CLASSIFICATION OF ROOT SYSTEMS

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ABSTRACT. In this report, we will focus on roots system and study its relation with (split) semisimple Lie algebras, and using concepts like Dynkin diagrams to find connections and correspondences between them. In the first section we will look into the background and define the roots of Lie algebras. We will then define root system and Dynkin diagram and study their properties in the second section. Finally, we will look into the connections between the three languages: (simple) Lie algebras, (irreducible) root systems, and (connected) Dynkin diagrams. This will allow us to classify Dynkin diagrams, and use it as a tool to classify the root systems and corresponding semisimple Lie algebras.

1. BACKGROUND

In this section, we would follow [1] and give a brief overview of the k-vector space structure we care about for a field k, its relation to Lie algebra, and the role of roots in its structure.

Definition 1.1 (Cartan Subalgebra). Let $\mathfrak{g}$ be a semisimple Lie algebra, then we say $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Cartan subalgebra* if it is self-normalizing, i.e., if for all $X \in \mathfrak{h}$ we have $Y \in \mathfrak{g}$ satisfying $[X, Y] \in \mathfrak{h}$, then $Y \in \mathfrak{h}$.

Definition 1.2 (Split). A semisimple Lie algebra $\mathfrak{g}$ is split if the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is *split*, i.e., for all $H \in \mathfrak{h}$, $\mathrm{ad}_{\mathfrak{g}}(H)$ is triangularizable (as a matrix).

Notation 1.3. (a) Throughout the report, a Lie algebra is assumed to be split.
(b) A split semisimple Lie algebra is a pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a semisimple Lie algebra with $\mathfrak{h}$ is a split Cartan subalgebra of $\mathfrak{g}$.

Remark 1.4. (a) If $\mathfrak{g}$ is semisimple and $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, then $\mathrm{ad}_{\mathfrak{g}}(x)$ is semisimple for all $x \in \mathfrak{h}$. In particular, if $\mathfrak{g}$ is split, then $\mathrm{ad}_{\mathfrak{g}}(x)$ is diagonalizable for all $x \in \mathfrak{h}$.
(b) If k is algebraically closed, then every semisimple Lie algebra is split.
(c) Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be splitting Cartan subalgebras of $\mathfrak{g}$, then there exists an automorphism on $\mathfrak{g}$ that sends $\mathfrak{h}_1$ to $\mathfrak{h}_2$.

Let $(\mathfrak{g}, \mathfrak{h})$ be a split semisimple Lie algebra. The action of $\mathfrak{h}$ on any representation of $\mathfrak{g}$ is diagonalizable, so by the adjoint representation theorem, we obtain a Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where the action of $\mathfrak{h}$ preserves each $\mathfrak{g}_\alpha$ and acts on it by scalar multiplication by the linear functional $\alpha \in \mathfrak{h}^*$. That is, for any $h \in \mathfrak{h}$ and any $x \in \mathfrak{g}_\alpha$, we have

$$\mathrm{ad}_h(x) := [h, x] = \alpha(h) \cdot x.$$

In this sense, each α acts as an eigenvalues. Note that $\mathfrak{h} = \mathfrak{g}_0$, and therefore $\mathfrak{g}$ is a direct sum of $\mathfrak{g}_\alpha$'s.

Definition 1.5 (Root). Let $(\mathfrak{g}, \mathfrak{h})$ be a split semisimple Lie algebra, and consider the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

A *root* of $(\mathfrak{g}, \mathfrak{h})$ is a linear form $\alpha \in R$ on $\mathfrak{h}$, then $R := R(\mathfrak{g}, \mathfrak{h})$ is the set of roots of $(\mathfrak{g}, \mathfrak{h})$.

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Definition 1.6 (Killing Form)**.** The *Killing form* on $\mathfrak{g}$ is defined, for any two elements on $\mathfrak{g}$, to be the trace of the composition of their adjoint actions on $\mathfrak{g}$.

Proposition 1.7. *Let α, β be roots of $(\mathfrak{g}, \mathfrak{h})$ and let $B(\cdot, \cdot)$ be the Killing form of $\mathfrak{g}$ (or, in general, any non-degenerated invariant symmetric bilinear form on $\mathfrak{g}$). For $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{h}$, then*

(a) $[x, y] \in \mathfrak{h}$, *and*
(b) $B(h, [x, y]) = \alpha(h)B(x, y)$.

Proof. Under the assumption, $[x, y] \in \mathfrak{g}_{\alpha-\alpha} = \mathfrak{g}_0 = \mathfrak{h}$, and $B(h, [x, y]) = B([h, x], y) = B(\alpha(h)x, y) = \alpha(h)B(x, y)$. □

Theorem 1.8. *Let α be a root of $(\mathfrak{g}, \mathfrak{h})$.*

(a) *$\mathfrak{g}_\alpha$ has dimension 1 as a vector space.*
(b) *The vector subspace $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{h}$ has dimension 1. Moreover, there exists a unique element $H_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$.*

Proof.
(a) Let $h_\alpha \in \mathfrak{h}$ be the unique element such that $\alpha(h) \in B(h_\alpha, h)$ for all $h \in \mathfrak{h}$. By Proposition 1.7, $[x, y] = B(x, y)h_\alpha$ for all $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, and also note that $B(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$, therefore $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = kh_\alpha$.
(b) Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ be such that $B(x, y) = 1$. Therefore, $[x, y] = h_\alpha$. Note that $[h_\alpha, x] = \alpha(h_\alpha)x$ and $[h_\alpha, y] = -\alpha(h_\alpha)y$. One can show that $\alpha(h_\alpha) \neq 0$, then there exists a unique element $H_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$.

□

Remark 1.9. Therefore, H_α is the unique element with eigenvalue 2 on $\mathfrak{g}_\alpha$ and -2 on $\mathfrak{g}_{-\alpha}$. In fact the eigenvalues of H_α are always integers, and are symmetric about the origin in $\mathbb{Z}$.

