02/06

Let G be either a (discrete) finite group (Most illustrations) or compact Lie group.
H is always a (closed) subgroup of G.

① Bredon Cohomology

Definition:

A coefficient system is a presheaf (i.e., contravariant functor) $M \in \mathrm{Fun}(\mathcal{O}_G^{op}, Ab)$ on the orbit category.

Remark:

$$\text{Coefficient System } M \xrightarrow{\text{via Elmendorf}} \mathrm{Fun}(\mathcal{O}_G^{op}, Top) \underset{\Psi}{\overset{\Theta}{\rightleftarrows}} GTop$$

Eilenberg-Maclane G-space $K(G,n)$ $\xrightarrow{\text{represents}}$ $\boxed{\text{Bredon Cohomology}}$

$\boxed{\text{characterization}}$ $\tilde{H}^n_G(X;M) \cong [X, K(M,n)]_G$

Definition: Eilenberg-Maclane G-space $K(G,n)$ of type (M,n):

G-space X of the G-homotopy type of G-CW complex such that X^H is $K(M(G/H), n)$ $\forall H$, and $M = \pi_n \circ X^{(-)} : \mathcal{O}_G^{op} \to Grp$.

$\Rightarrow$ Let B be the bar construction, then $B^n \circ M : \mathcal{O}_G^{op} \to Top$, and $\Theta(B^n \circ M)$ is $K(M,n)$.

Remark:

For compact Lie group G, we adjust the definition to $M \in \mathrm{Fun}(h\mathcal{O}_G^{op}, Ab)$.

Example:

1. $G \in Ab$, the constant presheaf $\underline{G}$ evaluated as G is the constant coefficient system with coefficients in G.

2. Fix G-space X.

$$\begin{array}{ccc} \mathrm{Top} & & \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top}) \\ \downarrow & \Longrightarrow & \downarrow \\ \mathrm{Ab} & & \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Ab}) \end{array}$$

For $n \geq 2$:

$$\underline{\Pi_n}(X) : (G/H \to X^H) \longmapsto (G/H \longrightarrow \pi_n(X^H))$$

$$\underline{H_n}(X) : \text{-----------} \longmapsto (G/H \longrightarrow H_n(X^H))$$

Goal: define this cohomology explicitly.

<u>Definition:</u>

Chain Complex $\underline{C_*}(X) = \underline{H_n}(X_n, X_{n-1}; \mathbb{Z})$

as a choice of CW chain complexes of X^H $\forall G/H$.

$$G/H \longmapsto H_n((X^H)_n, (X^H)_{n-1}; \mathbb{Z}) = C_n^{CW}(X^H).$$

with differentials at G/H being CW chain complex differential for X^H, i.e., connecting morphism for tuple $((X^H)_n, (X^H)_{n-1}, (X^H)_{n-2})$.

$\Rightarrow$ nth Bredon cohomology

enriched as abelian groups

$$H_G^n(X; M) := H^n(\mathrm{Hom}_{\mathrm{Fun}(\mathcal{O}_G^{op}, \underline{\mathrm{Ab}})}(\underline{C_*}(X), M)).$$

abelian

simplified from ends.

abelian

$:= H^n(C_G^*(X))$.

$\Rightarrow$ chain complex of abelian groups

$\Rightarrow$ cohomology as abelian groups.

Slogan: Understand cohomology via fixed points and subgroup lattice.
Hard to Calculate.

e.g.,

$$\begin{array}{ccc} G & \overset{\text{antipodal}}{\curvearrowright} & X \\ \| & & \| \\ C_2 & & S^n \end{array}$$

G-CW complex structure: 2 k-cells for all $0 \le k \le n$. "switching cells".

$$\Rightarrow \underset{\underset{H_n((X_n)^H,(X_{n-1})^H;\mathbb{Z})}{\|}}{\underline{C}_k(S^n)(G/H)} = \begin{cases} \mathbb{Z}^2, & k \le n,\ G/H = G. \\ 0, & \text{otherwise.} \end{cases}$$

G acts by permuting coordinates on $\mathbb{Z}^2$, $(a,b) \to (b,a)$.

As $\underline{C}_k(S^n)(*) = 0$, with $k \le n$ we have

$$\underline{C}^k_G(S^n) = \mathrm{Hom}_G(\mathbb{Z}^2, M(C_2)) \cong M(C_2)$$

via C_2-action, $\binom{1}{0} \sim \binom{0}{1}$ in O_G^{op}.

