Descent Theory Notes

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We already know that flat morphisms preserve exactness, which is very helpful when we study algebraic geometry. For instance, for $\pi : X \to Y$, we obtain a functor $\pi^* : \mathbf{QCoh(Y)} \to \mathbf{QCoh(X)}$. Note that this functor is not exact in general. However, if π is flat, we recover exactness! For instance, in the affine case, we have a map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$, then the pullback is exactly the tensor functor $- \otimes_A B : \text{A-}\mathbf{mod} \to \text{B-}\mathbf{mod}$. Therefore, the action of flat morphism is really described by the pullback geometrically, which is a generalized version of restriction, while respecting the sheaf conditions.

Definition 1. Let $f : A \to B$ be a morphism of schemes. We say f is

- a fpqc morphism if f faithfully flat and quasi-compact;
- a fppf morphism if f is faithfully flat and locally of finite presentation. In particular, a fppf morphism is open.

1 Descent on Quasi-Coherent Sheaves

Let $g : S' \to S$ be a fpqc morphism, and define $S'' = S' \times_S S'$. For $i = 1, 2$, we denote $p_i : S'' \to S'$ to be projections. For $1 \leqslant i < j \leqslant 3$, we denote $p_{ij} : S' \times_S S' \times_S S' \to S''$ to be the projection on the (i, j)-factor.

Definition 2. Let $\mathscr{F}$ be a quasi-coherent sheaf on S'. An isomorphism $\sigma : p_1^*\mathscr{F} \to p_2^*\mathscr{F}$ that satisfies the cocycle condition $p_{13}^*(\sigma) = p_{23}^*(\sigma) \circ p_{12}^*(\sigma)$ is called a descent datum.

Remark. For $1 \leqslant i \leqslant 3$, denote $\pi_i : S' \times_S S' \times_S S' \to S'$ to be the projection to the ith factor. For the definition to make sense, we need to identify $p_{13}^*(\sigma), p_{23}^*(\sigma), p_{12}^*(\sigma)$ with morphisms $\pi_1^*\mathscr{F} \to \pi_3^*\mathscr{F}, \pi_2^*\mathscr{F} \to \pi_3^*\mathscr{F}$, $\pi_1^*\mathscr{F} \to \pi_2^*\mathscr{F}$, respectively, through the canonical natural transformations

$$p_{ij}^* \circ p_1^* \cong \pi_i^*, p_{ij}^* \circ p_2^* \cong \pi_j^*.$$

One can ask a fundamental question, namely the Descent Problem, which is answered via Theorem 5: given $g : S' \to S$, and let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves over S with $\varphi : \mathcal{F} \to \mathcal{G}$. We have data of a homomorphism $\varphi' : g^*\mathcal{F} \to g^*\mathcal{G}$, then when is $\varphi' = g^*\varphi$? We note that $g \circ p_1 = g \circ p_2 : S'' \to S$, so a necessary condition would be $p_1^*\varphi' = p_2^*\varphi'$. However, part a. of Theorem 5 gives a sufficient condition: if φ' descends, then it descends uniquely.

Part b. of Theorem 5 establish the existence of descent data, as well as the morphism between descent data. The descent of morphisms says that given two sets of descent data, with respective objects on the base, then giving a morphism on those objects is the same as giving a morphism on the cover with compatibility satisfied, i.e., covering data. Therefore, if we are given an object on a Zariski cover, along with a gluing morphism satisfying cocycle conditions, then the descent determines an (unique up to unique isomorphism) object downstairs.

Descent theory is crucial because it can be extended from Zariski topology to étale topology and smooth topology, which in turn are required for stacks. Of course, this requires more formalism, in particular one should construct the descent category.

Let $\mathcal{C}$ be a site, and let $\mathcal{F}$ be a fibered category over $\mathcal{C}$, i.e., a presheaf of categories over $\mathcal{C}$. Let $U = \{\alpha_i : U_i \to U\}$ be a covering on $\mathcal{C}$, then one can extend the notion of descent to this cover, where descent data become the gluing condition on the covering, c.f., [02ZC].