To formalize this idea using familiar geometric notions in the Euclidean space, we define the following concept.

Definition 1.10 (Reflection)**.** Let V be a finite-dimensional k-vector space. An endomorphism $s \in \operatorname{End}(V)$ is a *reflection* with respect to $0 \neq \alpha \in V$ if

(a) $s(\alpha) = -\alpha$, and
(b) there exists a hyperplane $W \subseteq V$ such that $s|_W = \operatorname{id}$.

Under this new language, let $\Omega_\alpha = \{\beta \in \mathfrak{h}^* : B(H_\alpha, \beta) = 0\}$ be a hyperplane, then we define W_α to be the reflection in the plane Ω_α with respect to the line spanned by α. More explicitly, we have

$$W_\alpha(\beta) = \beta - \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}\alpha = \beta - \beta(H_\alpha)\alpha.$$

Define $T_\beta \in \mathfrak{h}$ to be the element such that $B(T_\beta, H) = \beta(H)$ for all $H \in \mathfrak{h}$. This induces a Killing form on $\mathfrak{h}^*$ by $(\beta, \alpha) := B(T_\beta, T_\alpha)$. With this Killing form, we reinterpret the property of reflection as

$$W_\alpha(\beta) = \beta - \frac{2\,(\beta, \alpha)}{(\alpha, \alpha)}\alpha,$$

i.e., an orthogonality condition.

Remark 1.11. Comparing the above two equations regarding the reflection, we deduce that

$$\beta(H_\alpha) = \frac{2\,(\beta, \alpha)}{(\alpha, \alpha)}.$$

Moreover, because the eigenvalues of H_α are always integers, it is easy to show that $\beta(H_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer.

Fact 1.12. *The set of roots generate a lattice $\Lambda \subseteq \mathfrak{h}^*$ of rank $\dim(\mathfrak{h})$. In particular, the roots of $\mathfrak{g}$ span $\mathfrak{h}^*$ as a real subspace, on which the Killing form $\langle \cdot, \cdot \rangle := \beta(H_\alpha)$ is positive definite.*

Therefore, this determines $\mathfrak{h}^*$ to be a finite-dimensional Euclidean space $(V, \langle\ ,\ \rangle)$. We will now study the roots and root systems in this new language.

2. Root Systems and Dynkin Diagrams

Following the notations above, we will study the Lie algebra $\mathfrak{h}^*$ as a Euclidean space (e.g., $\mathbb{R}$ or $\mathbb{C}$ with the usual inner product space notion) with a Killing form $\langle \beta, \alpha \rangle := \beta(H_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$, i.e., $(V, \langle \cdot, \cdot \rangle)$. Therefore, the reflection operation would now be in the usual sense.

2.1. Root Systems.

Definition 2.1 (Root System)**.** A subset Φ of V is called a *(reduced) root system* if

(a) Φ is a finite subset of V that does not contain $0 \in V$, and spans V.
(b) For $\alpha \in \Phi$, we have $n \cdot \alpha \in \Phi$ if and only if $n = \pm 1$.
(c) For each $\alpha \in \Phi$, the reflection W_α in the hyperplane $\alpha^\perp$ maps Φ to itself. Therefore, $W_\alpha(\Phi) \subseteq \Phi$.
(d) For $\alpha, \beta \in \Phi$, $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer. In particular, the term $W_\alpha(\beta) - \beta$ is an integral multiple of α.

Remark 2.2. In particular, the set of roots of V (i.e., the eigenvalues of $\mathfrak{h}^*$) is a root system. More generally, Φ is a root system in $\sum_{\alpha \in \Phi} \mathbb{R}\alpha \subseteq \mathfrak{h}^*$.

The following Lemma 2.3 shows that the reflection W_α specified in Definition 2.1 is uniquely determined by α.

Lemma 2.3. *Let Φ be a finite generating subset of V. For any $0 \neq \alpha \in \Phi$, there exists at most one reflection s (c.f., Definition 1.10) that preserves Φ, i.e., $s(\alpha) = -\alpha$, and $s(\Phi) = \Phi$.*

Proof. Fix $0 \neq \alpha \in \Phi$. Let $G \subseteq \mathrm{Aut}(V)$ that makes Φ stable, i.e., for arbitrary $\varphi \in G$, $\varphi(\Phi) \subseteq \Phi$. Since Φ generates V, then we know $G \cong \mathcal{S}$ can be viewed as a subgroup of the symmetric group of Φ. Let s, s' be reflections of V such that $s(\alpha) = s'(\alpha) = -\alpha$ and that $s(\Phi) = s'(\Phi) = \Phi$. Therefore, $t = s's \in G$ and therefore has finite order. In particular, t fixes α and satisfies $t(x) \equiv x \pmod{k\alpha}$ for all $x \in V$. Therefore, we can express t via a linear form f on V such that $t(x) = x + f(x)\alpha$ for all $x \in V$ and $f(\alpha) = 0$. By induction on composition of t, we see $t^n(x) = x + nf(x)\alpha$ for all $x \in V$ for linear form $f(\alpha) = 0$. In particular, let n be the order of t in G, t^n becomes the identity map and therefore $nf(x)\alpha = 0$ for all $x \in V$, and so f has to be the zero map, i.e., $f \equiv 0$. However, taking this back to the equation, we see $t(x) = x$ for all $x \in V$ and therefore t is the identity map. In particular, we composed two reflection maps and obtain the identity map, which means the two reflection maps are the same by checking it elementwise. Therefore, $s = s'$, which shows the uniqueness of such reflection. $\square$

Notation 2.4. Throughout this report, a root system is always reduced. We denote a root system to be (V, Φ).

