Étale Cohomology Notes

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Background. Let $\mathbf{Sch}/X$ be the category of schemes over a base scheme X, and we consider $\mathbf{C}/X$ be a full subcategory of $\mathbf{Sch}/X$ that is closed under pullbacks. Recall that the big étale site on $\mathbf{C}/X$ is having $\mathbf{C}$ equipped with the topology where the covers are given by jointly surjective collections of morphisms $\{X_i \to X\}_{i\in I}$, such that each $X_i \to X$ is an étale morphism.

We are more interested in the small étale site on X, that is, the big étale site on the category of étale morphisms into S. To be precise, this is the site with underlying category $\mathbf{Et}/X$, whose objects are the étale morphisms $U \to X$ and whose arrows are the X-morphisms $U \to V$, with associated coverings as surjective families of étale morphisms in $\mathbf{Et}/X$. This will be the most important toy example throughout the talk. Other examples we may be interested in includes the small Zariski site, associated with open immersions, or the big flat site, associated with flat morphisms that are locally of finite type. Therefore, we denote $X_E = (\mathbf{C}/X)_E$ (respectively, $U_E = (\mathbf{C}/U)_E$ for morphism $U \to X$) when we are talking about one of these examples of sites.

We define $\mathbf{P}(X_E) = \mathbf{P}((\mathbf{C}/X)_E)$ to be the category of presheaves[1] on $(\mathbf{C}/X)_E$, where a morphism of presheaves is a morphism of functors. Therefore, for a morphism $\varphi : P \to Q$ of presheaves, it assigns an object $U \in \mathbf{C}/X$ to a homomorphism $\varphi(U) : P(U) \to Q(U)$ that commutes with the restrictions. Moreover, we define $\mathbf{S}(X_E) = \mathbf{S}((\mathbf{C}/X)_E)$ to be the category of sheaves on $(\mathbf{C}/X)_E$, where a morphism of sheaves is the same as a morphism of presheaves. The associated inclusion functor and sheafification functor are denoted by i and a, respectively, which gives an adjunction

$$\begin{array}{c} \mathbf{P}(X_E) \\ a \downarrow\uparrow i \\ \mathbf{S}(X_E) \end{array}$$

1 Introduction

To define cohomology, we need to define derived functors, which requires having enough injective objects in the category.

Lemma 1. $\mathbf{P}(X_E)$ has enough injectives.

Theorem 2. $\mathbf{S}(X_E)$ has enough injectives.

The proofs we present below make use of the following result from [BDH68].

Definition 3. A family of objects $(A_i)_{i\in I}$ of a category $\mathbf{A}$ is a family of generators if, given a monomorphism $B \to A$ in $\mathbf{A}$ that is not an isomorphism, there is some index $j \in I$ and a morphism $A_j \to A$ that does not factor through $B \to A$.

[1] That is, a contravariant functor with target in $\mathbf{Ab}$.

Example 4. The singleton $\{\mathbb{Z}\}$ is a family of generators for $\mathbf{Ab}$: whenever $i : B \hookrightarrow A$ is not an isomorphism of abelian groups, we construct

$$\begin{aligned} f_a : \mathbb{Z} &\to A \\ 1 &\mapsto a \end{aligned}$$

for some $a \in A$ such that $a \notin i(B)$. In particular f_a does not factor through i.

Lemma 5. Any abelian category that satisfies AB3* (with arbitrary products), AB5 (with arbitrary coproducts,[2] and filtered colimits are exact), and possesses a family of generators $(A_i)_{i \in I}$ has enough injectives.

Roughly speaking, this follows from the proof for the category of modules, which uses Baer's criterion.

Proof of Lemma 1. It suffices to show that $\mathbf{S}(X_E)$ has a family of generators. Suppose $\mathscr{C}$ and $\mathscr{D}$ are abelian categories where $\mathscr{C}$ is small, and $\mathscr{D}$ satisfies AB3 and has generators. Then $\operatorname{Hom}(\mathscr{C}, \mathscr{D})$ also satisfies AB3 and has generators. Let $\mathscr{C} = (\mathbf{C}/X)_E^{\mathrm{op}}$ and $\mathscr{D} = \mathbf{Ab}$, then $\mathbf{P}(X_E)$ has generators if $(\mathbf{C}/X)_E$ is small.

