Computing $\mathrm{TC}(\mathbb{F}_\mathrm{p})$

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Calculations for $\mathrm{TC}(\mathbb{F}_\mathrm{p})$ has traditionally been done at the level of classical orthogonal cyclotomic spectrum, c.f., [Mad94]. In this talk, we will discuss a new calculation done by [NS18] using ∞-categorical machinery and homotopy-invariant notions. We will mostly follow Section IV.4 of [NS18], and calculate the homotopy groups. Because of the amount of preliminaries required before doing the calculation, we only skim through a few important things that we will use. Throughout the talk, let $\mathbb{P}$ be the set of primes. We fix a prime $p \in \mathbb{P}$, and let $q \in \mathbb{P}$ be an arbitrary prime.

1 Preliminaries

Recall that we have established the following result.

Theorem 1.1 ([NS18], Theorem IV.4.4). The homotopy groups of $\mathrm{THH}(H\,\mathbb{F}_\mathrm{p})$ are given by

$$\pi_i(\mathrm{THH}(H\,\mathbb{F}_\mathrm{p})) = \begin{cases} \mathbb{F}_\mathrm{p}, & i = 2n \geqslant 0 \\ 0, & \text{otherwise} \end{cases}$$

which is the polynomial algebra generated by a single element u of degree 2 over $\mathbb{F}_\mathrm{p}$.

We will use this as an ingredient to compute $\mathrm{TC}(H\,\mathbb{F}_\mathrm{p})$. The following are the "key observations" we have seen which are necessary for our computation. Recall that we have defined cyclotomic spectra using Tate fixed points with respect to $\mathbb{T}$-actions. We will study a few more properties based on that.

Proposition 1.2 ([NS18], Proposition II.1.9). Let $(X, (\varphi_q)_{q\in\mathbb{P}})$ be a cyclotomic spectrum, then there is a functorial fiber sequence

$$\mathrm{TC}(X) \longrightarrow X^{h\mathbb{T}} \xrightarrow{(\varphi_q^{h\mathbb{T}} - \mathrm{can})_{q\in\mathbb{P}}} \prod_{q\in\mathbb{P}} (X^{tC_q})^{h\mathbb{T}}$$

where

$$\varphi_q^{h\mathbb{T}} : X^{h\mathbb{T}} \to (X^{tC_q})^{h\mathbb{T}},$$

and

$$\mathrm{can} : X^{h\mathbb{T}} \simeq (X^{hC_q})^{h(\mathbb{T}/C_q)} \simeq (X^{hC_q})^{h\mathbb{T}} \to (X^{tC_q})^{h\mathbb{T}}$$

where the middle equivalence comes from the qth power map $\mathbb{T}/C_q \simeq \mathbb{T}$.

Proof. Note that the fiber of a difference map is equivalent to the equalizer of the two maps. Using the equivalence of $\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{CycSp}, \mathrm{Sp}) = \mathrm{Fun}^{\mathrm{Lex}}(\mathrm{CycSp}, \mathcal{S})$, we can directly check this on the level of mapping spaces using a descrption of the lax equalizer. □

Remark. A lot of statements today will involve assuming a spectrum being bounded(-below). This is really a technical condition that gives us enough properties on the spectrum, for instance,

- we may apply Tate orbit lemma and Tate fixed point lemma under these conditions,
- a few arguments in [NS18] then uses the same technique under these conditions: assume X is bounded, then reduce to $X = HM$ as Eilenberg-Maclane spectrum, then studying the torsion.

Lemma 1.3 ([NS18], Lemma II.4.2). Suppose X is a bounded-below spectrum with $\mathbb{T}$-action, then $(X^{tC_p})^{h\mathbb{T}}$ is p-complete, and $X^{t\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$ is the p-completion of $X^{t\mathbb{T}}$.