Definition 3. A descent data of $\mathcal{F}$ is a collection of pair (X_i, σ_{ij}) with respect to the covering U, where $X_i \in \mathcal{F}(U_i)$ and $\sigma_{ij} : p_1^*(X_i) \to p_2^*(X_j)$ in $\mathcal{F}(U_i \times_U U_j)$ are isomorphisms satisfying the cocycle condition $p_{13}^*(\sigma_{13}) = p_{23}^*(\sigma_{23}) \circ p_{12}^*(\sigma_{12})$. This gives a category of descent data of $\mathcal{F}$ with respect to the covering U.

For each $X \in U$, we can construct a descent data over U of $\mathcal{F}$, where objects are $(\alpha_i^* X, \sigma_{ij})$ for $\sigma_{ij} : p_1^* \alpha_i^* X \cong p_2^* \alpha_j^* X$.

Remark. Given a fpqc covering by a map, we can refine it into a covering by fpqc maps from a Zariski covering. First note that given Zariski descent (with respect to a Zariski covering) and descent along faithfully flat maps of affine schemes, we obtain a descent along fpqc morphisms, c.f., Conrad's handout, page 1. Now by Zariski descent and descent along fpqc morphisms, we obtain a fpqc descent (with respect to a fpqc covering), c.f., [Vis04], step 4 of Lemma 2.60. That is, if we start with a morphism $f : S' \to S$ where S' is fpqc and S is affine, then one can construct fpqc morphisms $f_i : S_i \to S$ as restrictions on some subschemes S_i of S'. By descent along those fpqc morphisms and the Zariski descent on target, we obtain fpqc descent on the given fpqc covering.

Remark. To compare the coverings, we have

fpqc	weak
fppf	↑
smooth	**covering**
étale	↓
Zariski	strong

Also c.f., Johan's post and [03NV].

Example 4. Let $X = \bigcup X_i$ be an open cover, and define $Y = \coprod X_i$ to be the disjoint union of this cover, hence there is a defined surjective morphism $p : Y \to X$. Now consider an arbitrary function f on X, we are looking for functions f' of Y so that they "descend" to f on X. That is, given f on X, when pulling back via the surjection p, we obtain them so that they are "isomorphic" to f' on Y, i.e., when restricting on the covering via pullback we get the same thing. However, for this to work, we need to study the gluing condition on the covering, in particular the subset of the form $Y \times_X Y$, where the intersection can be interpreted as subsets of different parts of the disjoint union. Therefore, a descent asks for uniqueness/recoverability of gluing along the pullback (as a notion of restriction). This can be further generalized to vector bundles.

Theorem 5.

a. Let $\mathscr{F}$ (respectively, $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_S$-module, and let $\mathscr{F}'$ and $\mathscr{F}''$ (respectively, $\mathscr{G}'$ and $\mathscr{G}''$) be its inverse images on S' and S''. Then

$$\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{G}) \xrightarrow{g^*} \operatorname{Hom}_{\mathscr{O}_{S'}}(\mathscr{F}', \mathscr{G}') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \operatorname{Hom}_{\mathscr{O}_{S''}}(\mathscr{F}'', \mathscr{G}'')$$

is exact.

Remark. That is, g^* is injective and $\mathrm{im}(g^*) = \{x \in \mathrm{Hom}_{\mathscr{O}_{S'}}(\mathscr{F}', \mathscr{G}') : p_1^*(x) = p_2^*(x)\}$. Unpacking this, suppose $u' \in \mathrm{Hom}_{\mathscr{O}_{S'}}(\mathscr{F}', \mathscr{G}')$ satisfies $p_1^*(u) = p_2^*(u)$, then there exists a unique $u \in \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{G})$ such that $g^* u = u'$.

b. Suppose $\mathscr{F}'$ is a quasi-coherent sheaf on S' such that there exists a descent datum σ, then there exists a quasi-coherent $\mathscr{O}_S$-module $\mathscr{F}$ and an isomorphism $\tau : g^*\mathscr{F} \cong \mathscr{F}'$ such that the diagram

$$\begin{array}{ccc} p_1^* g^* \mathscr{F} & \xrightarrow{p_1^*\tau} & p_1^* \mathscr{F}' \\ \cong\downarrow & & \cong\downarrow\sigma \\ p_2^* g^* \mathscr{F} & \xrightarrow[p_2^*\tau]{} & p_2^* \mathscr{F}' \end{array}$$

commutes. Moreover, such $\mathscr{F}$ is unique up to unique isomorphism.