Definition 2.5 (Dual Root System)**.** Given a root system (V, Φ), and let $\alpha \in \Phi$ be a root. Then a *coroot* α^* of α is $\alpha^* = \frac{2}{(\alpha,\alpha)}\alpha$. This gives a corresponding subset $\Phi^* \subseteq V^*$ in the dual vector space that is also a root system. In particular, we call it the *dual root system* (V^*, Φ^*), with its elements as the coroots of $(\mathfrak{g}, \mathfrak{h})$.

Remark 2.6. (a) If we identify the Cartan subalgebra $\mathfrak{h} \cong \mathfrak{h}^{**}$, then by Lemma 2.3 we know $H_\alpha = \alpha^*$ for all $\alpha \in R$, and this identifies the coroots as well.
(b) By Lemma 2.3, the reflection $W_\alpha := W_{\alpha,\alpha^*}$ is uniquely determined by α, so $W_\alpha(x) = x - \langle \alpha^*, v \rangle \cdot \alpha$ for all $x \in V$.

Remark 2.7. In Definition 2.1, property (b) is often called *reduced*; property (d) is what determines the geometry of set of roots in V.

Example 2.8 ([3])**.** Suppose we are working over $\mathbb{C}$ as a $\mathbb{R}$-vector space. Let θ be the angle between α and β as vectors, then

$$\langle \beta, \alpha \rangle = 2\cos(\theta)\frac{||\beta||}{||\alpha||}.$$

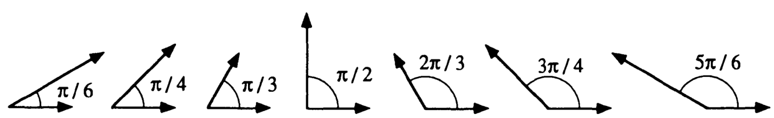
Therefore,

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\cos^2(\theta)$$

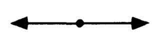
is an integer satisfying $0 \leq \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \leq 4$. Without loss of generality, say $||\beta|| \geq ||\alpha||$, i.e., $|\langle \beta, \alpha \rangle| \geq |\langle \alpha, \beta \rangle|$, then we have the following table. In particular, the case when $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4$ is trivial, since this implies $\beta = \pm\alpha$, i.e., they are colinear.

$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\langle \beta, \alpha \rangle$	4	3	2	1	0	-1	-2	-3	-4
$\langle \alpha, \beta \rangle$	1	1	1	1	0	-1	-1	-1	-1
$\frac{\lVert\beta\rVert}{\lVert\alpha\rVert}$	2	$\sqrt{3}$	$\sqrt{2}$	1		1	$\sqrt{2}$	$\sqrt{3}$	2

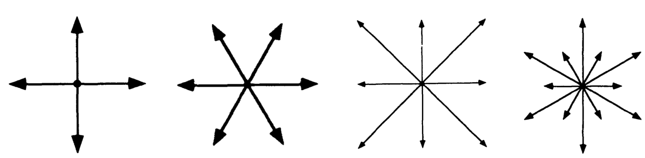
Therefore this gives a geometric understanding of the possible relations between two roots. For instance, following figure from [3] illustrates the non-trivial cases.



Example 2.9. The rank of the root system is the rank of the corresponding Lie algebra. Therefore, it would be easy to classify root systems with low rank. Moreover, by property (c) of Definition 2.1, the angle between two roots must be the same for any pair of adjacent roots in a 2-dimensional root system. Therefore, the root systems of rank 1 would have to be the following, denoted by A_1:



The root systems of rank 2 are $A_1 \times A_1$, A_2, B_2, and G_2, and are generated by angles of $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$, respectively.



However, the task of classifying root systems of higher ranks would be much more difficult.

The following proposition shows that we can construct a substructure out of a root system in a simple way.

Proposition 2.10 ([1], Section VI.1.2, Proposition 4)**.** *Let $X \subseteq \Phi$ and let V_X be the vector subspace of V generated by X, then $\Phi \cap V_X$ is a root system in V_X, and the canonical bijection from $\Phi \cap V_X$ to the dual root system is identified with the restriction of the map $\alpha \mapsto \alpha^*$ to $\Phi \cap V_X$.*

Proof. This is obvious by identifying the relation between coroots and roots. $\square$

Corollary 2.11. *Let (V, Φ) be a root system and let V' be a subspace V. Suppose W is the vector subspace of V' generated by $\Phi \cap V'$, then $(W, \Phi \cap V')$ is a root system.*

Proof. Take $X = \Phi \cap V'$. □

Definition 2.12 (Weyl Group)**.** A morphism between root systems $(V, \Phi) \to (V', \Phi')$ is just a map of vector space $V \to V'$ that maps $\Phi \to \Phi'$. Let $\mathrm{Aut}(V, \Phi)$ be the set of automorphisms on the root system (V, Φ), i.e., an isomorphism $f : V \to V$ on the vector space V such that $f(\Phi) = \Phi$. The *Weyl group* of Φ is the subgroup $W(\Phi) \subseteq \mathrm{Aut}(V, \Phi)$ consisting of reflections W_α for $\alpha \in \Phi$.

Proposition 2.13. *The action of $W(\Phi)$ preserves Φ, i.e., if $\alpha \in \Phi$, then $w \cdot \alpha \in \Phi$ for all $w \in W(\Phi)$.*

Proof. For any $\alpha \in \Phi$, the invertible operator W_α on $\mathfrak{g}$ gives

$$W_\alpha = e^{\mathrm{ad}_{\mathrm{X}_\alpha}} e^{-\mathrm{ad}_{\mathrm{Y}_\alpha}} e^{\mathrm{ad}_{\mathrm{X}_\alpha}}.$$