Corollary 1.4. Let $A = H\mathbb{F}_p \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$ be a connective ring spectrum, then we note that $\mathrm{THH}(A)$ is a cyclotomic spectrum with underlying spectrum bounded-below. Moreover, there is a fiber sequence

$$\mathrm{TC}(A) \longrightarrow \mathrm{THH}(A)^{h\mathbb{T}} \xrightarrow{\varphi_p^{h\mathbb{T}} - \mathrm{can}} \mathrm{THH}(A)^{t\mathbb{T}}$$

Proof. Taking the fiber sequence in Proposition 1.2, it suffices to show that $\mathrm{THH}(A)^{tC_q} \simeq 0$ for $q \neq p$.[1] Once we know this, then

$$\begin{aligned}\prod_{q\in\mathbb{P}}(\mathrm{THH}(A)^{tC_q})^{h\mathbb{T}} &= (\mathrm{THH}(A)^{tC_p})^{h\mathbb{T}} \\ &= (\mathrm{THH}(A)^{t\mathbb{T}})^{\wedge}_p \text{ by Lemma 1.3} \\ &= \mathrm{THH}(A)^{t\mathbb{T}}\end{aligned}$$

where the last equality follows from the following observation. If a spectrum X is p-complete, then so is $X^{h\mathbb{T}}$ by closure of p-completeness under limits. Moreover, if X is bounded below, then $X^{h\mathbb{T}}$ is also p-complete: this can be proven by writing X as the limit of $\tau_{\leqslant n}X$ and therefore reducing to the case where X is concentrated in degree 0. As a direct consequence, $X^{t\mathbb{T}}$ is p-complete as well under these assumptions. This is the case for $X = A$.

Remark. The proof in [NS18] uses the fact that $\mathrm{TC}(A)$ is p-completed, which achieves the same goal: by taking p-completion on the fiber sequence, we get that all primes $q \neq p$ vanishes as units, therefore the only relevant information left is from the prime p.

Therefore we get the said fiber sequence. □

2 Computations

It is clear what we need to do from here: set $A = H\mathbb{F}_p$, and we will compute the homotopy groups of $\mathrm{THH}(A)^{h\mathbb{T}}$ and $\mathrm{THH}(A)^{t\mathbb{T}}$. Using the fiber sequence above, it then suffices to study the image of both maps occurring in the difference map.

Proposition 2.1 ([NS18], Proposition IV.4.6). We have

$$\pi_i \mathrm{THH}(A)^{h\mathbb{T}} = \begin{cases}\mathbb{Z}_p, & i = 2n \\ 0, & \text{otherwise}\end{cases}$$

In particular, for $i \geqslant 0$,

$$\pi_{2i} \mathrm{THH}(A)^{h\mathbb{T}} = \mathbb{Z}_p \cdot \tilde{u}^i, \quad \pi_{-2i} \mathrm{THH}(A)^{h\mathbb{T}} = \mathbb{Z}_p \cdot v^i$$

for some choices of elements $\tilde{u}, v$, such that $\tilde{u}v = p$.

Proof. We will study the convergent homotopy fixed-point spectral sequence

$$E_2^{ij} = H^i(B\mathbb{T}, \pi_{-j}\mathrm{THH}(A)) \Rightarrow \pi_{-i-j}\mathrm{THH}(A)^{h\mathbb{T}}$$

in cohomological grading, associated to

$$\mathrm{THH}(A) \to \mathrm{THH}(A)^{h\mathbb{T}} \to B\mathbb{T} \simeq \mathbb{C}P^\infty$$

By Theorem 1.1, the coefficient ring is non-trivial only when j is even, but also note that the cohomology of $B\mathbb{T} = \mathbb{C}P^\infty$ is given by a generator of even degrees, so the non-trivial things happen when i is even, therefore all

[1] This can be justified by this or [NS18, Lemma I.2.9]. Alternatively, see the remark below.

contributions are in even total degree $i+j$. This shows that the spectral sequence collapses at E_2-page. Therefore, we have

$$\pi_{-n}(\mathrm{THH}(A)^{h\mathbb{T}}) = \bigoplus_{i+j=n} H^i(B\mathbb{T}, \pi_{-j}\,\mathrm{THH}(A)). \tag{2.2}$$

In particular, the morphism $\pi_*\,\mathrm{THH}(A)^{h\mathbb{T}} \to \pi_*\,\mathrm{THH}(A)$ is surjective (taking $i=0$). We conclude that there exists

- some element $\tilde{u} \in \pi_2\,\mathrm{THH}(A)^{h\mathbb{T}}$ with image $u \in \pi_2\,\mathrm{THH}(A)$, and
- for the canonical map $\mathbb{C}P^1 \to B\mathbb{T} \simeq \mathbb{C}P^\infty$, we note that the first Chern class $c_1 \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ is the generator that got pulled back to the fundamental class of $\mathbb{C}P^1$. Since the spectral sequence degenerates, the summand of Equation (2.2) $H^2(B\mathbb{T};\pi_0\,\mathrm{THH}(A))$ lifts to some element $v \in \pi_{-2}\,\mathrm{THH}(A)^{h\mathbb{T}}$ with image the generator of $H^2(B\mathbb{T},\pi_0\,\mathrm{THH}(A)) = \mathbb{F}_p$ that pulls back to the orientation class of $\mathbb{C}P^1$.