Proof. Using the quasi-compact property, one can reduce the statement to affine case, i.e., $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$, where A' is a faithfully flat A-algebra.

a. Since $\mathscr{F}$ and $\mathscr{G}$ are sheaves associated to A-modules M and N, respectively, then this is a purely homological algebra statement. This shows that when objects are pullbacks from the base, we have descent for morphisms.

b. Suppose such $\mathscr{F}$ exists, then its universality comes from the universal property of the kernel/equalizer in part a. when reduced to commutative algebra. The existence is also a homological algebra statement. The key is that every descent datum determines a unique object on the base.

Therefore, if we get an arbitrary pair of descent data where each admits a solution, then we have descents for the morphisms, and the solution to any descent is now unique up to unique isomorphism. □

Example 6. Part b of Theorem 5 fails if the covering map g is only faithfully flat, but neither fppf nor fpqc. For instance, take $g : \coprod_p \mathrm{Spec}(\mathbb{Z}_p) \to \mathrm{Spec}(\mathbb{Z})$.

Remark. For affine schemes, the descent for morphism holds, i.e., recover descent from pullback, is just the sheaf condition for $\mathrm{Hom}(-, X)$ over a fpqc (or fppf) cover, which is automatically true for schemes X. Therefore, descents are just generalizations of sheaf conditions. Moreover, the descent conditions are just notions that are local in the fpqc topology. See also [02YJ].

Lemma 7. Let $g : S' \to S$ be a fpqc morphism, and suppose $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_S$-module, then $g^*\mathscr{F}$ is locally of finite type (respectively, locally of finite presentation/locally free of finite rank) if and only if $\mathscr{F}$ is.

Proof.

($\Leftarrow$): this is a standard fact.

($\Rightarrow$): again, these are local properties so one can reduce the statement down to the affine case, i.e., set $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$, where A' is a faithfully flat A-algebra.

The locally of finite type property of schemes transform into the finitely-generated property of modules over rings. Let M be an arbitrary A-module, then we can write $M = \varinjlim_\lambda M_\lambda$ where each M_λ is a finitely-generated A-submodule of M, therefore $M \otimes_A A' = \varinjlim_\lambda M_\lambda \otimes_A A'$. Now $M \otimes_A A'$ is an A'-module, therefore it is finitely-generated. In particular, there exists some λ such that $M_\lambda \otimes_A A' \to M \otimes_A A'$ is surjective. Since A' is faithfully flat over A, then $M_\lambda \to M$ is surjective and therefore $M_\lambda = M$. Therefore, M is finitely-generated as an A-module.

The locally of finite presentation property of schemes transform into the finite presentation property of modules over rings. Again, let M be an arbitrary A-module, then $M \otimes_A A'$ has finite presentation as A'-module, therefore M is finitely-generated as an A-module. There now exists a short exact sequence

$$0 \longrightarrow R \longrightarrow L \longrightarrow M \longrightarrow 0$$

where L is a free A-module of finite rank. Since A' is faithfully flat over A, then we have another short exact sequence

$$0 \longrightarrow R \otimes_A A' \longrightarrow L \otimes_A A' \longrightarrow M \otimes_A A' \longrightarrow 0$$

But $M \otimes_A A'$ has finite presentation, therefore $R \otimes_A A'$ is finitely-generated as A'-module, and therefore R is finitely-generated as A-module, i.e., quotient of some free A-module. Therefore, M has finite presentation.

We omit the proof for local freeness.

□

2 Descent on Properties of Morphisms

The construction above gives a prototype for descents as a category, especially concerned with sheaves. We can also study the effect of descent on properties of morphisms, i.e., when is a property of morphism fpqc-local?

Definition 8. Let $f : X \to S$ be a morphism of schemes.

- We say f is surjective (respectively, injective) if it is surjective (respectively, injective) on the underlying topological space.
- We say f is universally injective if for any morphism $S' \to S$ of schemes, the base-change $f' : X \times_S S' \to S'$ of f is injective.
- We say f is radiciel if f is injective, and for any $x \in X$, the residue field $k(x)$ of X at x is a purely inseparable algebraic extension of the residue field $k(f(x))$ of Y at $f(x)$.

