Notes on Differential Equations

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Preliminaries

This document is the notes based on the following UCLA courses:

- MATH 134 (Section 1): Dynamical systems and chaos, taught by Dr. Jiajun Tong in Winter 2021, with textbook *Nonlinear dynamics and Chaos* by Strogatz,
- MATH 135 (Section 2): Ordinary differential equations, taught by Dr. Jiajun Tong in Fall 2020, with textbook *Differential Equations with Applications and Historical Notes* by Simmons,
- MATH 136 (Section 3): Partial differential equations, taught by Professor Marcus Roper in Spring 2022, with textbook *Partial Differential Equations: An Introduction* by Strauss and *Partial Differential Equations: An Introduction to Theory and Applications* by Shearer and Levy.

It is recommended that one should take first course in differential equations (c.f. MATH 33B at UCLA), which explains basic concepts as well as techniques in solving simple differential equations.

1 Dynamical Systems

1.1 Introduction to Dynamical Systems

As an introduction to dynamical systems, we first look through a few examples.

Example 1.1.1 (Population of rabbits)**.** Suppose there is a group of rabbits living in a certain habitat, then we can try to model the population of rabbits using certain equations. Let x_n be the number of rabbits in year n.

- In a naive way, we can simply consider $x_{n+1} = ax_n$ for $a > 0$.
- To make the model more precise, we may consider the capacity of the environment:

$$x_{n+1} = x_n(a - bx_n)$$

  for $a, b > 0$. This takes into account that the population growth slows down because of maximum capacity of the environment.

One should notice that the model is discrete in time (meaning the values of n we are taking can only be integers, as opposed to a certain moment in time), and is described by iterated maps $x_{n+1} = f(x_n)$ for some function f.

Example 1.1.2 (Harmonic oscillator)**.** Consider a mass with weight m tied down from the ceiling by a spring, which causes vertical oscillations. Let us denote $x = x(t)$ to be the vertical displacement of the mass at time t. By Newton's second law of motion, we know $F = ma$, where a is the acceleration with the same direction as the force F. Note that $\frac{dx}{dt}$ now represents the vertical velocity, and so $\frac{d^2x}{dt^2}$ represents the vertical acceleration, which is just a in our case. Because of energy conservation, we know the oscillation represents a transformation between gravitational potential energy (of the mass) and the elastic potential energy (of the spring). We can now consider a few models from here:

- Suppose we neglect friction, then by Hooke's law, we know the restoring force is directly proportional to the displacement, so if we denote $k > 0$ as the spring constant, then we can write

$$m\frac{d^2x}{dt^2} = -kx.$$

- Suppose we do not neglect friction, then we note that the friction is directly proportional to the velocity, so for some constant $b > 0$, we can improve our previous model as

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}.$$

One should note that this is a system of ordinary differential equations (ODEs), and is continuous in time, that is, we can consider the displacement at any point in time.

Example 1.1.3 (Heat conduction along a rod)**.** If we think of a rod as one-dimensional, then we can model $u(x,t)$, the temperature at position x and time t, using the heat equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t).$$

One should note that this is a system of partial differential equations (PDEs), and is continuous in time.

Remark 1.1.4. All these examples can be viewed as dynamical systems in the broadest sense. But in this chapter, we are only going to focus on those described by ODE systems.

We now try and formulate a dynamical system. Consider the systems described by a system of ODEs:

$$\begin{cases} \dot{x}_1(t) &= f_1(t, x_1(t), \dots, x_n(t)) \\ \dot{x}_2(t) &= f_2(t, x_1(t), \dots, x_n(t)) \\ &\vdots \\ \dot{x}_n(t) &= f_n(t, x_1(t), \dots, x_n(t)) \end{cases} \tag{1.1.1}$$

Here

- n is a finite integer,
- t is a time variable,
- $x_i = x_i(t)$ for $t = 1, \dots, n$ are scalar functions, which altogether keep a record of the state of the system,
- $\dot{x}_i := \frac{dx_i}{dt}$ is defined to be the change of rate variable of x_i with respect to time t,
- and f_i's are given functions described by $f_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

Moreover, one can abbreviate Equation (1.1.1) with vector-valued functions

$$\vec{x}(t) := \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

and

$$\dot{\vec{x}}(t) := \vec{f}(t, \vec{x}(t))$$

where $\vec{f}: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a multivariable function, or simply

$$\dot{x}(t) = f(t, x(t)).$$

With this new language, we can rewrite a higher-order ODE into a first-order ODE system.