When the site is not small, [AGV72] gives a similar argument using Grothendieck universe. □

Proof of Theorem 2. It suffices to show that $\mathbf{S}(X_E)$ has a family of generators. The easiest way to go about this is just to sheafify the generators from the proof of Lemma 1. We now give a more explicit construction. Again, we run the argument for small site. For any object $f : U \to X$ in $\mathbf{C}/X$, define the sheaf $\mathbb{Z}_U = f_!\mathbb{Z}$ for the constant sheaf $\mathbb{Z}$ on U_E, then

$$\operatorname{Hom}_X(\mathbb{Z}_U, F) \cong \operatorname{Hom}_U(\mathbb{Z}, F|_U) \cong F(U).$$

To find a family of generators, we pick one sheaf $\mathbb{Z}_U$ for each of sufficiently many isomorphism classes of objects of $\mathbf{C}/X$.[3] To see why this suffices, if $i : \mathcal{G} \hookrightarrow \mathcal{F}$ is not an isomorphism, then there exists some U such that $\mathcal{G}(U) \subsetneq \mathcal{F}(U)$ and there exists some element $\sigma \in \mathcal{F}(U) \backslash \mathcal{G}(U)$. Now $\varphi : \mathbb{Z}_U \to \mathcal{F}$ corresponding to σ does not factor through i. □

With enough injectives, we may define right derived functors on left exact functors $\mathbf{S}(X_E) \to \mathscr{A}$ where $\mathscr{A}$ is abelian. We will use the following useful homological algebra fact multiple times. For a proof, see [015Z].

Lemma 6. A functor that admits an exact left adjoint preserves injectives.

Example 7.

1. The global section functor

$$\begin{aligned} \Gamma(X, -) : \mathbf{S}(X_E) &\to \mathbf{Ab} \\ \mathcal{F} &\mapsto \mathcal{F}(X) \end{aligned}$$

is left exact.

Definition 8. For any sheaf $\mathcal{F}$, we define the nth cohomology group $H^n(X_E, \mathcal{F})$ of X_E to be the nth right derived functor $H^n(X_E, -) := R^n\Gamma(X, -)$ of site X_E with values in $\mathcal{F}$. That is, for a sheaf $\mathcal{F}$, we pick an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

then applying $\Gamma(X, -)$ gives a complex

$$\Gamma(X, I^0) \longrightarrow \Gamma(X, I^1) \longrightarrow \Gamma(X, I^2) \longrightarrow \cdots$$

which is not necessarily exact, and $H^n(X_E, \mathcal{F})$ calculates its nth cohomology group.

[2] Therefore, it contains all colimits by existence of quotient in abelian category.
[3] One should also be mindful of cardinality: we need to show that this family is actually a set.

2. For any object $U \to X$ in $\mathbf{C}/X$, the functor

$$\begin{aligned} U : \mathbf{S}(X_E) &\to \mathbf{Ab} \\ \mathcal{F} &\mapsto \mathcal{F}(U) \end{aligned}$$

is left exact. Its nth right derived functor is written as $H^n(U, \mathcal{F})$.[4] In particular, given the object $\varphi : U \to X$, the inverse image functor $\varphi^* : \mathbf{S}(X_E) \to \mathbf{S}(U_E)$ is exact and preserves injectives, c.f., Lemma 6. Now φ^* acts as the restriction, therefore the composition $\Gamma(U, -) \circ \varphi^*$ is just $\Gamma(U, -)$.

3. The inclusion functor

$$i : \mathbf{S}(X_E) \hookrightarrow \mathbf{P}(X_E)$$

is left exact as the right adjoint of the sheafification functor a. The nth right derived functor is denoted $\underline{H}^n(X_E, \mathcal{F})$. As we will see by Theorem 17, $\underline{H}^n(X_E, \mathcal{F})$ is the presheaf defined by $U \mapsto H^n(U_E, \mathcal{F}|_U)$.

4. Fix a sheaf $\mathcal{F}_0$ on X_E, then the functor $\mathrm{Hom}_{\mathbf{S}}(\mathcal{F}_0, -)$ is left exact, and its nth right derived functor is $R^n \mathrm{Hom}_{\mathbf{S}}(\mathcal{F}_0, -) = \mathrm{Ext}^n_{\mathbf{S}}(\mathcal{F}_0, -)$.