We will use the following two facts.

- The homotopy fixed-point spectral sequence above is in fact multiplicative. For a proof of this, see for example, Theorem 6.1 of [Dug03].
- Lemma 2.3 below, whose proof is a standard calculation in Hochschild homology, which we omit.

Lemma 2.3 ([NS18], Lemma IV.4.7)**.** The image of $p \in \pi_0\,\mathrm{THH}(A)^{h\mathbb{T}}$ in $H^2(B\mathbb{T},\pi_2\,\mathrm{THH}(A))$ is given by uv.

Therefore, consider the case of Equation (2.2) where $n=0$, which is

$$\pi_0(\mathrm{THH}(A)^{h\mathbb{T}}) = \bigoplus_{i \text{ even}} H^i(B\mathbb{T}, \pi_i\,\mathrm{THH}(A)),$$

then this induces an abutment filtration[2] $\{F^p\pi_0(\mathrm{THH}(A)^{h\mathbb{T}})\}_{p\geqslant 0}$, which coincides with the Nygaard filtration or even filtration[3] in the case of the homotopy fixed-point spectral sequence.

Remark. The spectral sequence comes with a natural filtration on the abutment $\pi_0\big(\mathrm{THH}(A)^{h\mathbb{T}}\big)$ given by the cohomological degree. In particular, we define the p-th filtered piece as

$$F^p\pi_0\big(\mathrm{THH}(A)^{h\mathbb{T}}\big) = \mathrm{im}\left(\bigoplus_{s\geqslant 2p} E_\infty^{s,-s} \longrightarrow \pi_0\big(\mathrm{THH}(A)^{h\mathbb{T}}\big)\right).$$

Because the spectral sequence collapses, we can identify

$$F^p\pi_0\big(\mathrm{THH}(A)^{h\mathbb{T}}\big) = \bigoplus_{j\geqslant p} H^{2j}\big(B\mathbb{T};\pi_{2j}\,\mathrm{THH}(A)\big).$$

The associated graded pieces of the filtration are given by

$$\mathrm{Gr}^p\,\pi_0\big(\mathrm{THH}(A)^{h\mathbb{T}}\big) = \frac{F^p\pi_0\big(\mathrm{THH}(A)^{h\mathbb{T}}\big)}{F^{p+1}\pi_0\big(\mathrm{THH}(A)^{h\mathbb{T}}\big)}.$$

Since

$$F^p\pi_0\big(\mathrm{THH}(A)^{h\mathbb{T}}\big) = \bigoplus_{j\geqslant p} H^{2j}\big(B\mathbb{T};\pi_{2j}\,\mathrm{THH}(A)\big),$$

we have

$$\mathrm{Gr}^p\,\pi_0\big(\mathrm{THH}(A)^{h\mathbb{T}}\big) \cong H^{2p}\big(B\mathbb{T};\pi_{2p}\,\mathrm{THH}(A)\big).$$

[2]Apparently people just call this the induced filtration of spectral sequence, c.f., [012K] of Stacks project.
[3]For more information on Nygaard filtration, see Theorem 7.2 of [BMS19] or Section 6 of [Mat22].

This gives the following picture where the dotted lines denote the said filtration.

$$
\begin{array}{c|ccccccc|c}
 & & & & & & & & i \\
 & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & 4 \\
 & & & & & & & & 3 \\
 & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & 2 \\
 & & & & & & & & 1 \\
 & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & 0 \\
\hline
j & -6 & -5 & -4 & -3 & -2 & -1 & 0 &
\end{array}
$$