Example 1.1.5. For instance, suppose we are given the second-order ODE

$$m\ddot{x} = -kx - b\dot{x}$$

from Example 1.1.2. Define $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, then we have $\dot{x}_1 = x_2$ and $\dot{x}_2 = \ddot{x}(t) = -\frac{k}{m}x - \frac{b}{m}\dot{x} = -\frac{k}{m}x_1 - \frac{b}{m}x_2$. Therefore, we just turn our second-order ODE into a system of first-order ODEs

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 \end{cases}.$$

Definition 1.1.6. Consider a dynamical system of the form

$$\dot{x}(t) = f(t, x(t)).$$

Suppose f is independent of t, i.e., $\dot{x}(t) = f(x(t))$, then we say the dynamical system is *autonomous*. Otherwise, we say the system is non-autonomous.

Actually, we can also rewrite non-autonomous systems into autonomous ones.

Example 1.1.7. Consider the system

$$\begin{cases} \dot{x}_1(t) = f_1(t, x_1(t), x_2(t)) \\ \dot{x}_2(t) = f_2(t, x_1(t), x_2(t)) \end{cases},$$

then let $x_3(t) = t$, so $\dot{x}_3(t) = 1$, then we can write

$$\begin{cases} \dot{x}_1(t) = f_1(x_3(t), x_1(t), x_2(t)) \\ \dot{x}_2(t) = f_2(x_3(t), x_1(t), x_2(t)) \\ \dot{x}_3(t) = 1 \end{cases},$$

which is an autonomous system.

Definition 1.1.8. Consider a dynamical system

$$\dot{x} = f(t, x).$$

We say the system is *linear* if $f(t, \cdot)$ is a linear function with respect to x, i.e.,

1. for all $y_1, y_2 \in \mathbb{R}^n$, then

$$f(t, y_1 + y_2) = f(t, y_1) + f(t, y_2),$$

2. for all $\lambda \in \mathbb{R}$ and all $y \in \mathbb{R}^n$, then

$$f(t, \lambda y) = \lambda f(t, y).$$

Otherwise, the dynamical system is not a linear system, and we say it is non-linear.

Remark 1.1.9. Linear systems admit the form

$$\dot{x}(t) = A(t)x(t)$$

where $A(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a matrix-valued function.

Example 1.1.10. We reconsider the system

$$m\ddot{x} = -kx - b\dot{x}$$

from Example 1.1.2. If we denote $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, then we have

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 \end{cases},$$

which is linear and autonomous. Moreover, we can write the system as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Example 1.1.11 (Swing of a pendulum)**.** Suppose we have a pendulum hanging below the ceiling, with a mass of weight m and string of length L. We may try to model the angle θ between the swinging pendulum and its fixed state (i.e., vertical), that is, constructing a function $\theta(t)$. With gravitational constant g, we have

$$\ddot{\theta}(t) + \frac{g}{L}\sin(\theta(t)) = 0.$$

Now we rewrite this into a system of first-order ODEs by writing $\theta_1(t) = \theta(t)$ and $\theta_2(t) = \dot{\theta}(t)$, so

$$\begin{cases} \dot{\theta}_1 = \theta_2 \\ \dot{\theta}_2 = -\frac{g}{L}\sin(\theta_1) \end{cases},$$

which gives a non-linear system.

