# Bounds on Simple Hexagonal Lattice

Jiantong Liu

Lattice Number Subproject, Polymath Jr 2022

August 14, 2022

# Overview

# Outline

# Definitions

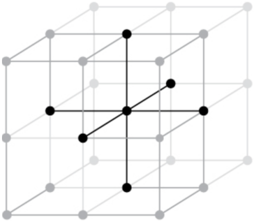
## Definition (Cubic Lattice)

For vectors $x = \langle 1, 0, 0 \rangle$, $y = \langle 0, 1, 0 \rangle$, and $z = \langle 0, 0, 1 \rangle$, their $\mathbb{Z}$-linear combinations form the cubic lattice, i.e.

$$\mathbb{L}^3 = \{a \langle 1, 0, 0 \rangle + b \langle 0, 1, 0 \rangle + c \langle 0, 0, 1 \rangle \mid a, b, c \in \mathbb{Z}\}.$$

Alternatively, we can define it as

$$\mathbb{L}^3 = (\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}).$$



# Definitions

## Definition (Simple Hexagonal Lattice/sh-lattice)

For vectors $x = \langle 1, 0, 0 \rangle$, $y = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle$, and $w = \langle 0, 0, 1 \rangle$, we define the simple hexagonal lattice as all $\mathbb{Z}$-linear combinations of $x, y$, and $w$, i.e.

$$sh = \{a \langle 1, 0, 0 \rangle + b \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle + c \langle 0, 0, 1 \rangle \mid a, b, c \in \mathbb{Z}\}.$$

For convenience, we also define $z = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle$. Note that $z = y - x$.

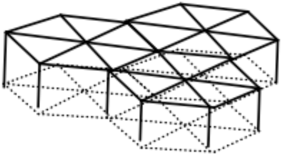
# Definitions

## Definition (Simple Hexagonal Lattice/sh-lattice)

For vectors $x = \langle 1, 0, 0 \rangle$, $y = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle$, and $w = \langle 0, 0, 1 \rangle$, we define the simple hexagonal lattice as all $\mathbb{Z}$-linear combinations of $x, y$, and $w$, i.e.

$$sh = \{a \langle 1, 0, 0 \rangle + b \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle + c \langle 0, 0, 1 \rangle \mid a, b, c \in \mathbb{Z}\}.$$

For convenience, we also define $z = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle$. Note that $z = y - x$.



# Definitions

## Main Focus

The relationship between specific data of cubic lattice and of sh-lattice.

# Definitions

## Main Focus

The relationship between specific data of cubic lattice and of sh-lattice.

## Definition (Lattice Knot)

A polygon in a lattice $\mathcal{A}$ is called an $\mathcal{A}$-lattice knot if it is closed and non-intersecting.

# Definitions

## Main Focus

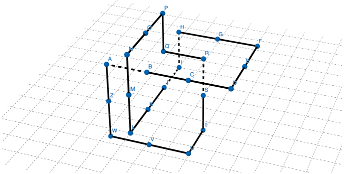
The relationship between specific data of cubic lattice and of sh-lattice.

## Definition (Lattice Knot)

A polygon in a lattice $\mathcal{A}$ is called an $\mathcal{A}$-lattice knot if it is closed and non-intersecting.

## Definition (Stick)

An $\alpha$-stick in the $\alpha$ direction is a maximal segment of a lattice knot.



# Definitions

Any cubic lattice or sh-lattice knot can be represented by a string of the appropriate sticks. For example, a sh-lattice can be represented by

$$x^{\alpha_1(x)}y^{\alpha_1(y)}z^{\alpha_1(z)}w^{\alpha_1(w)}\ldots x^{\alpha_n(x)}y^{\alpha_n(y)}z^{\alpha_n(z)}w^{\alpha_n(w)}\ldots$$

# Definitions

Any cubic lattice or sh-lattice knot can be represented by a string of the appropriate sticks. For example, a sh-lattice can be represented by

$$x^{\alpha_1(x)} y^{\alpha_1(y)} z^{\alpha_1(z)} w^{\alpha_1(w)} \dots x^{\alpha_n(x)} y^{\alpha_n(y)} z^{\alpha_n(z)} w^{\alpha_n(w)} \dots$$

## Definition (Stick Number)

The stick number of a knot $K$ in a lattice $\mathcal{A}$, denoted as $s_{\mathcal{A}}(K)$, is the minimal number of sticks required to construct a representation of the knot $K$ in a given lattice.