Theorem 9. Let $f : X \to S$ be a morphism of schemes, then the following are equivalent:

1. for any algebraically closed field K, the map $X(K) \to S(K)$ induced by f is injective. Here $X(K) = \mathrm{Hom}(\mathrm{Spec}(K), X)$ and $S(K) = \mathrm{Hom}(\mathrm{Spec}(K), S)$ are the K-points in X and in S, respectively;
2. f is universally injective;
3. f is radiciel;
4. the diagonal morphism $\Delta : X \to X \times_S X$ is surjective.

Remark. This is not true for general algebraic spaces, c.f., [0480].

Proof Sketch. We briefly cover why being universally injective is equivalent to the other three conditions.

1. and 2. Note that every point in an S-scheme can be embedded in an algebraically closed field, so we just have to check the injection over there.

3. and 2. To have non-zero separability degree is just saying when we base-change to some algebraic closure, for any point x we get $[k(x) : k(f(x))]_{\text{sep}}$ new points in the fiber. We need to avoid this, i.e., having purely inseparable extensions, to obtain an injection.

4. and 2. Recall that injection of sets is equivalent to surjection of diagonal map. Even though we are working over base-change, we expect the surjectivity of the diagonal scheme map to be "insensitive"to base-change.

□

Corollary 10. A universally injective morphism of schemes $X \to S$ is separated.

Proof. Recall that $X \to S$ is separated if and only if the image of Δ is closed in $X \times_S X$, then we conclude by Theorem 9. □

There are other properties described in [01S2]. The following remark describes some intuition developed from this blog post.

Remark (Why Radiciel?)**.** Roughly speaking, a morphism of schemes $f : X \to S$ is étale if it is locally of finite presentation (finiteness), and is both flat (functoriality) and unramified (geometric). The slogan being, étale morphisms are "locally isomorphisms" (in étale topology, where they are regarded as generalization of open sets.

First note that giving an open embedding $j : U \hookrightarrow X$ is the same as giving the image $j(U) \subseteq X$. Therefore, open embeddings are determined by set-theoretic properties. To be precise, we have

Theorem 11. Let $f : X \to S$ be a morphism locally of finite presentation, then f is an open embedding if and only if for any $g : Y \to S$ such that $g(Y) \subseteq f(X)$, there exists a unique S-morphism $g' : Y \to X$ such that $fg' = g$.

Hence, it is nice to see that open embeddings, "the nice maps", are étale. Radiciel morphisms then resolve the question of when étale morphisms are open embeddings: if and only if they are radiciel.

Theorem 12. Consider a pullback square

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

where g is surjective. Then

a. f is surjective if and only if f' is;

b. if f' is injective, then so is f;

c. f is universally injective if and only if f' is;

d. if, in addition, that g is quasi-compact, then f is quasi-compact if and only if f' is.

Proof. For any $s' \in S'$, we have $f'^{-1}(s') \cong f^{-1}(g(s')) \otimes_{k(g(s'))} k(s')$. Recall that the projection $f'^{-1}(s') \to f^{-1}(g(s'))$ is surjective, and $k(s')$ is faithfully flat over $k(g(s'))$. Therefore, we know $f^{-1}(s') \neq \varnothing$ if and only if $f^{-1}(g(s')) \neq \varnothing$. Since g is surjective, this proves a. If f' is injective, then $f'^{-1}(s')$ is at most a singleton, therefore $f^{-1}(g(s'))$ is at most a singleton, so f is injective, which proves b. c. now follows from b. Finally, we prove d.

Suppose f is quasi-compact, and let $V' \subseteq S'$ be an affine open subset such that $g(V')$ is contained in an affine open subset V of S. Let us write $f'^{-1}(V) = \bigcup_{i=1}^{n} U_i$ for affine open subsets $U_i \subseteq X$, then $f'^{-1}(V') = \bigcup_{i=1}^{n} (V' \times_V U_i)$ is a finite union of affine open subsets of $X \times_S S'$.

Suppose f' is quasi-compact, and let $V \subseteq S$ be a quasi-compact open subset. Since g is surjective, then $V = g(g^{-1}(V))$, therefore $f^{-1}(V) = f^{-1}(g(g^{-1}(V))) = g'(f'^{-1}g^{-1}(V))$. Since g and f' are both quasi-compact, then $f'^{-1}(g^{-1}(V))$ is quasi-compact, hence $f^{-1}(V) = g'(f'^{-1}g^{-1}(V))$ is quasi-compact, therefore f is quasi-compact. □

Theorem 13. Consider a pullback square

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

where g is fpqc, then f is of finite type if and only if f' is.