Remark 1.1.12 (Linear superposition principle of linear systems)**.** For a linear system

$$\dot{x} = f(t, x),$$

if two functions $x^{(1)}(t)$ and $x^{(2)}(t)$ both satisfy this system, then for any $c_1, c_2 \in \mathbb{R}$, the function $c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$ also satisfies this system. Indeed,

$$\begin{aligned}\frac{d}{dt}(c_1 x^{(1)}(t) + c_2 x^{(2)}(t)) &= c_1 \frac{d}{dt} x^{(1)}(t) + c_2 \frac{d}{dt} x^{(2)}(t) \\ &= c_1 f(t, x^{(1)}(t)) + c_2 f(t, x^{(2)}(t)) \\ &= f(t, c_1 x^{(1)}(t) + c_2 x^{(2)}(t))\end{aligned}$$

by linearity of f.

Remark 1.1.13. Non-linear systems are harder to study in general. A useful perspective of studying qualitative behavior of non-linear system without solving the equation is the geometric method. We consider the system

$$\dot{x} = f(t, x).$$

If we think of $x(t)$ as the position of a particle at time t and $f(t, x)$ as the flow or the wind field, then the system above describes the particle movement with the flow. In particular, $x(t)$ draws a curve in a space (e.g., $\mathbb{R}^n$). This curve is call a trajectory, and the space is called the phase space.

Example 1.1.14. We consider the pendulum in Example 1.1.11 again:

$$\begin{cases}\dot{\theta}_1 = \theta_2 \\ \dot{\theta}_2 = -\frac{g}{L}\sin(\theta_1)\end{cases}$$

In particular, $(\theta_1(t), \theta_2(t))$ moves along a curve in $\mathbb{R}^2$, which corresponds to a trajectory in the phase space $\mathbb{R}^2$. Note that there are two different curves in this phase space, since there are two different initial datum.

If we draw trajectories corresponding to many different initial states, we obtain something called a phase portrait, which is a collection of trajectories that shows dynamics of particles starting from all possible positions.

1.2 Flows on the Line

1.2.1 Fixed Points and Stability

In this section, we start studying the flows on $\mathbb{R}$. For starters, we consider an autonomous 1-dimensional system

$$\dot{x} = f(x)$$

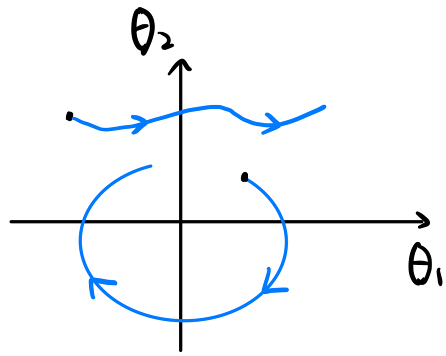


FIGURE 1.1.1: Trajectories corresponding to two different initial datum

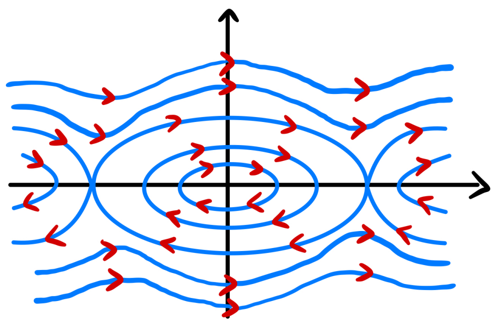


FIGURE 1.1.2: Phase portrait

where $x = x(t)$ and $f : \mathbb{R} \to \mathbb{R}$.

Remark 1.2.1. Such one-dimensional problem may be solvable by using techniques like separation of variables: since $\frac{dx}{dt} = f(x)$, then $\frac{dx}{f(x)} = dt$, so there is $dF(x) = dt$ for some F, and it suffices to find such F.

We may just want to study qualitative properties of the system, for example:

1. Given $x(0) = x_0$, does $x(t)$ increase or decrease, or does it oscillate?

2. What happens to $x(t)$ as $t \to \infty$?

3. For different values of x_0, how can we classify the behavior of $x(t)$ as $t \to \infty$?

Note that answering these questions does not necessarily require knowledge of explicit solutions.

We now look back at the geometric perspective (c.f. Remark 1.1.13), where we interpret the ODE as a vector field:

- $x(t)$ is the position of a virtual particle,
- $\dot{x}(t)$ is the velocity of the particle,
- $f(x)$ is the flow in the phase space.