# Definitions

Any cubic lattice or sh-lattice knot can be represented by a string of the appropriate sticks. For example, a sh-lattice can be represented by

$$x^{\alpha_1(x)}y^{\alpha_1(y)}z^{\alpha_1(z)}w^{\alpha_1(w)}\dots x^{\alpha_n(x)}y^{\alpha_n(y)}z^{\alpha_n(z)}w^{\alpha_n(w)}\dots$$

**Definition (Stick Number)**

The stick number of a knot $K$ in a lattice $\mathcal{A}$, denoted as $s_{\mathcal{A}}(K)$, is the minimal number of sticks required to construct a representation of the knot $K$ in a given lattice.

Therefore, the stick number of a knot can be determined by summing the number of sticks in each direction. For example, the stick number in a sh-lattice is the sum of $x$-, $y$-, $z$-, and $w$-sticks.

# Definitions

## Definition (Knot Type)

Given a knot $K$, a knot type $[K]$ is the equivalent class of all knots $K'$ that are equivalent to $K$ under ambient isotopy.

# Definitions

## Definition (Knot Type)

Given a knot $K$, a knot type $[K]$ is the equivalent class of all knots $K'$ that are equivalent to $K$ under ambient isotopy.

## Definition (Stick Number of a Knot Type)

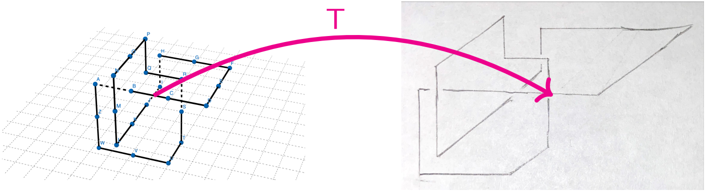
The stick number of a knot type $[K]$ in a given lattice $\mathcal{A}$, denoted as $s_{\mathcal{A}}[K]$, is the minimal number of sticks required to construct any representation $K$ of knot type $[K]$ in the lattice $\mathcal{A}$, i.e. $s_{\mathcal{A}}[K] = \min_{K' \in [K]} s_{\mathcal{A}}(K')$.

# Outline

# Main Tool: Linear Transformation

$$T = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T$ sends $x$-sticks, $y$-stick, and $z$-sticks in the cubic lattice to $x$-sticks, $y$-sticks, and $w$-sticks in the sh-lattice, respectively.



# $T$-invariant Properties

# $T$-invariant Properties

- Knot type

# $T$-invariant Properties

- Knot type
- Stick number

# $T$-invariant Properties

- Knot type
- Stick number
- Order of sticks

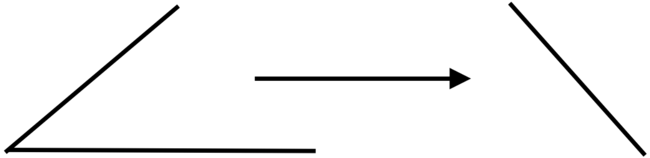
# $T$-invariant Properties

- Knot type
- Stick number
- Order of sticks
- Length of each stick

# Motivation

## Intuition

For a given knot type $[K]$, its stick number in sh-lattice should be less than its stick number in cubic lattice.