- If $M(C_2) = \mathbb{Z}$, e.g., $M = \underline{\mathbb{Z}}$, by trivial G-module structure of $\mathbb{Z}$, generators of $\mathbb{Z}^2$ are fixed in $M(C_2)$.

$$\Rightarrow \underline{C}^k_G(S^n) = \begin{cases} \mathbb{Z}, & k \le n \\ 0, & k > n. \end{cases}$$

Study the local degree.

Say attaching map φ_1 of a k-cell has $\deg(\varphi_1) = 1$, then $\varphi_2 = \overset{\text{antipodal generator}}{g} \cdot \varphi_1$ has degree $(-1)^{k+1}$. antipodal degree

$$\Rightarrow \text{Local degree } \delta_k : \underline{C}^k_G \longrightarrow \underline{C}^{k+1}_G$$
$$x \longmapsto (1 + (-1)^{k+1})x.$$

Should have the same cohomology as $\mathbb{R}P^n = S^n/C_2$!

$$H^k_G(S^n; M) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \text{ odd}. \\ C_2 & k \text{ even}, k \le n \\ 0 & \text{otherwise} \end{cases}$$

("Proof by example").

Lemma:

If G acts freely on CW complex X, then

$$H^*_G(X;M) \cong H^*(X/G; M(G/e))$$

for any coefficient system M.

Axiomatic Characterization

A general G-equivariant cohomology theory of pairs $H^*_G(X,A;M)$ satisfies

- invariance under weak equivalences.
- long exact sequence of $(X,A;M)$
- excision, i.e., $X = A \cup B \Rightarrow H^*_G(X/A;M) \cong H^*_G(B/(A\cap B);M)$
- additivity, i.e., $X = \bigvee_i X_i \Rightarrow H^*_G(X;M) \cong \prod_i H^*_G(X_i;M)$.
- dimension, i.e., let $H \subseteq G$ be a (closed) subgroup, then

$$H^*(G/H;M) = \begin{cases} M(G/H), & * = 0 \\ 0, & * \neq 0 \end{cases}$$

"orbits as points"

e.g. Bredon; Borel satisfies everything except dimension axiom!

② Borel Cohomology

Definition:

Given a G-space X, the Borel construction is $EG \times_G X$. "homotopy orbit space"
as balanced product, i.e., quotient by diagonal G-action.

$\Rightarrow EG \times_G X \cong (EG \times X)/\sim_{\text{diagonal action}}$ $(y, gx) \sim (yg, x)$

The Borel Cohomology of X is $H^*_G(X) := H^*(EG \times_G X)$.

Viewing this as a homotopy orbit space, we need to define homotopy fixed points $X^{hG} := \mathrm{Map}(EG, X)^G$.

Remark:

1. For abelian group G, $H^*_G(X; \underline{A}) \cong H^*(X/G; A)$.

   $\Rightarrow H^*_G(EG \times X; \underline{A}) \cong H^*(EG \times_G X; A)$.

2. $H^*(EG \times_G *) \cong H^*(BG)$.

   $\Rightarrow H^*_G(X)$ is an $H^*(BG)$-module.

3. $\begin{matrix} EG \\ \downarrow \\ * \end{matrix} \Longrightarrow X^G \longrightarrow X^{hG}$.

③ Smith Theory.

Theorem:

Let G be a finite p-group, X be a finite CW complex, where X is a $\mathbb{F}_p$-cohomology sphere of dimension n, then either $X^G = \emptyset$ or X^G is a $\mathbb{F}_p$-cohomology sphere of dimension $m \leq n$.

Key Reduction: let $G = C_p$.

($\exists\ H \triangleleft G \Rightarrow G/H \cong C_p \Rightarrow X^H$ is a C_p-space.)

Proof Using Bredon:

Find coefficient systems to recover cohomologies $H^n(X)$, $H^n(X^G)$, $H^n((X/G)/G)$. Take SES on coefficient systems and get LES on $H^*_G(X)$. Use a rank argument.

Proof Using Borel:

Look at fibration $X \longrightarrow EG \times_G X \longrightarrow BG$ and take SS

$$H^*(BG, H^*(X)) = H^*(BG) \otimes H^*(X) \Longrightarrow H^*(EG \times_G X)$$

Collapses due to dimension /or section of fixed point.