Example 1.2.2. Consider the equation $\dot{x} = \sin(x)$ on $\mathbb{R}$. If we draw the figure of $y = \dot{x}$, we obtain

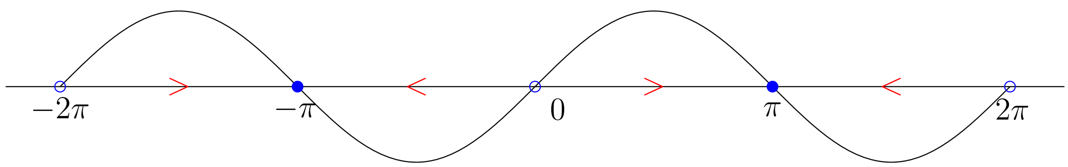


FIGURE 1.2.1: Phase space of the sine function

The data on the phase space is given by 1) the flow on $\mathbb{R}$, 2) fixed points (also known as equilibrium points/steady states), and 3) stability of fixed points.

- The flow is to the right whenever $f(x) > 0$, and the flow is to the left whenever $f(x) < 0$.
- A point $x_* \in \mathbb{R}$ is called a *fixed point* of the system if $f(x_*) = 0$.
- A fixed point x_* is called *(locally) stable* if there is a small neighborhood of x_* such that all solutions of the system starting in that neighborhood will converge to x_*, i.e., small disturbances damp out in time. A (locally) stable fixed point is also called an attractor or a sink.

Remark 1.2.3. A fixed point x_* is (locally) stable if and only if there exists $\delta > 0$ such that the solution $x(t)$ of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

satisfies $|x(t) - x_*| \xrightarrow[t\to\infty]{} 0$ whenever $x_0 \in (x_* - \delta, x_* + \delta)$.

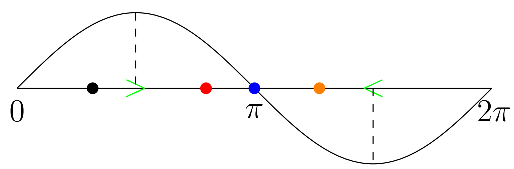
Definition 1.2.4. If small disturbance with respect to a fixed point x_* grows in time, then x_* is called an *unstable fixed point*, i.e., repeller, source, which is to say $|x(t) - x_*|$ increases in time no matter how small $|x_0 - x_*|$ is.

Definition 1.2.5. A graph collecting the three datum on the phase plane is called a phase portrait of $\dot{x} = f(x)$.

Remark 1.2.6. The phase portrait helps determine qualitative behavior of the solution (starting from different initial datum). Let us consider the system

$$\begin{cases} \dot{x} = \sin(x) \\ x(0) = x_0 \end{cases}$$

originated from Example 1.2.2. Because of the periodicity of sine function, we only have to consider how the flow behaves when the initial data x_0 is in $[0, 2\pi)$. From Figure 1.2.1, we note that the solid dots are stable fixed points and the circles are unstable fixed points, where the arrows indicate the direction of the flow. We now look at the sine function on this interval,



and we will consider how the behavior of four different points (black, red, blue, orange) on this interval can be represented by the phase portrait. We then obtain a phase portrait as in Figure 1.2.2.

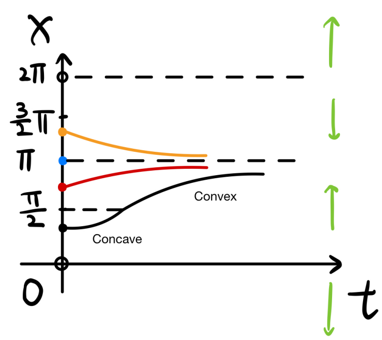


FIGURE 1.2.2: Phase Portrait of Example 1.2.2

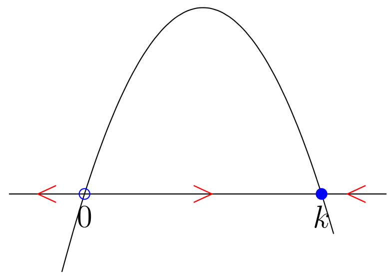
Note that the phase portrait describes two information in particular: 1) the monotonicity of $x(t)$, and 2) the convexity of $x(t)$. Moreover, we should be able to extend the diagram on the entire interval $\mathbb{R}$ by periodicity.