Suppose $H \in \mathfrak{h}$ satisfies $\langle \alpha, H \rangle = 0$, then $[H, X_\alpha] = \langle \alpha, H \rangle X_\alpha = 0$, so H and X_α commutes, hence ad_H and ad_{X_α} also commute, and similarly ad_H and ad_{Y_α} also commutes. Therefore, given that $\langle \alpha, H \rangle = 0$, W_α would commute with ad_H, thus $W_\alpha \,\mathrm{ad}_H\, W_\alpha^{-1} = \mathrm{ad}_H$ with $\langle \alpha, H \rangle = 0$. Moreover, by the adjoint action on $\mathfrak{g}$, we have $W_\alpha \,\mathrm{ad}_{H_\alpha}\, W_\alpha^{-1} = -\mathrm{ad}_{H_\alpha}$, then for all $H \in \mathfrak{h}$, there is $W_\alpha \,\mathrm{ad}_H\, W_\alpha^{-1} = \mathrm{W}_\alpha \cdot \mathrm{H}$. If β is any root and X is associated to β, then $W_\alpha^{-1}(X) \in \mathfrak{g}$ satisfies

$$\begin{aligned}
\mathrm{ad}_H(W_\alpha^{-1}(X)) &= W_\alpha^{-1}(W_\alpha \,\mathrm{ad}_H\, W_\alpha^{-1})(X) \\
&= W_\alpha^{-1} \,\mathrm{ad}_{W_\alpha \cdot H}(X) \\
&= \langle \beta, W_\alpha \cdot H \rangle \, W_\alpha^{-1}(X) \\
&= \left\langle W_\alpha^{-1}\beta, H \right\rangle W_\alpha^{-1}(X).
\end{aligned}$$

This shows that the set of roots is invariant under each reflection W_α, and therefore it is invariant under W. □

Corollary 2.14. *The Weyl group $W(\Phi)$ is finite.*

Proof. Since the roots of $\mathfrak{g}$ span $\mathfrak{h}$, then each $w \in W(\Phi)$ is determined by its action on Φ. Also, note that w sends Φ onto Φ, then W can be embedded as a subgroup $W \hookleftarrow S_n$ as a permutation group on the roots Φ. □

Remark 2.15 ([5], p.51)**.** Any reflection of the Weyl group $W(\Phi)$ sends a system of simple roots to another system of simple roots. In particular, the Weyl group $W(\Phi)$ acts simply transitively on the set of systems of Φ.

Fact 2.16 ([1], Section VII.3.2, Theorem 1)**.** *If k is algebraically closed, there exists a normal subgroup of* $\mathrm{Lie}(\mathfrak{g})$ *that acts transitively on the set of Cartan subalgebras of $\mathfrak{g}$.*

Proposition 2.17. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h}_1, \mathfrak{h}_2$ be splitting Cartan subalgebras of $\mathfrak{g}$. Then there exists an isomorphism from $\mathfrak{h}_1^*$ to $\mathfrak{h}_2^*$ that sends $R(\mathfrak{g}, \mathfrak{h}_1)$ to $R(\mathfrak{g}, \mathfrak{h}_2)$.*

Proof. Let $\bar{k}$ be the algebraic closure of k. For $(\mathfrak{g}, \mathfrak{h}_i)$ for $i = 1, 2$, consider a split semisimple Lie algebra $(\mathfrak{g}', \mathfrak{h}'_i)$ on $\bar{k}$, defined by $\mathfrak{g}' = \mathfrak{g} \otimes_k \bar{k}$ and $\mathfrak{h}'_i = \mathfrak{h}_i \otimes_k \bar{k}$. Therefore, the root system $R(\mathfrak{g}', \mathfrak{h}'_i)$ is the image of $R(\mathfrak{g}, \mathfrak{h}_i)$ under the map

$$\begin{aligned}
\mathfrak{h}_i^* &\to \mathfrak{h}_i^* \otimes_k \bar{k} \cong \mathfrak{h}_i'^* \\
\lambda &\mapsto \lambda \otimes 1
\end{aligned}$$

By Fact 2.16, there exists an automorphism of $\mathfrak{g}'$ that maps $\mathfrak{h}'_1$ to $\mathfrak{h}'_2$, which induces an isomorphism

$$\begin{aligned}
\varphi : \mathfrak{h}_1'^* &\xrightarrow{\sim} \mathfrak{h}_2'^* \\
R(\mathfrak{g}', \mathfrak{h}'_1) &\mapsto R(\mathfrak{g}', \mathfrak{h}'_2)
\end{aligned}$$

Therefore, the restriction $\varphi|_{\mathfrak{h}_1^*}$ maps $R(\mathfrak{g}, \mathfrak{h}_1)$ to $R(\mathfrak{g}, \mathfrak{h}_2)$, hence maps $\mathfrak{h}_1^*$ to $\mathfrak{h}_2^*$. □

Remark 2.18. The root system of $(\mathfrak{g}, \mathfrak{h})$ depends, up to isomorphism, on $\mathfrak{g}$ but not on $\mathfrak{h}$. In this sense, many constructions in this report are unique.

Definition 2.19 (Irreducible)**.** Let (V, Φ) be a root system. We say Φ is *irreducible* if we cannot write (V, Φ) as $V = V_1 \oplus V_2$ and $\Phi = \Phi_1 \cup \Phi_2$ for any root systems (V_1, Φ_1) and (V_2, Φ_2).

Lemma 2.20. *Let (V, Φ) be a root system. If V is a k-vector space that can be written as a direct sum of vector spaces $V_1, \dots, V_r$, and define $\Phi_i = \Phi \cap V_i$, then the following are equivalent:*

(1) The V_i's are stable under $W(\Phi)$.
(2) $\Phi \subseteq V_1 \cup \cdots \cup V_r$.
(3) For all i, Φ_i is a root system in V_i, and $\Phi = \Phi_1 \oplus \cdots \oplus \Phi_r$.

Proof. (1) $\Rightarrow$ (2): Suppose V_i's are stable under $W(\Phi)$. Let $\alpha \in \Phi$ and let H be the kernel of α^*, then each V_i would be a direct sum of a subspace of H and a subspace of $k\alpha$. Therefore, one of the V_i's must contain $k\alpha$, hence $\alpha \in V_1 \cup \cdots \cup V_r$, therefore $\Phi \subseteq V_1 \cup \cdots \cup V_r$ as desired.