Let us look at the images of elements $p^i \in \pi_0 \operatorname{THH}(A)^{h\mathbb{T}}$ for $i \geqslant 1$. For $i = 1$, this is a direct application of Lemma 2.3: the image of p in the first step $H^2(B\mathbb{T}, \pi_2 \operatorname{THH}(A))$ of the abutment filtration, given by $uv \in \mathbb{F}_p$. But since this spectral sequence is multiplicative, we deduce that the image of p^i for any $i \geqslant 1$ is

$$u^i v^i \in H^{2i}(B\mathbb{T}, \pi_{2i} \operatorname{THH}(A)) = \mathbb{F}_p$$

in the ith step of the abutment filtration. By Equation (2.2), the image of each power of p generates each summand of $\pi_0(\operatorname{THH}(A)^{h\mathbb{T}})$, hence elements of it are just of the form $\sum\limits_{i\geqslant 0} a_i p^i$, hence

$$\pi_0 \operatorname{THH}(A)^{h\mathbb{T}} = \mathbb{Z}_p$$

is the p-adic integers. For positive-degree homotopy groups, the result follows by multiplying by powers of $\tilde{u}$; for negative-degree homotopy groups, the result follows by multiplying by powers of v. Up to multiplication by a unit, we get that $\tilde{u}v = p$. $\square$

Remark. This in particular shows that

$$\pi_* \operatorname{THH}(A)^{h\mathbb{T}} = \mathbb{Z}_p[\tilde{u}, v]/(\tilde{u}v - p).$$

Corollary 2.4 ([NS18], Corollary IV.4.8)**.**

$$\pi_i \operatorname{THH}(A)^{t\mathbb{T}} = \mathbb{Z}_p[v^{\pm 1}]$$

for $v \in \pi_{-2}(\operatorname{THH}(A)^{h\mathbb{T}})$ in Proposition 2.1. In particular, the canonical map can induces injections

$$\pi_i \operatorname{THH}(A)^{h\mathbb{T}} \simeq \mathbb{Z}_p \to \pi_i \operatorname{THH}(A)^{t\mathbb{T}} \simeq \mathbb{Z}_p$$

for all even $i \in \mathbb{Z}$. If $i \leqslant 0$, it is an isomorphism, while if $i = 2j \geqslant 0$, it has image $p^j\mathbb{Z}_p$.

Proof. We will mimic the same argument in Proposition 2.1, and also proving things degreewise. We take the Tate spectral sequence

$$E_2^{ij} = \pi_{-i}(H\pi_{-j} \operatorname{THH}(A))^{t\mathbb{T}} \Rightarrow \pi_{-i-j} \operatorname{THH}(A)^{t\mathbb{T}}$$

which is deduced from the usual Tate spectral sequence

$$\hat{H}^i(\mathbb{T}, \pi_{-j} \operatorname{THH}(A)) \Rightarrow \pi_{-i-j} \operatorname{THH}(A)^{t\mathbb{T}}$$

and the fact that for any G-module M,

$$\pi_*(HM^{tG}) \cong \hat{H}^{-*}(G; M).$$

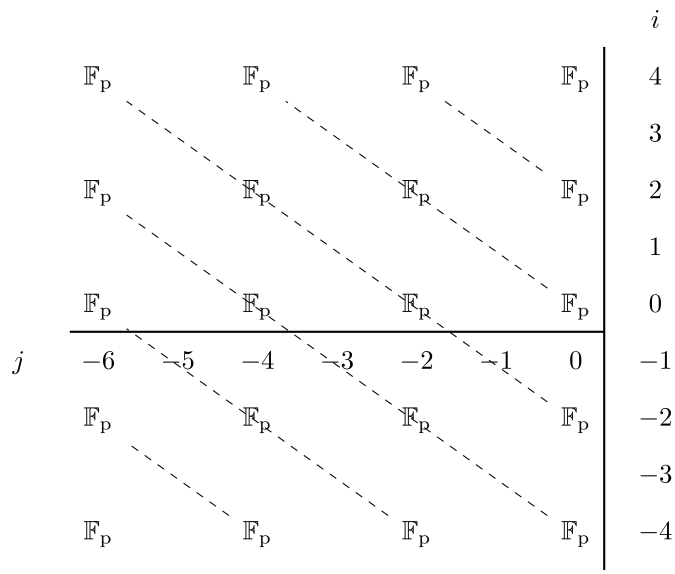
Again, this spectral sequence is multiplicative and concentrated in even total degree. Again, the spectral sequence collapses at E_2-page, therefore we have

$$\pi_{-n}\,\mathrm{THH}(A)^{t\mathbb{T}} \cong \bigoplus_{n=i+j} \pi_{-i}(H\pi_{-j}\,\mathrm{THH}(A))^{t\mathbb{T}}.$$