We now look at a few examples of phase portraits.

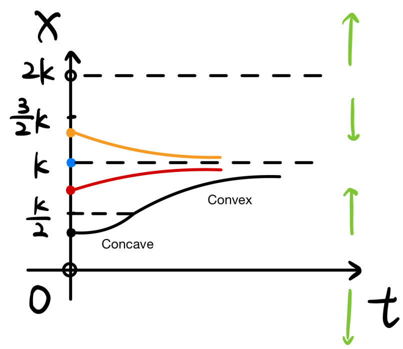
Example 1.2.7 (Population Growth)**.** Consider the logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{k}\right)$$

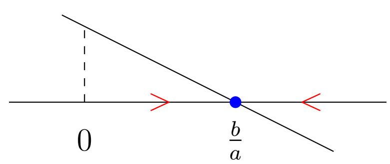
where $r > 0$ is the growth rate and $k > 0$ is the carrying capacity. By drawing the graph of the function, we have



and with phase portrait



Example 1.2.8 (Global Stability)**.** Consider the equation $\dot{x} = b - ax$ where $a, b > 0$. The graph of the function looks like



In this case, we say the point $\left(\frac{b}{a}, 0\right)$ is a *globally stable fixed point*, i.e., a fixed point x_* such that the solution $x(t)$ of $\dot{x} = f(x)$ with $x(0) = x_0$ converges to x_* for all values of x_0.

Remark 1.2.9. Sometimes a function could be too hard to plot, e.g., $\dot{x} = x - \cos(x)$, so in this case we would have to plot the graphs of x and $\cos(x)$ separately, so that we determine the sign of $\dot{x}$ as a whole.

1.2.2 Linear Stability Analysis

In this section, we introduce a more quantitative way of determining stability of fixed point. This is crucial because sometimes we can not simply plot a figure. In particular, we consider the problem formulated as follows:

Problem 1.2.10. Consider the system

$$\dot{x} = f(x)$$

where f is C^2 on $\mathbb{R}$, i.e., continuously twice differentiable, so f, f', f'' are all continuous on $\mathbb{R}$. Suppose x_* is known to be a fixed point, i.e., $f(x_*) = 0$. How do we determine the stability of x_* using analytic methods?

We approach this problem by performing linearization of $f(x)$ around the point x_*: determining stability of x_* boils down to study the growth and/or decay of $|x(t) - x_*|$ given $|x(0) - x_*| = |x_0 - x_*| \ll 1$, i.e., small enough. Therefore, we define $\eta(t) = x(t) - x_*$ to be the deviation from the equilibrium and consider the behavior of this function. Then we note that

$$\dot{\eta}(t) = \dot{x}(t) = f(x) = f(x_* + \eta(t)).$$

Here $\eta(t)$ is considered to be small (at least for some time) because $\eta(0)$ is assumed to be small.

By performing Taylor expansion of f at x_*, we obtain

$$f(x_* + \eta(t)) = f(x_*) + \eta(t)f'(x_*) + O(\eta^2),$$

where $O(\eta^2)$ is the higher-order error, bounded above (thus controlled) by $C|\eta^2|$ for some coefficient C that only depends on f and x_* (therefore not on η).

Remark 1.2.11 (Taylor expansion in its general form)**.** By Taylor expansion, we have

$$f(x) = f(x_*) + (x - x_*)f'(x_*) + \frac{(x - x_*)^2}{2!} f''(x_*) + \cdots + \frac{(x - x_*)^k}{k!} f^{(k)}(x_*) + \cdots .$$

For the purpose of approximation, we would view all terms from second-order and beyond as higher-order errors. By our assumption, $f(x_*) = 0$, and note that if $|\eta| \ll 1$ is small enough and $f'(x_*) \neq 0$, then the error term described by $O(\eta^2)$ should be negligible compared to $\eta(t)f'(x_*)$.