(2) $\Rightarrow$ (3): Suppose $\Phi \subseteq V_1 \cup \cdots \cup V_r$, then Φ_i generates V_i for all $i = 1, \dots, r$, so by Corollary 2.11 we know (V_i, Φ_i) is a root system for all $i = 1, \dots, r$. In particular, we have $V = \bigoplus_{i=1}^{r} V_i$ and $\Phi = \bigcup_{i=1}^{r} \Phi_i$.

(3) $\Rightarrow$ (1): Let $\alpha \in \Phi_i$. Fix $j \neq i$, then suppose the kernel of α^* contains V_j, therefore the reflection W_α now induces an identity map on V_j. Moreover, $k\alpha \subseteq V_i$, therefore V_i is stable with respect to reflection W_α. In particular, this shows that $W(\Phi) \cong \prod_{i=1}^{r} W(\Phi_i)$. ☐

Therefore, the study of root systems comes down to the study of irreducible root systems through decomposition of direct sum.

Definition 2.21 (Simple Roots)**.** A subset $\mathcal{B} \subseteq \Phi$ is a *system of simple roots* if any $\alpha \in \Phi$ can be expressed as a linear combination

$$\alpha = \sum_{\beta \in \mathcal{B}} n_\beta \cdot \beta$$

for some uniquely determined $n_\beta \in \mathbb{Z}$, such that either either all $n_\beta \geq 0$ or all $n_\beta \leq 0$. In particular, $\mathcal{B}$ becomes a *basis* for V.

Definition 2.22 (Positive/Negative Roots)**.** Consider the Euclidean space $(V, \langle \cdot, \cdot \rangle)$ with a root system (V, Φ). Since Φ is finite, then there exists $v \in V$ such that $\langle v, \alpha \rangle \neq 0$ for all $\alpha \in \Phi$. According to this value, $\Phi^+ = \{\alpha \in \Phi : \langle v, \alpha \rangle > 0\}$ is called the set of *positive roots*, and $\Phi^- = \{\alpha \in \Phi : \langle v, \alpha \rangle < 0\}$ is called the set of *negative roots*.

Remark 2.23.
- $\Phi^- = -\Phi^+$.
- Let $\mathfrak{g}_+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ and $\mathfrak{g}_- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$, then $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are nilpotent subalgebras of $\mathfrak{g}$. Moreover, $\mathfrak{b} = \mathfrak{g}_+ \oplus \mathfrak{h}$ is a solvable subalgebra of $\mathfrak{g}$, such that $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{g}_+$.
- A system of simple roots $\mathcal{B}$ can be regarded as a subset of Φ^+ without loss of generality. Moreover, a simple root would be one that cannot be written as a sum of to positive roots.

Theorem 2.24. *Let (V, Φ) be a root system and let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be a system of simple roots of Φ. Then*

(1) for any $\alpha_i \neq \alpha_j \in \mathcal{B}$, $\langle \alpha, \beta \rangle \leq 0$.
(2) $\Phi^+ \subseteq \mathbb{Z}_{\geq 0}\alpha_1 + \cdots + \mathbb{Z}_{\geq 0}\alpha_n$.
(3) $\mathcal{B}$ is a basis of V. In particular, every root system has a basis.
(4) for any $\alpha \in \mathcal{B}$, $W_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$.
(5) if $v' \in V$ is another vector such that $\langle v, \alpha \rangle \neq 0$ for any $\alpha \in \Phi$ and $\mathcal{B}'$ is the associated system of simple roots, then there exists $s \in W(\Phi)$ such that $s(\mathcal{B}) = \mathcal{B}'$. Moreover, this element $s \in W(\Phi)$ is unique.

Proof. (1) Assume $\alpha, \beta \in \Phi^+$, where $\alpha \neq \pm\beta$ and $\langle \alpha, \beta \rangle \geq 0$, then $0 \leq \langle \alpha, \beta \rangle \langle \alpha, \beta \rangle \leq 3$, so either $\langle \alpha, \beta \rangle = 0$, $\langle \alpha, \beta \rangle = 1$, or $\langle \beta, \alpha \rangle = 1$. If $\langle \beta, \alpha \rangle = 1$, then $W_\alpha() = \beta - \langle \beta, \alpha \rangle \alpha = \beta - \alpha \in \Phi$. If $\beta - \alpha \in \Phi+$, then $\beta = \alpha + (\beta - \alpha)$ is not simple; if $\beta - \alpha \in \Phi^-$, then $\alpha = \beta + (\alpha - \beta)$ is not simple. Similar contradiction happens if $\langle \alpha, \beta \rangle = 1$, therefore this forces $\langle \alpha, \beta \rangle = 0$.

(2) Suppose $\alpha \in \Phi^+$ is either 1) $\alpha \in \mathcal{B}$ or 2) $\alpha = \beta + \gamma$ where $\beta, \gamma \in \Phi^+$. If 2) happens, then $\langle v, \alpha \rangle = \langle v, \beta \rangle + \langle v, \gamma \rangle$ where all of them are positive, then both $\langle v, \beta \rangle, \langle v, \gamma \rangle < \langle v, \alpha \rangle$. Proceeding inductively, we note that α has so be a sum of simple roots, and this consequently gives a linear combination as desired.

(3) Note that $\mathcal{B}$ spans V. Suppose this is not a basis, then there exists disjoint non-empty subsets $I, J \subseteq \{1, \dots, n\}$ and positive scalars μ_i such that $\gamma = \sum_{i \in I} \mu_i \alpha_i = \sum_{j \in J} \mu_j \alpha_j$. Then $0 \leq \langle \gamma, \gamma \rangle = \sum_{\substack{i \in I \\ j \in J}} \mu_i \mu_j \langle \alpha_i, \alpha_j \rangle \leq 0$, so $\gamma = 0$, therefore $0 < \langle v, \gamma \rangle = 0$, contradiction.