Proof. This uses the same idea as in Lemma 7. □

Corollary 14. Consider a pullback square

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

where g is fpqc, then f is of quasi-finite if and only if f' is.

Lemma 15. Let $g : Y' \to Y$ be flat and let $Z \subseteq Y$ be a subset. Assume there exists a quasi-compact morphism $f : X \to Y$ such that $Z = f(X)$, then $g^{-1}(\bar{Z}) = \overline{g^{-1}(Z)}$.

Corollary 16. Let $g : Y' \to Y$ be quasi-compact and flat, and let Z' be a closed subset of Y' such that $Z' = g^{-1}(g(Z'))$, then $Z' = g^{-1}(\overline{g(Z')})$. Moreover, the subspace topology on $g(Y')$ induced from Y coincides with the quotient topology induced from Y'.

Proof. The first statement follows from Lemma 15. To prove the second statement, since $g : Y' \to Y$ is continuous, then every subset of $g(Y')$ that is closed with respect to the subspace topology induced from Y is closed with respect to the quotient topology induced from Y'. Conversely, let $Z \subseteq g(Y')$ be closed with respect to the quotient topology induced from Y', then $g^{-1}(Z')$ is closed. By the first statement, we know $g^{-1}(Z) = g^{-1}(\overline{g(g^{-1}(Z))}) = g^{-1}(\bar{Z})$. Therefore, $Z = \bar{Z} \cap g(Y')$, i.e., Z is closed with respect to the topology induced from Y. □

Corollary 17. Assume $g : Y' \to Y$ is fpqc, then the topology on Y is the same as the quotient topology induced from Y'.

Corollary 18. Let $g : S' \to S$ be a fpqc morphism, and denote $S'' = S' \times_S S'$. Let $O(S), O(S'), O(S'')$ be the set of open subsets in S, S', S'', respectively, and $F(S), F(S'), F(S'')$ be the set of closed subsets in S, S', S'', then the sequences

$$O(S) \longrightarrow O(S') \underset{p_2^{-1}}{\overset{p_1^{-1}}{\rightrightarrows}} O(S'')$$

and

$$F(S) \longrightarrow F(S') \underset{p_2^{-1}}{\overset{p_1^{-1}}{\rightrightarrows}} F(S'')$$

are exact.

Corollary 19. Assume $g : Y' \to Y$ be fpqc morphism. Let $Z \subseteq Y$ be the image of a quasi-compact morphism, then Z is locally closed if and only if $g^{-1}(Z)$ is.

Proof. Consider the pullback square

$$\begin{array}{ccc} g^{-1}(\bar{Z}) & \longrightarrow & \bar{Z} \\ \downarrow & & \downarrow \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

where $\bar{Z}$ has a closed scheme structure. Note that $g^{-1}(\bar{Z}) \to \bar{Z}$ is quasi-compact and faithfully flat. By Lemma 15, we have $g^{-1}(\bar{Z}) = \overline{g^{-1}(Z)}$. If $g^{-1}(Z)$ is locally closed, then by definition it is open in $g^{-1}(\bar{Z})$, now by Corollary 17 we know Z is open in $\bar{Z}$, therefore Z is locally closed. □

Theorem 20. Consider a pullback square

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[g]{} & S \end{array}$$

where g is fpqc.

a. If f' is an open mapping (respectively, closed mapping/quasi-compact embedding/homeomorphism), then so is f.

b. f is universally open (respectively, universally closed/universally a homeomorphism) if and only if f' is.

c. f is separated (respectively, proper) if and only if f' is.

Proof.

a. If f' is open or closed, this follows from Corollary 17. If f' is a quasi-compact embedding, then f is injective and quasi-compact by Theorem 12. For any closed subset $Z \subseteq X$, we know by Lemma 15 that

$$g^{-1}(\overline{f(Z)}) = \overline{g^{-1}(f(Z))} = \overline{f'(g'^{-1}(Z))}.$$

Therefore,

$$\begin{aligned} g'^{-1}(Z) &= f'^{-1}(\overline{f'(g'^{-1}(Z))}) \text{ since } f' \text{ is an embedding} \\ &= f'^{-1}(g^{-1}(\overline{f(Z)})) \\ &= g'^{-1}(f^{-1}(\overline{f(Z)})) \end{aligned}$$

Since g' is surjective, then $f^{-1}(\overline{f(Z)}) = Z$. Finally, if f' is a homeomorphism, then f is a surjective embedding by Theorem 12.

b. By definition.

c. If f is separated, then this follows from part a. Recall that f being proper is just being of finite type, separated, and universally closed, so the statement follows from Theorem 13 and part a. and b.