Combining these to the equation regarding η, we find that

$$\dot{\eta}(t) \approx \eta(t)f'(x_*),$$

so the linear ODE $\dot{\eta}(t) = \eta(t)f'(x_*)$, known as the linearization about x_*, describes the dominating behavior of $\eta(t)$ (when $|\eta| \ll 1$ is small and $f'(x_*) \neq 0$). Therefore, we can solve that

$$\eta(t) = \eta(0)e^{f'(x_*)t}.$$

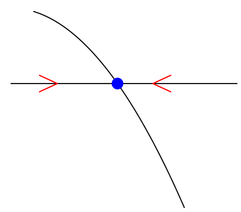
This tells us that

- if $f'(x_*) > 0$, then $\eta(t)$ grows exponentially;
- if $f'(x_*) < 0$, then $\eta(t)$ decays to 0 exponentially.

In conclusion,

- if $f'(x_*) > 0$, then x_* is unstable;
- if $f'(x_*) < 0$, then x_* is (locally) stable.

Remark 1.2.12. Comparing this to the graphic approach we described before, this implies that the fixed point is stable if and only if the figure looks like



locally.

Remark 1.2.13 (Indication of growth/decay rate)**.** Note that the size of $f'(x_*)$ shows the growth/decay rate of $\eta(t)$, since we know $\eta(t) = \eta(0)e^{f'(x_*)t}$. In fact, this shows how stable/unstable the fixed point x_* is. We would call $\frac{1}{|f'(x_*)|}$ the *characteristic time scale*, which is the time scale over which $\eta(t)$ experiences considerable changes, e.g., $\eta(t)$ gets doubled/halved. In particular, by the equation above, this occurs only when $f'(x_*)t$ is of order 1.

Example 1.2.14. Consider the equation

$$\dot{x} = \sin(x)$$

from Example 1.2.2. We note that the fixed points of the equation are the solutions of the equation $\sin(x) = 0$, which is just $x = k\pi$ for $k \in \mathbb{Z}$. We now try to determine their stability: for $f(x) = \sin(x)$, we have $f'(x) = \cos(x)$, so

$$f'(x_*) = f'(k\pi) = \cos(k\pi) = (-1)^k.$$

Therefore, $x = k\pi$ is locally stable if k is odd, and is unstable if k is even.

Example 1.2.15. Consider the equation

$$\dot{x} = rx\left(1 - \frac{x}{k}\right)$$

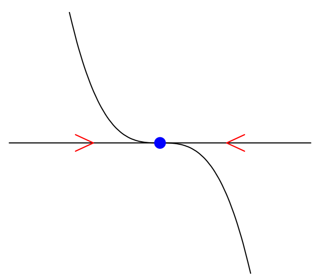
for $r, k > 0$ from Example 1.2.7. It is easy to note that $x = 0$ and $x = k$ are the fixed points. For the equation $f(x) = rx\left(1 - \frac{x}{k}\right)$, we have

$$f'(x) = r\left(1 - \frac{2x}{k}\right),$$

so $f'(0) = r > 0$, which means $x = 0$ is unstable, and $f'(k) = -r$, which means $x = k$ is locally stable. Moreover, at both positions, the characteristic time scale of the fixed point x_* is $\frac{1}{|f'(x_*)|} = \frac{1}{r}$.

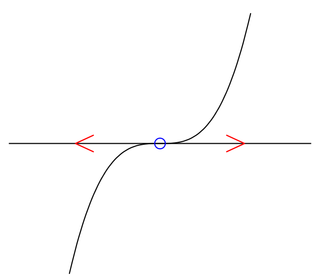
Example 1.2.16. Note that all the studies above are regarding $f'(x_*) \neq 0$. We would now consider the behavior when the derivative is 0 at the fixed point. Through this example, we would show that the behavior now would be unpredictable.