# Outline

# Upper Bound on Stick Number

**Proposition**

*For any knot type [K], $s_{sh}[K] \leq s_L[K]$, where $s_{sh}$ is the stick number of [K] in the simple hexagonal lattice and $s_L$ is the stick number of [K] in the cubic lattice.*

# Upper Bound on Stick Number

**Proposition**

*For any knot type* $[K]$, $s_{sh}[K] \leq s_L[K]$, *where* $s_{sh}$ *is the stick number of* $[K]$ *in the simple hexagonal lattice and* $s_L$ *is the stick number of* $[K]$ *in the cubic lattice.*

**Question**

But when can we make this bound strict, i.e. substitute a pair of $x$- and $y$-stick to a $z$-stick in sh-lattice?

# Upper Bound on Stick Number

**Lemma**

*Project a knot [K] in the cubic lattice down to the xy-plane. Suppose we have a x- and y-stick of equal length connected in an L-shape. Connect the endpoints to create an isosceles right triangle. If there are no z-sticks within the triangle intersecting the z-level on which these sticks lie, then we can replace them with a z-stick in the hexagonal lattice after applying* $T$.

# Upper Bound on Stick Number

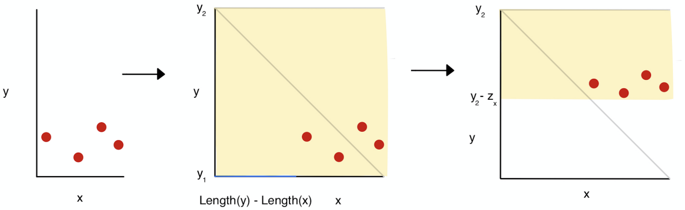
**Theorem**

For any knot type $[K]$, $s_{sh}[K] < s_L[K]$.

# Upper Bound on Stick Number

**Theorem**

For any knot type $[K]$, $s_{sh}[K] < s_L[K]$.



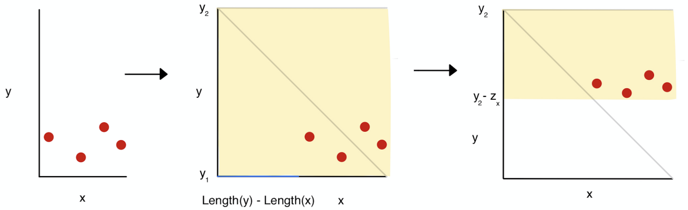

Perform the following operations:

- Lengthen the shorter stick so the lengths are equal
- Compress all sticks inside the highlighted area above the diagonal
- ...

# Upper Bound on Stick Number

**Theorem**

For any knot type $[K]$, $s_{sh}[K] < s_L[K]$.



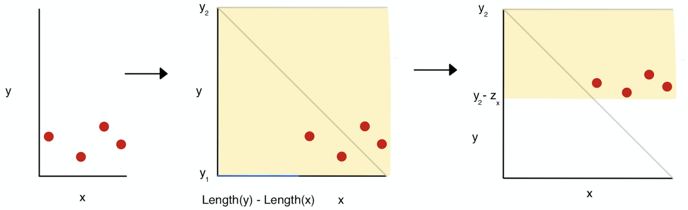

Perform the following operations:

- *"But how can you guarantee that those sticks are still on the lattice?"*
- *"That's the neat part – you don't!"*

# Upper Bound on Stick Number

**Theorem**

For any knot type $[K]$, $s_{sh}[K] < s_L[K]$.




Perform the following operations:

- Lengthen the shorter stick so the lengths are equal
- Compress all sticks inside the highlighted area above the diagonal
- Expand entire knot to fit back into the lattice
- Apply $T$ and previous lemma

# Upper Bound on Edge Length

**Definition (Edge, Edge Length)**

An edge of a polygon in a lattice is a unit-length segment of the polygon between two points in the lattice. The edge length of a polygon in a lattice is the total number of edges in the polygon.

We denote $e_{\mathcal{A}}[K]$ to be the minimal edge length of knot type $[K]$ in the $\mathcal{A}$ lattice.