(4) Under an ordered basis, we assume $\alpha = \alpha_1$. Let $\beta \neq \alpha$ be in Φ^+, then $\beta = \sum_{i=1}^{n} m_i \alpha_i$ where $m_i \in \mathbb{Z}_{\geq 0}$. Since $\beta \neq \alpha$, then there exists $j \geq 2$ such that $m_j > 0$. Therefore, $W_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = (m_1 - \langle \beta, \alpha \rangle)\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n \in \Phi$, and since $m_j > 0$ for some j, then $\alpha \neq W_\alpha(\beta) \notin \Phi^-$, so $W_\alpha(\beta) \in \Phi^+ \setminus \{\alpha\}$.

(5) Let Φ be associated with $\mathcal{B}$ and Φ' be associated with $\mathcal{B}'$, then $\Phi = \Phi^+ \cup \Phi^- = \Phi'^+ \cup \Phi'^-$. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ be the Weyl vector and let $\sigma \in W(\Phi)$ be such that $\langle \sigma(v'), \rho \rangle$ is maximal. Therefore, for any $\alpha \in \mathcal{B}$, we have

$$\begin{aligned} \langle \sigma(v'), \rho \rangle &\geq \langle W_\alpha \sigma(v'), \rho \rangle \\ &= \langle \sigma(v'), W_\alpha(\rho) \rangle \\ &= \langle \sigma(v'), \rho - \alpha \rangle \\ &= \langle \sigma(v'), \rho \rangle - \left\langle v', \sigma^{-1}(\alpha) \right\rangle \end{aligned}$$

and therefore $\left\langle v', \sigma^{-1}(\alpha) \right\rangle \geq 0$. In particular, $\sigma^{-1}(\mathcal{B}) \subseteq \Phi'^+$, and so $\sigma^{-1}(\Phi^\pm) = \Phi'^\pm$. □

2.2. Dynkin Diagrams.

Definition 2.25 (Cartan Matrix)**.** Fix an ordering $(\alpha_1, \dots, \alpha_r)$ of the simple roots $\mathcal{B}$ of (V, Φ). The *Cartan matrix* of Φ is $(\langle \alpha_i, \alpha_j \rangle)_{i,j}$.

Lemma 2.26. *Let Φ and Φ' be root systems corresponding to Cartan matrices C and C', respectively. Then $\Phi \cong \Phi'$ if and only if $C \sim C'$.*

Proof. Let $\mathcal{B}$ and $\mathcal{B}'$ be root bases that defines C and C', respectively. Suppose we have an isomorphism $\varphi : \Phi \to \Phi'$ between the root systems. Note that $\varphi(\mathcal{B})$ becomes a base of Φ', then there exists $w \in W(\Phi')$ such that $\varphi(\mathcal{B}) = w(\mathcal{B}')$. Now $\mathcal{B}$ and $\varphi(\mathcal{B})$ define the same Cartan matrix C, and the Cartan matrix of $w(\mathcal{B}')$ is equivalent to C' of $\mathcal{B}'$, thus $C \sim C'$.

Now suppose $C \sim C'$, then by reordering the simple roots, we have $C = C'$. Suppose they are defined by $\mathcal{B} = \{\alpha_1, \dots, \alpha_l\}$ and $\mathcal{B}' = \{\alpha'_1, \dots, \alpha'_l\}$, then $\langle \alpha_i, \alpha_j \rangle = \left\langle \alpha'_i, \alpha'_j \right\rangle$ for all i, j. Let $\varphi : V \to V'$ be the linear map defined by $\varphi(\alpha_i) = \alpha'_i$ for all i, then by definition this defines a vector space isomorphism such that $\varphi(\mathcal{B}) = \mathcal{B}'$ and $\langle \alpha, \beta \rangle = \langle \varphi(\alpha), \varphi(\beta) \rangle$ for all $\alpha, \beta \in \Phi$. It now suffices to show that $\varphi(\Phi) = \Phi'$.

Suppose $v \in V$ and $\alpha_i \in \mathcal{B}$, then $\langle v, \alpha_i \rangle = \langle \varphi(v), \alpha'_k \rangle$ because of the definition of φ and because $\langle \cdot, \cdot \rangle$ is linear in the first slot. Therefore, $\varphi(W_{\alpha_i}(v)) = \varphi(v) - \langle v, \alpha_i \rangle \alpha' = W_{\alpha'_i}(\varphi(v))$. Therefore, the image under φ of the orbit of $v \in V$ under the Weyl group $W(\Phi)$ is contained in the orbit of $\varphi(v)$ under $W(\Phi')$. By definition, $\varphi(\Phi) \subseteq \Phi'$. Similarly, we have $\varphi^{-1}(\Phi') \subseteq \Phi$, hence $\varphi(\Phi) = \Phi'$, as desired. □

Fact 2.27. *The Cartan matrix has all entries as integers, such that*

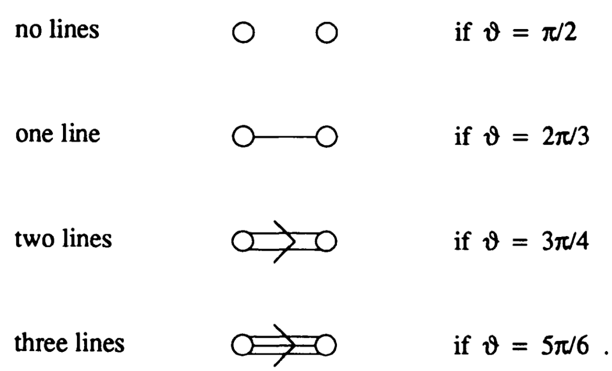
- $C_{ii} = 2$ *for all* i.