(a) Consider $\dot{x} = -x^3$,



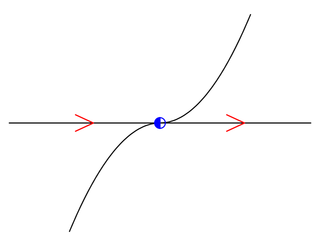
then the only fixed point is $x_* = 0$. Therefore, the only fixed point is locally stable and therefore globally stable.

(b) Consider $\dot{x} = x^3$,



then the only fixed point is $x_* = 0$. Therefore, the only fixed point is unstable.

(c) Consider $\dot{x} = x^2$,



then the only fixed point is $x_* = 0$. As one notice from the notation, this fixed point is neither (locally) stable nor unstable. We call such fixed points *half-stable*.

(d) Consider $\dot{x} = 0$,

then every point on $\mathbb{R}$ is a fixed point. We call such fixed points *neutral*.

Remark 1.2.17. To see why this is the case, note that when $f'(x_*) = 0$, we are not able to omit the error term $O(\eta^2)$ to derive $\dot{\eta} \approx \eta(t)f'(x_*)$ because the error term is not small enough compared to $\eta(t)f'(x_*)$.

1.2.3 Existence and Uniqueness

In this section, we discuss the fundamental question when studying differential equations, described below.

Problem 1.2.18. Consider the dynamical system (often called an *initial value problem* (IVP)) given by

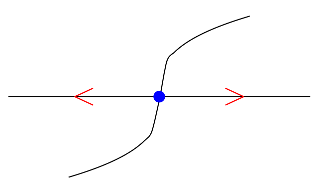
$$\begin{cases} \dot{x} &= f(x) \\ x(0) &= x_0 \end{cases}.$$

- Does the system have a solution? (Existence)
- Is the solution unique (if it exists)? (Uniqueness)

Example 1.2.19 (Non-uniqueness)**.** Consider the IVP

$$\begin{cases} \dot{x} &= x^{\frac{1}{3}} \\ x(0) &= 0 \end{cases}.$$

Note that the fixed point is at $x_* = 0$, which gives $f'(0) = \infty$, so 0 is very unstable.



We claim that

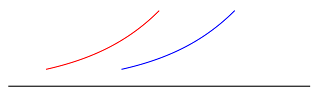
- $x(t) \equiv 0$ is a solution (which is obvious), and

- $$\tilde{x}(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \left(\frac{2}{3}t\right)^{\frac{3}{2}}, & \text{if } t > 0 \end{cases}$$
  is also a solution of the IVP.

Indeed,

$$\tilde{x}'(t) = \frac{3}{2}\left(\frac{2}{3}t\right)^{\frac{1}{2}} \cdot \frac{2}{3} = \left(\frac{2}{3}t\right)^{\frac{1}{2}} = \tilde{x}^{\frac{1}{3}}.$$

Actually, the IVP has infinitely many solutions: given any $\alpha > 0$, the function $x_\alpha(t) := \tilde{x}(t-\alpha)$ is also a solution.



Theorem 1.2.20 (Existence & Uniqueness)**.** Suppose $f(x)$ and $f'(x)$ are continuous on (a, b) and $x_0 \in (a, b)$. Then the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$$

admits a unique solution on the time interval $t \in (t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$ that only depends on x_0 and $f(x)$ on (a, b).

Remark 1.2.21 (Solution lifespan)**.** The theorem states that the solution may not exist for all time t, therefore requires the interval given by $\delta > 0$. Indeed, consider

$$\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 > 0 \end{cases}.$$

By separation of variables,

$$x(t) = \frac{x_0}{1 - tx_0},$$

so $x(t) \to \infty$ as $t \to x_0^{-1}$. Therefore, the function blows up, i.e., the solution goes to $\pm\infty$ in finite time.

However, if we think back to Example 1.2.7 given by

$$\dot{x} = rx\left(1 - \frac{x}{k}\right),$$

how can we make sure the solutions behave qualitatively like that? For example,

- do solutions touch/cross each other?

- are there oscillating solutions?