# Upper Bound on Edge Length

## Proposition

*$T$ preserves edge length.*

# Upper Bound on Edge Length

**Proposition**

*$T$ preserves edge length.*

**Proposition**

*For knot type $[K]$ in the simple hexagonal lattice, $e_{sh}[K] < e_L[K]$, where $e_{sh}[K]$ is the edge length of $[K]$ in the simple hexagonal lattice and $e_L[K]$ is the edge number of $[K]$ in the cubic lattice.*

# Upper Bound on Edge Length

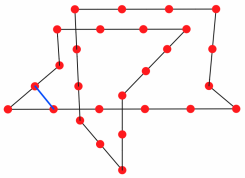
**Proposition**

*$T$ preserves edge length.*

**Proposition**

*For knot type $[K]$ in the simple hexagonal lattice, $e_{sh}[K] < e_L[K]$, where $e_{sh}[K]$ is the edge length of $[K]$ in the simple hexagonal lattice and $e_L[K]$ is the edge number of $[K]$ in the cubic lattice.*

Find a corner and reduce the $x$-stick and $y$-stick to a $z$-stick; no $z$-stick should be within the triangle in cubic lattice, so we get the corner we need



# Outline

# Lower Bound on Stick Number and Edge Length

**Proposition**

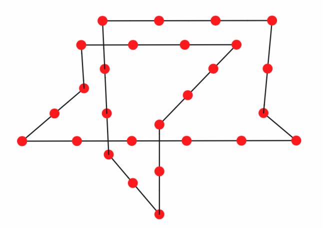
*For any given knot $K$, we have* $s_{sh}(K) \geq 2\sqrt{s_L(K) + \frac{9}{4}} - 3$

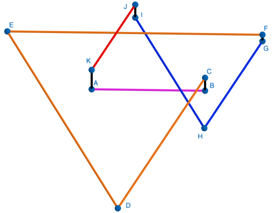
**Proposition**

*For a given non-trivial knot $K$, we have* $e_{sh}(K) \geq \frac{3e_L(K)+20}{8}$.

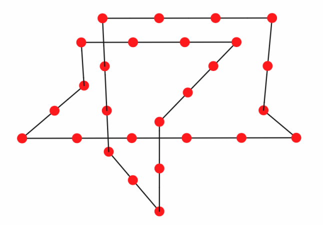
# Outline

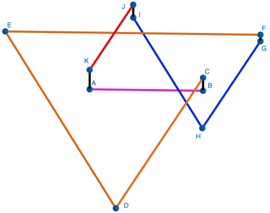
# Future Work





# Future Work






**Theorem (Huh & Oh, 2010)**

*The only non-trivial knot types $[K]$ with $s_L([K]) \leq 14$ are $3_1$ and $4_1$.*

**Conjecture**

The only non-trivial knot types $[K]$ with $s_{sh}([K]) \leq 11$ are $3_1$ and $4_1$.

# Future Work

- Is there a maximum (respectively, minimum) $\alpha > 0$ such that $s_{sh}(K) \leq Cs_L(K)^\alpha$ (respectively, $s_{sh}(K) \geq Cs_L(K)^\alpha$) for constant $C > 0$?
- Find a function $f$ such that for any stick number $x$ in the sh-lattice there is $f(x) \geq dist_{sh}([K])$. Here

$$dist_{sh}([K]) = \min_{K' \in [K]} \max_{x,y \in K'} d_{K'}(x, y)$$

where $d_{K'}(x, y)$ is the distance between two points $x, y \in K'$ along the knot $K'$.

# References


Bailey, Ryan, et al.
Stick Numbers in the Simple Hexagonal Lattice
*Involve, a Journal of Mathematics* 8.3 (2015): 503-512.

Huh, Youngsik, and Seungsang Oh
Lattice Stick Numbers of Small Knots
*Journal of Knot Theory and Its Ramifications* 14.07 (2005): 859-867.

Huh, Youngsik, and Seungsang Oh
Knots with Small Lattice Stick Numbers
*Journal of Physics A: Mathematical and Theoretical* 43.26 (2010): 265002.

Mann, Casey E., Jennifer C. Mcloud-Mann, and David P. Milan
The Stick Number for the Simple Hexagonal Lattice
*Journal of Knot Theory and Its Ramifications* 21.14 (2012): 1250120.