5. Fix sheaves $\mathcal{F}_0, \mathcal{F}_1$ on X_E, and let $\underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{F}_1)$ to be the sheaf defined by $U \mapsto \mathrm{Hom}(\mathcal{F}_0|_U, \mathcal{F}_1|_U)$. The functor

$$\begin{aligned} \underline{\mathrm{Hom}}(\mathcal{F}_0, -) : \mathbf{S}(X_E) &\to \mathbf{S}(X_E) \\ \mathcal{F} &\mapsto \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{F}) \end{aligned}$$

is left exact with nth right derived functor $\underline{\mathrm{Ext}}^n(\mathcal{F}_0, \mathcal{F})$.

6. The direct image functor $\pi_* : \mathbf{S}(X'_{E'}) \to \mathbf{S}(X_E)$ of a (continuous)[5] map $\pi : X'_{E'} \to X_E$ is left exact, and its nth right derived functor $R^n\pi_*$ is called a (higher) direct image functor.

As an overview, we briefly look through connections étale cohomology has with other cohomology theories.

Fact 9. Let X be a coherent sheaf, then the étale cohomology of $\mathcal{F}$ is exactly the coherent sheaf cohomology of $\mathcal{F}$ (with respect to Zariski topology), c.f., [03DW].

Fact 10. If $X = \mathrm{Spec}(K)$ is the spectrum of a field, recall that we know the sheaf category $\mathbf{S}(X_{\text{ét}})$ on étale site is isomorphic to the (discrete) category of G-modules, where $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. More explicitly, for a G-module M, there is a sheaf $\mathcal{F}_M$ whose sections over a finite separable extension K'/K are given by $M^{G'}$ for $G' = \mathrm{Gal}(K^{\mathrm{sep}}/K')$. Moreover, the functor $\Gamma(X, -)$ is just the fixed point functor $(-)^G$ from G-modules to $\mathbf{Ab}$. Therefore, $H^*(X, \mathcal{F}_M) = H^*(K, M)$. This correspondence allows us to study étale cohomology using Galois cohomology.

1. Suppose $\mathcal{F}$ corresponds to a G-module M, then $\Gamma(X, \mathcal{F}) \cong M^G$ (as derived functors), therefore we have an isomorphism

$$H^n(X, \mathcal{F}) \cong H^n(G, M) := H^n(K, M)$$

that connects the two cohomology theories. (See [Fu11], Proposition 5.7.8.) In particular, this implies Hilbert's Theorem 90.

2. If $\mathcal{F}$ and $\mathcal{G}$ correspond to M and N, then $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_G(M, N)$ and $\mathrm{Ext}^n_{\mathbf{S}}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^n_G(M, N)$.

[4] Here $\mathcal{F}$ should be interpreted as $\mathcal{F}|_U$: see Theorem 17.

[5] A morphism of sites is usually considered as a continuous functor, i.e., preserving all existing small limits.

3. If $\mathcal{F}$ and $\mathcal{G}$ correspond to M and N, then $\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})$ corresponds to
$$\bigcup_H \mathrm{Hom}_{\mathbf{Ab}}(M,N)^H = \bigcup_H \mathrm{Hom}_H(M,N)$$
for open normal subgroups $H \lhd G$.[6]

Fact 11. Let X be a quasi-compact scheme such that any finite subset of X is contained in some affine open set, and let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$, then there is a natural isomorphism $\check{H}^n(X_{\text{ét}},\mathcal{F}) \cong H^n(X_{\text{ét}},\mathcal{F})$ for all n between Čech cohomology and étale cohomology.

There are also connections with cohomology relative to complex topology, flat cohomology, etc.

2 Flasque Sheaf on Sites

Recall that a sheaf $\mathcal{F}$ on a topological space X is flasque if the restriction $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for any U. Moreover, recall that when studying sheaf cohomology, the usual definition agrees with the one defined by the flasque resolution. This makes sense because flasque sheaves are acyclic, i.e., cohomology vanishes in positive degrees. The analogue holds in étale cohomology, that is, one can calculate étale cohomology using flasque sheaves.

Definition 12. A sheaf $\mathcal{F}$ on a site X_E is flasque if the étale cohomology $H^n(U,\mathcal{F}) = 0$ for all $U \in X_E$ and all $n > 0$.

Once we understand Čech cohomology, identification of flasque sheaves sheaves somewhat agree on both theories.

Theorem 13. Let $\mathcal{F}$ be a sheaf on the small étale site $X_{\text{ét}}$, the following are equivalent.