□

Remark. Radiciel morphisms are stable under composition, products and base-change.

Lemma 21. Let $f : S' \to S$ be a fpqc morphism, and let $S'' = S' \times_S S'$. For any scheme Z, the sequence

$$\mathrm{Hom}(S, Z) \longrightarrow \mathrm{Hom}(S', Z) \rightrightarrows \mathrm{Hom}(S'', Z)$$

is exact.

Proof. This is a consequence of Theorem 5 and Corollary 17. □

Corollary 22. Let $f : S' \to S$ be a fpqc morphism and $S'' = S' \times_S S'$. Suppose X and Y are S-schemes, then

$$\mathrm{Hom}_S(X, Y) \longrightarrow \mathrm{Hom}_{S'}(X \times_S S', Y \times_S S') \rightrightarrows \mathrm{Hom}_{S''}(X \times_S S'', Y \times_S S'')$$

is exact.

Proof. By Lemma 21, the sequences

$$\mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(X \times_S S', Y) \rightrightarrows \mathrm{Hom}(X \times_S S'', Y)$$

and

$$\mathrm{Hom}(X, S) \longrightarrow \mathrm{Hom}(X \times_S S', S) \rightrightarrows \mathrm{Hom}(X \times_S S'', S)$$

are exact, therefore

$$\mathrm{Hom}_S(X, Y) \longrightarrow \mathrm{Hom}_S(X \times_S S', Y) \rightrightarrows \mathrm{Hom}_S(X \times_S S'', Y)$$

is exact. This is the desired sequence. □

Corollary 23. Let $f : X \to Y$ be an S-morphism, and let $f' : X' \to Y'$ be the base-change of f, then f is an isomorphism if and only if f' is.

Proof. If f' is an isomorphism, then the images of g' in $\mathrm{Hom}_{S'}(Y', X') \rightrightarrows \mathrm{Hom}_{S''}(Y \times_S S'', X \times_S S'')$ are both the inverse of the base-change $f'' : X \times_S S'' \to Y \times_S S''$ of g, thus g' lies in the kernel of $\mathrm{Hom}_{S'}(Y', X') \rightrightarrows \mathrm{Hom}_{S''}(Y \times_S S'', X \times_S S'')$. By Corollary 22, there exists an S-morphism $g : Y \to X$ with base-change g'. Since $g'f' = \mathrm{id}_{X'}$ and $\mathrm{Hom}_S(X, X) \to \mathrm{Hom}_{S'}(X', X')$ is injective, then $gf = \mathrm{id}_X$. Similarly, $fg = \mathrm{id}_Y$, thus f is an isomorphism. □

Corollary 24. Let $f : X \to Y$ be an S-morphism, and let $f' : X' \to Y'$ be the base-change of f, then f is a closed (respectively, open/quasi-compact) immersion if and only if f' is.

3 Descent on Schemes

Let $f : S' \to S$ be a fpqc morphism, $S'' = S' \times_S S'$, $S''' = S' \times_S S' \times S'$, with $p_1, p_2 : S'' \to S'$, $p_{12}, p_{13}, p_{23} : S''' \to S''$ as projections.

Definition 25. Let X' be an S'-scheme. Any S''-isomorphism $\sigma : p_1^*X' \cong p_2^*X'$ satisfying cocycle condition $p_{13}^*(\sigma) = p_{23}^*(\sigma) \circ p_{12}^*(\sigma)$ is called a descent datum for X', where p_i^*X' is defined by the pullback

$$\begin{array}{ccc} p_i^*X' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ S'' & \xrightarrow[p_i]{} & S' \end{array}$$

Theorem 26. Let X' be an affine S'-scheme with descent datum $\sigma : p_1^*X' \cong p_2^*X'$, then there exists an affine S-scheme X and S'-isomorphism $\tau : g^*X \cong X'$ such that the diagram

$$\begin{array}{ccc} p_1^*g^*X & \xrightarrow{p_1^*\tau} & p_1^*X' \\ \cong\downarrow & & \downarrow\sigma \\ p_2^*g^*X & \xrightarrow[p_2^*\tau]{} & p_2^*X' \end{array}$$

commutes, where $g^*X = X \times_S S'$. Such X is unique up to unique isomorphism.