- *If $i \neq j$, then $C_{ij} \leq 0$, and $C_{ij} = 0$ if and only if $C_{ji} = 0$.*

One can illustrate the Cartan matrix using graphs called Dynkin diagrams.

Definition 2.28 (Dynkin Diagram)**.** Let Φ be a root system and $\mathcal{B}$ be a system of simple roots. The *Dynkin diagram* of Φ is a graph with $\mathcal{B}$ as its vertices. The vertices are connected in the following way: for any vertices α and β, they are connected by $\langle \alpha^*, \beta \rangle \cdot \langle \beta^*, \alpha \rangle$ edges. If $\langle \alpha^*, \beta \rangle > \langle \beta^*, \alpha \rangle$, then all the edges are directed from α to β.

Remark 2.29. Therefore, we can draw the relation between two vertices by the angle in between two roots. For instance, here is an illustration from [3].



In particular,

- if there is one edge between roots, then the roots have the same length;
- if there are multiple edges, then the arrow is directed from the longer root to the shorter root.

Remark 2.30. We call the undirected version of a Dynkin diagram is called the *Coxeter graph.*

3. Connections between Lie Algebras, Root Systems, and Dynkin Diagrams

3.1. Irreducible Root Systems and Simple Lie Algebra.

Lemma 3.1. *Let $\mathfrak{g}$ be a simple Lie algebra, then (V, Φ) is an irreducible root system.*

Proof. Suppose not, then we obtain a decomposition $\Phi = \Phi_1 \cup \Phi_2$ such that they are orthogonal components. Take $\alpha \in \Phi_1$ and $\beta \in \Phi_2$, then $\langle \alpha + \beta, \alpha \rangle \neq 0$ and $\langle \alpha + \beta, \beta \rangle \neq 0$. If $\alpha + \beta$ is a root, then it must be in either Φ_1 or Φ_2, but we note that it is not orthogonal to either α or β, which means it is not a root. We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$. Therefore, the subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ generated by all $\mathfrak{g}_\alpha$ for $\alpha \in \Phi_1$ is centralized by all $\mathfrak{g}_\beta$ for $\beta \in \Phi_2$. In particular, since the center $Z(\mathfrak{g}) = 0$, then $\mathfrak{g}'$ is a proper subalgebra of $\mathfrak{g}$. Moreover, since $\mathfrak{g}'$ is normalized by all such $\mathfrak{g}_\alpha$'s, then it is normalized by all roots in Φ, hence normalized by $\mathfrak{g}$. In particular, this shows that $\mathfrak{g}'$ is a proper non-zero ideal of $\mathfrak{g}$, however that means $\mathfrak{g}$ is not a simple Lie algebra, contradiction. □

Lemma 3.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Suppose we have the decomposition $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$ into simple Lie algebras, then $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$ is the Cartan subalgebra of $\mathfrak{g}_i$. Moreover, the corresponding root system of $\mathfrak{h}_i$ embeds into Φ, i.e., as a subset $\Phi_i \subseteq \Phi$, such that $\Phi = \bigcup_{i=1}^{n} \Phi_i$.*

Proof. By the Killing form on $\mathfrak{g}$, we know the decomposition of $\mathfrak{g}$ ends up in orthogonal subalgebras. For any $h \in \mathfrak{h}$ and any $x_i \in \mathfrak{g}_i$, we know the Lie bracket distributes $[h, \sum_{i=1}^{n} x_i] = \sum_{i=1}^{n} [h, x_i]$ and therefore we have a decomposition $\mathfrak{h} = \bigoplus_{i=1}^{n} \mathfrak{h}_i$ where $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$. If $\alpha \in \Phi_i$, then α can be extended as a linear function on $\mathfrak{h}$ by letting $\alpha(\mathfrak{h}_j) = 0$ for all $j \neq i$. Therefore, α becomes a root of $\mathfrak{g}$ with $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_i$. On the other hand, if $\alpha \in \Phi$, then there exists some i such that $[\mathfrak{h}_i, \mathfrak{g}_\alpha] \neq 0$, then $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_i$. Looking at the Killing form, this shows that $\Phi = \bigcup_{i=1}^{n} \Phi_i$ is a proper disjoint union, with $\Phi_i = \{\alpha \in \Phi : \alpha(\mathfrak{h}_i) \neq 0\} \neq \varnothing$. In particular, this shows that a non-simple Lie algebra has a reducible root system. ☐

Theorem 3.3 ([3], Exercise 21.6)**.** *Let (V, Φ) be a root system, then Φ is irreducible if and only if the semisimple Lie algebra $\mathfrak{g}$ is simple.*

Proof. This is a direct consequence of Lemma 3.1 and Lemma 3.2. Alternatively, apply Lemma 2.20 and Maschke's Theorem. ☐

Corollary 3.4. *An irreducible decomposition of root systems is unique.*

3.2. **Irreducible Root Systems and Connected Dynkin Diagrams.**

Theorem 3.5. *Two root systems are isomorphic (c.f., Definition 2.12) if and only if their Dynkin diagrams are the same.*

Proof. We know from Lemma 2.26 that isomorphic root systems have similar Cartan matrices, and it is easy to see that the entires of a Cartan matrix define the Dynkin diagram. Therefore, given similar Cartan matrices, we obtain the same Dynkin diagram by relabeling the simple roots. Now given a Dynkin diagram, this recovers the information $\langle \alpha_i, \alpha_j \rangle$ for all $i \leq j$, and therefore determines the entire Cartan matrix via Lemma 2.26. ☐

Theorem 3.6. *A root system (V, Φ) is irreducible if and only if its corresponding Dynkin diagram is connected.*

Proof. Suppose Φ is reducible, then $\Phi = \Phi_1 \cup \Phi_2$. Let $\mathcal{B}$ be a system of simple roots. Therefore, we have $\mathcal{B} = (\mathcal{B} \cap \Phi_1) \cup (\mathcal{B} \cap \Phi_2)$, and the vertices associated to elements in $\mathcal{B} \cap \Phi_1$ are not connected to those vertices associated to elements in $\mathcal{B} \cap \Phi_2$. Therefore, that means the corresponding Dynkin diagram is not connected, by construction.