1. $\mathcal{F}$ is flasque.
2. The Čech cohomology $\check{H}^n(U,\mathcal{F}) = 0$ for all $U \in X_E$ and all $n > 0$.
3. Given any étale covering $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha\in I}$ in $X_{\text{ét}}$, we have $\check{H}^1(\mathfrak{U},\mathcal{F}) = 0$. (This in turn implies $H^1(U,\mathcal{F}) = 0$.)
4. Given any étale covering $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha\in I}$ in $X_{\text{ét}}$, we have $\check{H}^n(\mathfrak{U},\mathcal{F}) = 0$ for all $n > 0$.

Remark. Note that Theorem 13 does not hold on general sites! Given arbitrary sheaves on some site, *the* cohomology should be replaced by abelian sheaf cohomology, then the desired comparison statement is described in [07A1]. In general sites, the definition of a flasque sheaf is taken to be part 2. of Theorem 13.

The proof of Theorem 14 follows from Lemma III.1.8 in [Mil80].

Theorem 14. Let T be the collection of all flasque sheaves on X_E, and let $f \in \{\Gamma(X,-), H^0(U,-), \pi_*\}$. Then T contains all injective objects of $\mathbf{S}(X_E)$, and every element of T is f-acyclic, i.e., $R^n f(t) = 0$ for all $n > 0$ and $t \in T$.

Example 15. For an object given by $\pi : U \to X$, the inverse image functor $\pi^* : \mathbf{S}(X_E) \to \mathbf{S}(U_E)$ has a right adjoint, namely the direct image functor π_*. Therefore, π_* preserves injectives.

[6]Therefore H always has the conjugation action on the hom group: for $\sigma \in H$ and $f : M \to N$, we have $\sigma(f) = \sigma f \sigma^{-1}$.

Lemma 16. Let $\pi : U \to X$ be an object, then the inverse image functor $\pi^* : \mathbf{S}(X_E) \to \mathbf{S}(U_E)$, i.e., the restriction on U, preserves injectives.

Proof. We will deal with the case where the site is just the small étale site, that is, $U_E = U_{\text{ét}}$ and $X_E = X_{\text{ét}}$. (The general case requires a different construction of extension by zero, c.f., [Mil80], II.3.18. However, the same argument works out.)

We define a functor $\pi_! : \mathbf{S}(U_{\text{ét}}) \to \mathbf{S}(X_{\text{ét}})$ using extension by zero: for any sheaf $\mathcal{F}$ on $U_{\text{ét}}$ and any geometric point $\bar{x} \to X$, we have

$$(\pi_!\mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}}, & x \in U \\ 0, & x \notin U \end{cases}$$

In particular, $\pi_!$ is an exact functor, and also the left adjoint of π^*. By Lemma 6 we are done. □

Theorem 17. For any sheaf $\mathcal{F}$ over X_E and any object given by $\pi : U \to X$ in X_E, the groups $H^n(U, \mathcal{F})$ and $H^n(U_E, \mathcal{F}|_U)$ are canonically isomorphic.

Proof. Given $\pi : U \to X$, we know the inverse image functor π^* is exact and we are done by Lemma 16. □

Corollary 18. The restriction functor defined by $\mathcal{F} \mapsto \mathcal{F}|_U$ preserves flasque sheaves.

Lemma 19. For a morphism $\pi : X'_{E'} \to X_E$ of sites, the direct image functor π_* preserves flasque sheaves. Moreover, suppose π^* is exact (which is often times true), then by Lemma 6 we know π_* preserves injectives.

Proof. We have a factorization

$$\begin{array}{ccc} \mathbf{S}(X_E) & \xrightarrow{\quad\Gamma(X,-)\quad} & \mathbf{Ab} \\ & {\scriptstyle -|_U} \searrow \qquad \nearrow {\scriptstyle \Gamma(U,-)} & \\ & \mathbf{S}(U_E) & \end{array}$$

Note that $-|_U$ is exact, then the statement follows from the Grothendieck spectral sequence. □

For the remaining of the section, we elaborate on geometric points a bit, and give an alternative proof of Theorem 2. As we will see later, the geometric points on the small étale site are good analogues of points on a topological space.