Proof. Let $f' : X' \to S'$ be the structure morphism, and set $\mathscr{A}' = f'_*\mathscr{O}_{X'}$, then $\mathscr{A}'$ is a quasi-coherent $\mathscr{O}_{S'}$-algebra, so by Theorem 5 we construct quasi-coherent $\mathscr{O}_S$-module $\mathscr{A}$. Set $X = \mathrm{Spec}(\mathscr{A})$. All required properties illustrated in Theorem 5 descend from quasi-coherent sheaves. □

Corollary 27. Let $f : X \to Y$ be an S-morphism and let $f' : X' \to Y'$ be the base-change of f, then f is affine if and only if f' is.

Corollary 28. Let $f : X \to Y$ be an S-morphism and let $f' : X' \to Y'$ be the base-change of f, then f is integral (respectively, finite/finite and locally free) if and only if f' is.

Proposition 29. A morphism $f : X \to Y$ is quasi-finite if and only if it is quasi-compact, separated, and the canonical Y-morphism $X \to \mathrm{Spec}(f_*\mathscr{O}_X)$ is an open immersion.

Corollary 30. Let $f : X \to Y$ be an S-morphism and let $f' : X' \to Y'$ be the base-change of f, then f is quasi-affine if and only if f' is.

Proof. If f' is quasi-affine, then f' is quasi-compact and separated, therefore f is quasi-compact and separated. The base-chnage of the canonical Y-morphism $X \to \mathrm{Spec}(f_*\mathscr{O}_X)$ can be identified with the canonical Y'-morphism $X' \to \mathrm{Spec}(f'_*\mathscr{O}_X)$, which is an open immersion. By Corollary 24, $X \to \mathrm{Spec}(f_*\mathscr{O}_X)$ is an open immersion, hence f is quasi-affine. □

Proposition 31. Let X' be a quasi-affine S''-scheme with descent datum $\sigma : p_1^*X' \cong p_2^*X'$, then there exists an affine S-scheme X and S'-isomorphism $\tau : g^*X \cong X'$ such that the diagram

$$\begin{array}{ccc} p_1^*g^*X & \xrightarrow{p_1^*\tau} & p_1^*X' \\ \cong\downarrow & & \downarrow\sigma \\ p_2^*g^*X & \xrightarrow[p_2^*\tau]{} & p_2^*X' \end{array}$$

commutes, where $g^*X = X \times_S S'$. Such X is unique up to unique isomorphism.

Proof. By Theorem 5, there exists a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}$ such that $g^*\mathscr{A} \cong f'_*\mathscr{O}_{X'}$. By Corollary 18, there exists an open immersion $X \hookrightarrow \mathrm{Spec}(\mathscr{A})$ whose base-change is the open immersion $X' \hookrightarrow \mathrm{Spec}(f'_*\mathscr{O}_{X'})$, then X has the required property. □

4 Passage to Limit

Finally, we establish results so that we can pass results on quasi-coherent algebras to the limit via a direct system.

Let S_0 be the base scheme, I be a direct set, and consider a direct system $(\mathscr{A}_\lambda, \varphi_{\lambda\mu})$ where A_λ's are quasi-coherent $\mathscr{O}_{S_0}$-algebras for $\lambda \in I$, and $\varphi_{\lambda\mu} : \mathscr{A}_\lambda \to \mathscr{A}_\mu$ are morphisms of $\mathscr{O}_{S_0}$-algebras for $\lambda \leqslant \mu \in I$. Now set $\mathscr{A} = \varinjlim_\lambda \mathscr{A}_\lambda$ and set $\varphi_\lambda : \mathscr{A}_\lambda \to \mathscr{A}$ be canonical morphisms. Set $S_\lambda = \operatorname{Spec}(\mathscr{A}_\lambda)$ and $S = \operatorname{Spec}(\mathscr{A})$, then they are S_0-schemes. Set $u_{\lambda\mu} : S_\mu \to S_\lambda$ be the S_0-morphisms induced by $\varphi_{\lambda\mu}$ and $u_\lambda : S \to S_\lambda$ be the S_0=morphisms induced by φ_λ. Moreover, objects over S_0 are denoted with subscript 0, and the corresponding object over S_λ (respectively, S) induced by base-change by the same symbol with subscript λ (respectively, without subscript).