Since the $\mathbb{T}$-action is trivial on $\mathbb{F}_p$, we note that

$$\pi_{-i}(H\pi_{-j}\,\mathrm{THH}(A))^{t\mathbb{T}} = \begin{cases} \mathbb{F}_p, & i \text{ even and } j \leqslant 0 \text{ even} \\ 0, & \text{else} \end{cases}$$

for any $i+j=n$. Therefore, we see that the canonical map $\mathrm{THH}(A)^{h\mathbb{T}} \to \mathrm{THH}(A)^{t\mathbb{T}}$ induces an inclusion of spectral sequences, and therefore we have an inclusion of filtration diagrams.



The abutment above showcases the injection in all degrees. Again, if $i+j=n\geqslant 0$, then the homotopy groups are given by the argument of Proposition 2.1 directly, and in these cases the induced map is an isomorphism by the inclusion. In the case where $i+j=n<0$, then we use the same multiplicity argument as in Proposition 2.1 where we multiply everything by v. $\square$

Proposition 2.5 ([NS18], Proposition IV.4.9)**.** Let

$$\varphi_p^{h\mathbb{T}} : \mathrm{THH}(A)^{h\mathbb{T}} \to \mathrm{THH}(A)^{t\mathbb{T}}$$

be the Frobenius map. For all even $i \in \mathbb{Z}$, the map

$$\pi_i\varphi_p^{h\mathbb{T}} : \pi_i\,\mathrm{THH}(A)^{h\mathbb{T}} \simeq \mathbb{Z}_p \to \pi_i\,\mathrm{THH}(A)^{t\mathbb{T}} \simeq \mathbb{Z}_p$$

is injective. If $i \geqslant 0$, this is an isomorphism, while if $i=-2j\leqslant 0$, the image is given by $p^j\mathbb{Z}_p$.

Proof. We note that the Frobenius map $\varphi_p : \mathrm{THH}(A)^{h\mathbb{T}} \to \mathrm{THH}(A)^{t\mathbb{T}}$ must be determined by

$$\tilde{u} \mapsto av^{-1}, \quad v \mapsto bv$$

for some $a, b \in \mathbb{Z}_p$, so by multiplicity of the map, we get that the Frobenius map assigns

$$p = \tilde{u}v \mapsto ab = av^{-1}bv,$$

therefore $ab = p$. This shows that the map must be injective degreewise. Moreover, since p is prime in $\mathbb{Z}_p$, either a is a unit or b is a unit, so the maps are isomorphisms either in positive or in negative degrees. Suppose, towards contradiction, that they are isomorphisms in negative degrees. By multiplicity of the map, it suffices to check on the level of π_{-2}. Note that there is a natural map $(-)^{h\mathbb{T}} \to \mathrm{id}$, which induces a morphism between two sequences given by the commutative diagram

$$\begin{tikzcd} \pi_{-2}\operatorname{THH}(A)^{h\mathbb{T}} \arrow[r, "\pi_{-2}\varphi_p^{h\mathbb{T}}", "\simeq"'] \arrow[d] & \pi_{-2}\operatorname{THH}(A)^{t\mathbb{T}} \arrow[r, "\pi_{-2}\pi^{t\mathbb{T}}"] \arrow[d] & \pi_{-2}A^{t\mathbb{T}} \arrow[d] \\ \pi_{-2}\operatorname{THH}(A) \simeq 0 \arrow[r, "\pi_{-2}\varphi_p"'] & \pi_{-2}\operatorname{THH}(A)^{tC_p} \arrow[r, "\pi_{-2}\pi^{tC_p}"'] & \pi_{-2}A^{tC_p} \end{tikzcd}$$

where each row is given by the universal property of THH and Corollary IV.2.4 of [NS18], where $\pi : \operatorname{THH}(A) \to A$ is a retract of $A \to \operatorname{THH}(A)$. We conclude that

$$\pi_{-2}\operatorname{THH}(A)^{t\mathbb{T}} \simeq \mathbb{Z}_p \cdot v \to \pi_{-2}A^{tC_p} \simeq \mathbb{F}_p$$