As discussed last time, the notion of a geometric point can be generalized in topos theory, in which it is just a geometric morphism $\mathbf{Set} \to T$ to some topos T: a geometric point of topos of a sober topological space (for instance, a scheme, or a locally Hausdorff space) really is just a stalk functor at points of the topological space.

Fact 20. Let $i : \bar{x} \to X$ be a geometric point of X. That is, $\bar{x}$ is the spectrum of a separably closed field $k(\bar{x})$ containing $k(x)$, and i is induced by the inclusion $k(x) \hookrightarrow k(\bar{x})$. Recall from Fact 10 that we have an equivalence of categories between $\mathbf{S}(\bar{x}_{\text{ét}})$ and $\mathbf{Ab}$. Suppose $\mathcal{F}$ is a sheaf on X_E, then $(i^*\mathcal{F})(\bar{x}) = \mathcal{F}_{\bar{x}}$ by definition. Therefore, given any morphism $\pi : Y \to X$ of schemes and a geometric point $i : \bar{y} \to Y$ of Y, we have

$$(\pi^*\mathcal{F})_{\bar{y}} = i^*(\pi^*\mathcal{F})(\bar{y}) = \mathcal{F}_{\bar{x}}$$

where we identify $\bar{x}$ to be the geometric point of X via $\bar{y} \xrightarrow{i} Y \xrightarrow{\pi} X$. In particular, this shows that π^* is exact and by Lemma 6 we know π_* preserves injectives.

Let us apply Fact 20 and give an alternative proof of Theorem 2.

Proof of Theorem 2. For any $x \in X$, we pick a geometric point $i_x : \bar{x} \to X$ with image x. Since $\mathbf{S}(\bar{x}_{\text{ét}})$ is isomorphic to $\mathbf{Ab}$, then it has enough injectives. For $\mathcal{F} \in \mathbf{S}(X_{\text{ét}})$ and $x \in X$, we have an embedding $i_x^*\mathcal{F} \cong \mathcal{F}_{\bar{x}} \to I(x)$ into some injective sheaf. By Fact 20, $i_{x*}(I(x))$ is injective. Since a product of injective objects is also injective, therefore $\prod_{x \in X} i_{x*}(I(x))$ is also an injective sheaf. Moreover, the canonical morphisms

$$\mathcal{F} \to \prod_{x \in X} i_{x*} i_x^* \mathcal{F}$$

and

$$\prod_{x \in X} i_{x*} i_x^* \mathcal{F} \to \prod_{x \in X} i_{x*}(I(x))$$

are monomorphisms. Their composition gives a monomorphism into an injective object, as desired. □

3 Computing Higher Direct Images

Recall that given a continuous map of sites $\pi : X'_{E'} \to X_E$, the nth higher direct image is $R^n\pi_*(-)$, the nth right derived functor of the direct image functor $\pi_* : \mathbf{S}(X'_{E'}) \to \mathbf{S}(X_E)$. We should think of $R^n\pi_*$ as describing the fibers of X' using the cohomology over X. For instance, when $X = \operatorname{Spec}(k)$ is a point, then $R^n\pi_*(-)$ gives the cohomology of the global section functor. In general, we patch together the cohomology of the fibers X'_x for all $x \in X$.

Proposition 21 ([Liu02], Proposition 5.2.34)**.** Let $f : X' \to X$ be a projective[7] morphism of schemes where X is locally Noetherian. Suppose $r = \sup_{x \in X} \dim(X'_x)$, then $R^n f_* \mathcal{F} = 0$ for all $n > r$ and every quasi-coherent sheaf $\mathcal{F}$ on X'.

The point being, we can describe the higher direct image functor of the direct image functor as sheafifications of base-changes.

Lemma 22. Let $\pi : X'_{E'} \to X_E$ be continuous morphism of sites and fix $\mathcal{F} \in \mathbf{S}(X'_E)$, then $R^n\pi_*\mathcal{F}$ is the sheafification of the presheaf defined by $U \mapsto H^n(U \times_X X', \mathcal{F}|_{U \times_X X'})$.