With this setting in mind, the passage to limit property says that, given an object over S_0, if its base-change to S has property $\mathcal{P}$, then its base-changes to S_λ have property $\mathcal{P}$ for sufficiently large λ.

Proposition 32.

a. S is the inverse limit of the inverse system $(S_\lambda, u_{\lambda\mu})$ in the category of schemes.

b. For any quasi-compact open subset U of S, there exists a quasi-compact open subset $U_\lambda \subseteq S_\lambda$ for some λ such that $u_\lambda^{-1}(U_\lambda) = U$.

c. The underlying topological space of S is the inverse limit of the inverse system $(S_\lambda, u_{\lambda\mu})$ in the category of topological spaces.

d. If S_0 is quasi-compact and $S = \varnothing$, then $S_\lambda = \varnothing$ for sufficiently large λ.

Remark. There are two situations we usually apply "passage to limit" to.

- Let A be a ring, and let $\{A_\lambda\}$ be a direct system of A-subalgebras finitely-generated by $\mathbb{Z}$, then $A = \varinjlim_\lambda A_\lambda$. Therefore, we can reduce problems over a base scheme $S = \operatorname{Spec}(A)$ to problems over Noetherian base schemes $S_\lambda = \operatorname{Spec}(A_\lambda)$.
- Let S_0 be an affine scheme, and let $x \in S_0$ be a fixed point. Let $\{S_\lambda\}$ be the inverse system of affine open neighborhoods of x in S_0, and let $S = \operatorname{Spec}(\mathscr{O}_{S_0,x})$, then $\mathscr{O}_{S_0,x} = \varinjlim_\lambda \Gamma(S_\lambda, \mathscr{O}_{S_\lambda})$. Therefore, we can prove that if the base-change to $\operatorname{Spec}(\mathscr{O}_{S_0,x})$ of an object over S_0 has property $\mathcal{P}$, then its base-change to a neighborhood of x has property $\mathcal{P}$ as well.

Proposition 33. Suppose S_0 is quasi-compact and quasi-separated.

a. Let X_0 and Y_0 be S_0-schemes such that $X_0 \to S_0$ is quasi-compact and quasi-separated, and that $Y_0 \to S_0$ is locally of finite presentation, then the canonical map

$$\varinjlim_\lambda \operatorname{Hom}_{S_\lambda}(X_\lambda, Y_\lambda) \to \operatorname{Hom}_S(X, Y)$$

is a bijection.

b. Suppose X_0 and Y_0 are S_0-schemes of finite presentation. If there is an S-isomorphism $f : X \to Y$, then for a sufficiently large λ, there exists an S_λ-isomorphism $f_\lambda : X_\lambda \to Y_\lambda$.

c. For any S-scheme X of finite presentation, there exists an S_λ-scheme X_λ of finite presentation for a sufficiently large λ such that $X \cong X_\lambda \times_{S_\lambda} S$.

Theorem 34. Let S_0 be quasi-compact and quasi-separated. Let $f_0 : X_0 \to S_0$ be a morphism of finite presentation. If f is a morphism of property $\mathcal{P}$, where $\mathcal{P}$ is one of the following, then for sufficiently large λ, f_λ also has property $\mathcal{P}$.

- Open immersion
- Closed immersion
- Separated
- Finite
- Affine
- Surjective
- Radiciel
- Immersion
- Quasi-affine
- Quasi-finite
- Proper

Theorem 35 (Zariski Main Theorem)**.** Let S be a Noetherian scheme, and let $f : X \to S$ be a separated quasi-finite morphism, then there exists a finite morphism $\bar{f} : \bar{X} \to S$ and an open immersion $j : X \hookrightarrow \bar{X}$ such that $f = \bar{f}j$.

Remark. Theorem 35 holds if we only assume S is quasi-compact and quasi-separated.

References

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