is the zero map. Taking the clockwise direction of the right square, we note that v got sent to a non-zero class in $\pi_{-2}A^{t\mathbb{T}}$ by the retraction. The vertical map really is $\mathbb{F}_p \to \mathbb{F}_p$, so it is either the zero map or an isomorphism. However, this map is really a comparison of cohomology of degree 2, which is given by an isomorphism of cohomologies with coefficients in $\mathbb{Z}$. This map is then generated by Chern class on the line bundle, therefore it must be non-trivial hence an isomorphism, and we recover the result with $\mathbb{F}_p$-coefficient. This is a contradiction. Therefore, the maps are isomorphisms in the positive degrees. (In this case, a is a unit in $\mathbb{Z}_p$.) The description of the injection is again given by the usual argument. □

Corollary 2.6 ([NS18], Corollary IV.4.10). We have

$$\pi_i \operatorname{TC}(A) = \begin{cases} \mathbb{Z}_p, & i = 0, -1 \\ 0, & \text{otherwise} \end{cases}$$

Proof. From Proposition 2.1, we know the second and third term has only even homotopy groups, so for each n we get an exact sequence

$$0 \longrightarrow \pi_{2n}(\operatorname{TC}(A)) \longrightarrow \pi_{2n}(\operatorname{THH}(A)^{h\mathbb{T}}) \xrightarrow{\operatorname{can} - \varphi_p^{h\mathbb{T}}} \pi_{2n}(\operatorname{THH}(A)^{t\mathbb{T}}) \longrightarrow \pi_{2n-1}(\operatorname{TC}(A)) \longrightarrow 0$$

Comparing Corollary 2.4 and Proposition 2.5, we know $\operatorname{can} - \varphi_p^{h\mathbb{T}}$ is the difference of an isomorphism and a map divisible by p between two copies of $\mathbb{Z}_p$. In particular, if $2|j| \neq 0$, the map on $\pi_{2j}(-)$ looks like some scalar multiple of $\mathrm{id} - p^{|j|}$, but multiplication by $p^{|j|}$ is already invertible in the p-complete case since p is invertible in $\mathbb{Z}_p$, therefore the difference is a non-trivial isomorphism and thereby inducing a levelwise isomorphism on homotopy groups. Moreover, for $n = 0$, $\varphi_p^{h\mathbb{T}}$ is the identity map as $\mathbb{Z}_p$-algebras, therefore the middle map is the zero map.

In particular, for $n = 0$ we have

$$0 \longrightarrow \pi_0(\operatorname{TC}(A)) \longrightarrow \pi_0(\operatorname{THH}(A)^{h\mathbb{T}}) \xrightarrow{0} \pi_0(\operatorname{THH}(A)^{t\mathbb{T}}) \longrightarrow \pi_{-1}(\operatorname{TC}(A)) \longrightarrow 0$$

which forces $\pi_{-1}(\operatorname{TC}(A)) \cong \pi_0(\operatorname{THH}(A)^{t\mathbb{T}}) \cong \mathbb{Z}_p$ and $\pi_0(\operatorname{TC}(A)) \cong \pi_0(\operatorname{THH}(A)^{h\mathbb{T}}) \cong \mathbb{Z}_p$. For $n \neq 0$, we have

$$0 \longrightarrow \pi_{2n}(\operatorname{TC}(A)) \longrightarrow \mathbb{Z}_p \xrightarrow{\cong} \mathbb{Z}_p \longrightarrow \pi_{2n-1}(\operatorname{TC}(A)) \longrightarrow 0$$

and therefore $\pi_{2n}(\operatorname{TC}(A)) = \pi_{2n-1}(\operatorname{TC}(A)) = 0$. We conclude with what we need. □

The calculation on the level of spectrum was done in the language of Witt vectors by [HM97, Theorem B], which we omit.

Theorem 2.7. Let k be a perfect field over $\mathbb{F}_p$, then

$$\operatorname{TC}(k) = H\mathbb{Z}_p \vee \Sigma^{-1}H(\operatorname{coker}(\mathrm{id} - \varphi_p)).$$

In particular, for finite field k, $\operatorname{coker}(\mathrm{id} - \varphi_p) = \mathbb{Z}_p$, therefore

$$\operatorname{TC}(\mathbb{F}_p) = H\mathbb{Z}_p \vee \Sigma^{-1}H\mathbb{Z}_p.$$

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