Proof. Let a be the sheafification functor, let $\pi_p : \mathbf{P}(X'_{E'}) \to \mathbf{P}(X_E)$ be the direct image presheaf, and let $i : \mathbf{S}(X'_{E'}) \hookrightarrow \mathbf{P}(X'_{E'})$ be the inclusion functor. One can check that both π_p and a are exact. However, this is not the case with i! As a remedy, fix an injective resolution I^* of $\mathcal{F}$, then by exactness we have $R^n\pi_*\mathcal{F} = H^n(a\pi i I^*) = a\pi_p H^n(iI^*) = a\pi_p(\underline{H}^n(\mathcal{F}))$. Now the presheaf we want is exactly $\pi_p(\underline{H}^n(\mathcal{F}))$. □

Theorem 23. Suppose $\mathcal{F}$ is a flasque sheaf, then $R^n\pi_*\mathcal{F} = 0$ for $n > 0$.

Proof. By definition, $H^n(U \times_X X', \mathcal{F}|_{U \times_X X'}) = 0$ for any flasque sheaves $\mathcal{F}$ and any base-change $U \times_X X'$. □

Moreover, this tells us that we can use flasque resolutions to compute higher direct images $R^n\pi_*$. Indeed, this follows from Theorem 14 since we know $R^1\pi_*\mathcal{F} = 0$ whenever $\mathcal{F}$ is flasque.

We also get passage to limits, i.e., étale cohomology commutes with inverse limits of schemes.

Lemma 24. Let $\mathbf{I}$ be a filtered category, and consider a contravariant functor

$$\begin{aligned} \mathbf{I} &\to \mathbf{Sch}/X \\ i &\mapsto X_i \end{aligned}$$

[7] More generally, this can be extended to proper morphisms, c.f., this post.

Assume that every X_i is quasi-compact, and that the maps $X_j \to X_i$ are affine. Now let $X_\infty = \varprojlim X_i$, and let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$, then denote $\mathcal{F}_n$ and $\mathcal{F}_\infty$ to be $\mathcal{F}$'s inverse images on X_n and X_∞, respectively. Then

$$\varinjlim H^m((X_n)_{\text{ét}}, \mathcal{F}_n) \cong H^m((X_\infty)_{\text{ét}}, \mathcal{F}_\infty).$$

This result allows us to compute higher direct images using étale cohomology.

Theorem 25. Let $\pi : Y \to X$ be quasi-compact, and let $\mathcal{F}$ be a sheaf on $Y_{\text{ét}}$. Suppose $\bar{x}$ is a geometric point of X such that $k(\bar{x})$ is the separable closure of $k(x)$. Denote $\tilde{X} = \operatorname{Spec}(\mathcal{O}_{X,\bar{x}})$ and $\tilde{Y} = Y \times_X \tilde{X}$, then we have a pullback diagram

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & X \end{array}$$

and let $\tilde{\mathcal{F}}$ be the inverse image of $\mathcal{F}$ on $\tilde{Y}$, then $R^n\pi_*(\mathcal{F})_{\bar{x}} \cong H^n(\tilde{Y}, \tilde{\mathcal{F}})$.

Proof. As always, reduce the proof to the case of affine subset U, now

$$\begin{aligned}(R^n\pi_*\mathcal{F})_{\bar{x}} &\cong \varinjlim_U H^n(U \times_X Y, \mathcal{F}|_{U\times_X Y}) \text{ by Lemma 22} \\ &\cong H^n(\varinjlim_U U \times_X Y, \mathcal{F}|_{U\times_X Y}) \text{ by Lemma 24} \\ &= H^n(Y \times_X \tilde{X}, \tilde{\mathcal{F}}).\end{aligned}$$

□

The Leray spectral sequence can be constructed from Theorem 14 and Lemma 19. The proof follows from the conditions for the Grothendieck spectral sequence.

Theorem 26.

1. Suppose $\pi : X'_{E'} \to X_E$ is a continuous morphism of sites, then there is a spectral sequence

$$E_2^{p,q} = H^p(X_E, R^q\pi_*\mathcal{F}) \Rightarrow H^{p+q}(X'_{E'}, \mathcal{F})$$

for any sheaf $\mathcal{F}$ on $X'_{E'}$.

2. For continuous morphisms $X''_{E''} \xrightarrow{\pi'} X'_{E'} \xrightarrow{\pi} X_E$, there is a spectral sequence

$$(R^p\pi_*)(R^q\pi'_*)\mathcal{F} \Rightarrow R^{p+q}(\pi\pi')_*\mathcal{F}$$

for any sheaf $\mathcal{F}$ on $X''_{E''}$.

In particular, if $\pi : X_{\text{Zar}} \to X_{\text{ét}}$ is the inclusion from small Zariski site to small étale site, the spectral sequence above allows us to compute étale cohomology in terms of Zariski cohomology.