Proposition 1.2.22. Suppose $f(x)$ and $f'(x)$ are continuous on $\mathbb{R}$, then for the 1-dimensional autonomous equation $\dot{x} = f(x)$,

1. all solutions are monotone,
2. all solutions converge to $\pm\infty$ or to a fixed point, i.e., $x(t) \to \pm\infty$ in infinite or finite time, or $x(t) \to x_*$ as $t \to \infty$, where x_* is a fixed point,
3. (Comparison Theorem) consider two solutions with different initial data, i.e., consider

$$\begin{cases} \dot{x} = f(x_1) \\ x_1(t_0) = x_{1,0} \end{cases}$$

and

$$\begin{cases} \dot{x} = f(x_2) \\ x_2(t_0) = x_{2,0} \end{cases}$$

for $x_{1,0} > x_{2,0}$, then $x_1(t) > x_2(t)$ whenever x_1 and x_2 are defined at time t.

Remark 1.2.23. Proposition 1.2.22 says a lot about the behavior of the solution:

- No oscillation/periodic solutions exist in such 1-dimensional autonomous systems.
- It is impossible for a solution to converge to some point that is not a fixed point.
- Solutions starting from different initial values do not cross or touch each other.
- It is impossible for a solution to stay steady for a while and then start to move, or vice versa.

1.2.4 Potentials

Consider the dynamical system

$$\dot{x} = f(x).$$

Definition 1.2.24. If $f(x) = -\frac{dV}{dx}$ for some $V = V(x)$, then V is called a *potential* for this dynamical system.

Remark 1.2.25. In the 1-dimensional case, for arbitrary function f, it is always possible to find such a $V(x)$. For instance, one can pick

$$V(x) = -\int_0^x f(s)ds.$$

Note that this definition transfers the dynamical system above to

$$\dot{x} = -\frac{dV}{dx}(x).$$

We say V is the potential, or the heigth or energy of the state.

Claim 1.2.26. $V(x(t))$ is non-increasing in t.

Proof. By the chain rule, we have

$$\begin{aligned}\frac{dV(x(t))}{dt} &= \frac{dV}{dx}(x(t)) \cdot \dot{x}(t) \\ &= \frac{dV}{dt}(x(t)) \cdot \left(-\frac{dV}{dx}(x(t))\right) \\ &= -\left(\frac{dV}{dx}\right)^2 \\ &\leq 0.\end{aligned}$$

□

That is to say, in the system

$$\dot{x} = -\frac{dV}{dx},$$

the particle moves downhill in terms of the potential V. Therefore, x_* is a fixed point of $\dot{x} = f(x)$ if and only if $f(x_*) = 0$ if and only if $\frac{dV}{dt}(x_*) = 0$. Hence, whenever the last equation holds, we say x_* is called an *equilibrium point* of V. This also gives a criterion for the stability:

1. x_* is a locally fixed point if and only if x_* is a local minimum of V, and
2. x_* is an unstable fixed point if and only if x_* is a local maximum of V.

We now look at an example to find the potential.

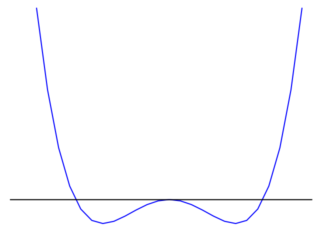
Example 1.2.27. Consider the function $\dot{x} = x - x^3$. If we write $f(x) = x - x^3 = -V'$, then

$$\begin{aligned}V(x) &= -\int_0^x f(s)ds + C \\ &= -\frac{1}{2}x^2 + \frac{1}{4}x^2 + C\end{aligned}$$

for some constant C. Without loss of generality, say $C = 0$, then we have

$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2.$$

Moreover, the equilibrium $f(x) = 0$ is given by $x = \pm 1, 0$. When $x = \pm 1$, we attain local minima, hence stable fixed points. When $x = 0$, we attain a local maximum, hence an unstable fixed point.



In particular, this is called a bistable system since it has two stable fixed points.

1.3 Bifurcations

2 Ordinary Differential Equations

3 Partial Differential Equations