Apply Localization Theorem. $H^*_G(X)$ has a free $H^*(BG)$-module structure, with generator at degree n. Check the dimensions.

Remark:

We know Borel Cohomology has a natural $H^*(BG)$-module structure. In particular,

$$H^*(B(\mathbb{Z}/p\mathbb{Z})^n; \mathbb{Z}/p\mathbb{Z}) \cong \Lambda(x_1, \dots, x_n) \otimes \mathbb{Z}/p\mathbb{Z}[y_1, \dots, y_n]$$

for $p \neq 2$, with $\beta(x_i) = y_i$.

④ $RO(G)$-gradings and Brown Representability

Recall:

$$\begin{tikzcd}
\text{Cohomology Theories (abelian groups)} \arrow[r, leftrightarrow, "\text{Brown}"] \arrow[d, Rightarrow, "\text{with } G\text{-action}"] & \Omega\text{-spectra} \\
\text{Equivariant Cohomology Theories (Mackey functors)} \arrow[r, leftrightarrow, "?"'] & G\text{-spectra}
\end{tikzcd}$$

$\Rightarrow$ RO(G)-graded Cohomology Theories $\xleftrightarrow{\text{Equivalence}}$ Category Sp^G of spectra.

<u>Definition</u>:

Given a group G and a ring R, the representation ring of G over R is the ring generated by isomorphism classes of finite-rank G-representations over R with

$$\begin{cases} [V \oplus W] = [V] \oplus [W] \\ [V \otimes W] = [V] \cdot [W] \end{cases}$$

"The" representation ring is $RO(G) := \mathbb{R}(G)$ over $\mathbb{R}$.

As a ring of representations $G \longrightarrow \mathbb{R}$.

Remark:

A RO(G)-graded cohomology is a collection of functors $\{E^{\alpha}\}_{\alpha \in RO(G)}$ with suspension isomorphism

$$E^{\alpha}(X) \cong E^{\alpha+V}(S^V \wedge X)$$

satisfying some axioms
wedge and cofiber

Hard to study! (Need it to be complete, i.e., with respect to a G-universe.)

Try to study $Ho(RO(G;\mathcal{U}))$ instead.

With suspension

$$\Sigma^W : Ho(RO(G;\mathcal{U})) \times Ho(GTop) \longrightarrow Ho(RO(G;\mathcal{U})) \times Ho(GTop)$$
$$(V, X) \longmapsto (V \oplus W, S^W \wedge X).$$

Definition:

An RO(G)-graded cohomology theory is a functor

$$E : Ho(RO(G;\mathcal{U})) \times Ho(GTop)^{op} \longrightarrow Ab$$
$$(V, X) \longmapsto E^V(X)$$

with isomorphisms $\sigma_W : E^V(X) \longrightarrow E^{V\oplus W}(S^W \wedge X)$
such that for each V, $E^V(-)$ satisfies wedge and cofiber axioms, and for each isometric isomorphism $\alpha : W \longrightarrow W'$, the diagram

$$\begin{tikzcd}
E^V(X) \arrow[r, "\sigma_W"] \arrow[d, "\sigma_{W'}"'] & E^{V\oplus W}(S^W \wedge X) \arrow[d, "(1\oplus\alpha, 1)"] \\
E^{V\oplus W'}(S^{W'} \wedge X) \arrow[r, "(1\oplus 1, \alpha)"'] & E^{V\oplus W'}(S^W \wedge X)
\end{tikzcd}$$

commutes.

Definition:

Let $E \in Sp^G$, then the E-cohomology is
$$E_G^V(X) := [S^V \wedge X, E]_G$$

This is $RO(G)$-graded!

To define a G-spectrum based on a $RO(G)$-graded cohomology theory, we need Neeman's version of Brown Representability.

Theorem: (Neeman)

Let $\mathcal{T}$ be a compactly-generated triangulated category, and $H : \mathcal{T}^{op} \longrightarrow Ab$ be homological. If $H(\coprod_{\lambda\in\Lambda} T_\lambda) \cong \prod_{\lambda\in\Lambda} H(T_\lambda)$, then H is representable.

Using Brown Representability to define Eilenberg-Maclane spectra, which gives the equivalence.