We also have a local-global spectral sequence of Ext functors, which relies on the following lemma.

Lemma 27. Suppose $\mathcal{F}_1, \mathcal{F}_2$ are sheaves on X_E, such that $\mathcal{F}_2$ is injective, then $\underline{\operatorname{Hom}}(\mathcal{F}_1, \mathcal{F}_2)$ is flasque.

Theorem 28. Suppose $\mathcal{F}_1, \mathcal{F}_2$ are sheaves on X_E, then there is a spectral sequence

$$H^p(X_E, \underline{\operatorname{Ext}}^q(\mathcal{F}_1, \mathcal{F}_2)) \Rightarrow \operatorname{Ext}^{p+q}(\mathcal{F}_1, \mathcal{F}_2).$$

Remark. One can identify $\underline{\operatorname{Ext}}^n(\mathcal{F}_1, \mathcal{F}_2)$ to be the sheafification of the presheaf defined by

$$U \mapsto \operatorname{Ext}^n_{\mathbf{S}(U_E)}(\mathcal{F}_1|_U, \mathcal{F}_2|_U).$$

4 Axiomatization

Recall that we have given the definition of étale cohomology in Definition 8. As a cohomology in algebraic geometry, there is no way we can run an axiomatized approach to formalize cohomology theories like Eilenberg-Steenrod axioms do. (For example, we can not implement additivity axiom in a nice way.) However, one can actually show that étale cohomology satisfies properties that are analogues of those axioms. For this section, we are mostly interested in the small étale site $X_E = X_{\text{ét}}$. Most statements made can be found in [Mil12], and therefore their proofs are omitted.

Since the étale cohomology is defined via right derived functors of the left exact functor $\Gamma(X, -)$, then one can run the usual theory of derived functors to show that

Lemma 29.

1. Étale cohomology does not depend on the choice of injective resolutions.
2. For any sheaf $\mathcal{F}$, we have $H^0(X_E, \mathcal{F}) = \Gamma(X, \mathcal{F})$.
3. Given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

there is a long exact sequence of cohomology

$$0 \longrightarrow H^0(X_E, \mathcal{F}') \longrightarrow H^0(X_E, \mathcal{F}) \longrightarrow H^0(X_E, \mathcal{F}'') \longrightarrow H^1(X_E, \mathcal{F}') \longrightarrow \cdots$$

In particular, $H^n(X_E, -)$ is a functor.

These results together show that $H^n(X_E, -)$ is uniquely determined (up to a unique isomorphism).

Lemma 30. Let $L = L_2 \circ L_1$ be a composition of left exact functors from abelian categories with enough injectives. If L_1 preserves injectives and $(R^n L_1)(X) = 0$ for some X, then $(R^n L)(X) = (R^n L_2)(L_1 X)$.

Proof. Choose an injective resolution $X \to I$ of X, and note that $L_1 X \to L_1 I$ is now an injective resolution of $L_1 X$. Therefore, both $(R^n L)(X)$ and $(R^n L_2)(L_1 X)$ give the nth cohomology of LI. □

Other than the functoriality shown in Lemma 29, one can show that the cohomology functor is functorial in the other variable as well.

Lemma 31. Consider a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

of sheaves on X and a morphism $\varphi : Y \to X$. We know $\varphi^* : \mathbf{S}(X_E) \to \mathbf{S}(Y_{E'})$ is exact by Fact 20, then there is also a long exact sequence

$$\cdots \longrightarrow H^n(Y_E, \varphi^*\mathcal{F}') \longrightarrow H^n(Y_E, \varphi^*\mathcal{F}) \longrightarrow H^n(Y_E, \varphi^*\mathcal{F}'') \longrightarrow \cdots$$

Moreover, φ can be extended to a morphism between the two exact sequences, which is uniquely determined by $H^0(X_E, \mathcal{F}) \to H^0(Y_E, \varphi^*\mathcal{F})$.

Lemma 32. Let $\varphi : Y \to X$ be a finite surjective radiciel morphism, and let $\mathcal{F}$ be a sheaf on X, then

$$H^n(Y, \mathcal{F}) \cong H^n(X, \varphi^*\mathcal{F}).$$

Let us start with the analogue of homotopy axiom. In this case, the idea of homotopy equivalence is realized as rational equivalence in our case, c.f., [02RV].

