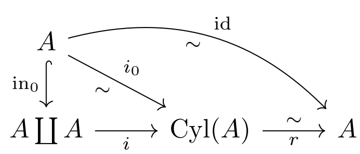
Now we form a pushout square



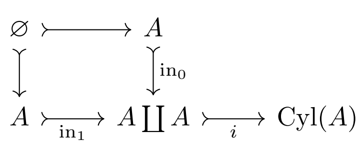
where H'' exists because of the pushout. Finally, we need to show that C actually gives a cylinder object. Consider



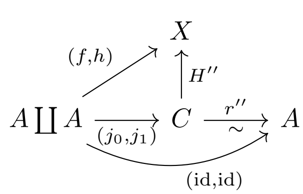
Since C gives a pushout square, then there exists r'', and it satisfies $r''j_0 = \mathrm{id} = r''j_1$. To show r'' is a weak equivalence, we just need to show that the map $\mathrm{Cyl}'(A) \to C$ is a weak equivalence by MC2. Consider the commutative diagram



Note that i is a cofibration since we have a good left homotopy. Recall that A is cofibrant, so we have a pushout square



By cobase-change, both in_0 and in_1 are cofibrations. Since i is a cofibration as well, so the compositions $i_0 = i\,\mathrm{in}_0$ and $i_1 = i\,\mathrm{in}_1$ are cofibrations too. In particular, $i_0, i_1 \in \mathcal{W} \cap \mathrm{Cof}$ are acyclic cofibrations. By the cobase-change, we know $\mathrm{Cyl}'(A) \to C$ is also an acyclic cofibration, as desired. This concludes the proof because we have a commutative diagram



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