We then consider the dimension axiom. Recall from Fact 10 that we have an isomorphism between $\mathbf{S}(X_{\text{ét}})$ and the category of G-modules for $G = \text{Gal}(K^{\text{sep}}/K)$. Therefore, since $(\mathcal{F}_{\bar{x}})^G = \Gamma(x, \mathcal{F})$, then the derived functors of $(-)^G$ and $\Gamma(x, -)$, therefore we have $H^n(x, \mathcal{F}) \cong H^n(G, \mathcal{F}_{\bar{x}})$. The dimension axiom now asks for

$$H^n(x, \mathcal{F}) = 0$$

for any sheaf $\mathcal{F}$ and $n > 0$. This is true if we take x to be a geometric point, i.e., the spectrum of a separably closed field! Thus, if we think of small étale sites as an analogy of topological spaces, then the notion of geometric points on étale sites mimics the function of a point on a topological space.

We now move on to the exactness axiom. This requires first studying the Ext groups.

Fact 33. Given a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

there exists long exact sequences of Ext functors on X

$$\cdots \longrightarrow \text{Ext}(\mathcal{F}_0, \mathcal{F}') \longrightarrow \text{Ext}(\mathcal{F}_0, \mathcal{F}) \longrightarrow \text{Ext}(\mathcal{F}_0, \mathcal{F}'') \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \text{Ext}(\mathcal{F}'', \mathcal{F}_0) \longrightarrow \text{Ext}(\mathcal{F}, \mathcal{F}_0) \longrightarrow \text{Ext}(\mathcal{F}', \mathcal{F}_0) \longrightarrow \cdots$$

Example 34. For constant sheaf $\mathbb{Z}$ and any other sheaf $\mathcal{F}$ on X, we have $\text{Hom}_X(\mathbb{Z}, \mathcal{F}) \cong \mathcal{F}(X)$ and therefore $\text{Hom}_X(\mathbb{Z}, -) \cong \Gamma(X, -)$. This gives

$$\text{Ext}^n(\mathbb{Z}, -) \cong H^n(X_{\text{ét}}, -).$$

We require a notion of cohomology with compact support as a counterpart. For any closed subscheme Z of X, take $U = X \backslash Z$, then for any sheaf $\mathcal{F}$ on $X_{\text{ét}}$, we define $\Gamma_Z(X, \mathcal{F})$ to be the kernel of $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$. The functor $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F})$ is left exact, and we define the nth cohomology with compact support in Z to be the nth right derived functor $H^n_Z(X, -)$. The exactness axiom now shows that we have a long exact sequence

$$\cdots \longrightarrow H^n_Z(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}) \longrightarrow \cdots$$

which is functorial in the pair $(X, X \backslash Z)$ and $\mathcal{F}$.

Finally, we can summarize the excision axiom as

Theorem 35. Let $\pi : X' \to X$ be an étale morphism and let $Z' \subseteq X'$ be a closed subscheme such that

1. $Z := \pi(Z')$ is closed in X, and the restriction $\pi|_{Z'}$ is an isomorphism of Z' onto Z, and
2. $\pi(X' \backslash Z') \subseteq X \backslash Z$,

then for any sheaf $\mathcal{F}$ on $X_{\text{ét}}$, the canonical map $H^n_Z(X_{\text{ét}}, \mathcal{F}) \to H^n_{Z'}(X'_{\text{ét}}, \mathcal{F}|_{X'})$ is an isomorphism for all n.

Example 36. Consider an étale morphism $\pi : X' \to X$ where $Z \subseteq X$ is a closed subscheme, along with a morphism $s : Z \to X'$ such that $\pi \circ s = \text{id}_Z$ and $\pi^{-1}(Z) = s(Z)$.

Corollary 37. Let $x \in X$ be a closed point and $\mathcal{F}$ be a sheaf on X, then there is an isomorphism

$$H^n_x(X, \mathcal{F}) \cong H^n_x(\text{Spec}(\mathcal{O}^h_{X,x}), \mathcal{F}).$$

Proof. By Theorem 35, for any étale neighborhood (U, u) of x such that u is the unique preimage of x in U, we have $H^n_x(X, \mathcal{F}) \cong H^n_u(U, \mathcal{F})$. Such étale neighborhoods are cofinal, so we can apply passage to limit. ☐

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