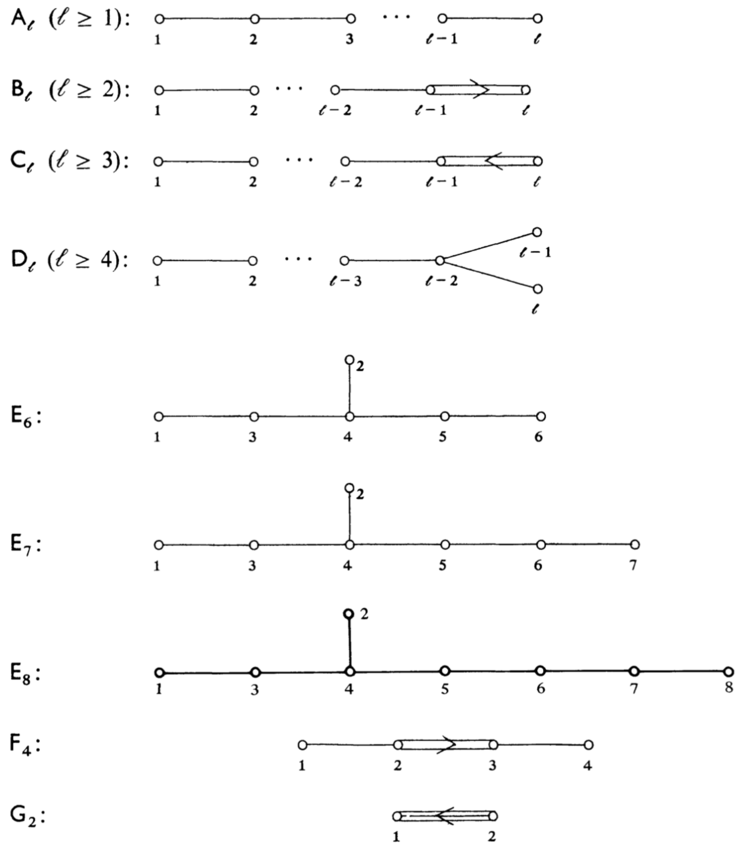
Now suppose the system of simple roots can be written as a disjoint union $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1, \mathcal{B}_2 \neq \varnothing$, and consider the orthogonal root systems $(V_1, \Phi_1), (V_2, \Phi_2)$ corresponding to $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. It now suffices to show that $\Phi = \Phi_1 \cup \Phi_2$. For any $\alpha \in \Phi_1$, then $W_\alpha|_{V_2} = \mathrm{id}$. By Lemma 2.3 and Theorem 2.24, there is $s \in W_1$ such that $s(\mathcal{B}_1) = -\mathcal{B}_1$, where $W_1 \subseteq W(\Phi)$ is the subgroup generated by W_α's where $\alpha \in \Phi_1$. By reordering, we have $\mathcal{B}_1 = \{\alpha_1, \ldots, \alpha_r\}$ and $\mathcal{B}_2 = \{\alpha_{r+1}, \ldots, \alpha_n\}$. Then any $\beta \in \Phi$ can be written as a linear combination $\beta = \sum_{i=1}^{n} m_i \alpha_i$ where $m_i \in \mathbb{Z}$, such that either $m_i \geq 0$ or $m_i \leq 0$ for all i, according to Theorem 2.24. But since s is such that $s(\beta) \in \Phi$ and $s(\mathcal{B}_1) = -\mathcal{B}_1$, then

$$s(\beta) = m_1'\alpha_1 + \cdots + m_r'\alpha_r + m_{r+1}\alpha_{r+1} + \cdots + m_n\alpha_n$$

and such that the coefficients $m_1', \ldots, m_r'$ is a reordering of $-m_1, \ldots, -m_r$. Therefore, either $m_1 = \cdots = m_r = 0$ or $m_{r+1} = \cdots = m_n = 0$. Therefore, either $\beta \in \Phi_1$ or $\beta \in \Phi_2$. ☐

3.3. **Classification of Connected Dynkin Diagrams.** By classifying the connected Dynkin diagrams, we would be able to classify the irreducible root systems and the simple Lie algebras by the corespondence above. Consequently, this would give us a general picture for classifying reduced root systems and split semisimple Lie algebras.

Theorem 3.7. *If (V, Φ) is an irreducible root system of rank l, then its Dynkin diagram must be one of the following:*



Proof. See [1] or [5]. This is done by first classifying the Coxeter graphs, and use Euclidean geometry to determine the lengths of roots. ☐

Example 3.8. The first four families of Dynkin diagrams (A_n, B_n, C_n, D_n) are well-studied. In particular, over $\mathbb{C}$, they correspond to $\mathfrak{sl}_{n+1}(\mathbb{C})$, $\mathfrak{so}_{2n+1}(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$, and $\mathfrak{so}_{2n}(\mathbb{C})$, respectively.

The other five classes $(E_6, E_7, E_8, F_4, G_2)$ are called exceptional Lie algebras.

Remark 3.9. Although the first four families of Dynkin diagrams (A_n, B_n, C_n, D_n) have restraints on the rank, one can extend all of them to $l \geq 1$. In particular,

- when $l = 1$, the case D_1 is degenerate, and $A_1 = B_1 = C_1$, which corresponds to $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C})$.
- when $l = 2$, we have $D_2 \cong A_1 \times A_1$ since $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$; we also have $B_2 \cong C_2$, since $\mathfrak{so}_5(\mathbb{C}) \cong \mathfrak{sp}_4(\mathbb{C})$.
- when $l = 3$, we have $D_3 \cong A_3$ since $\mathfrak{so}_6(\mathbb{C}) \cong \mathfrak{sl}_4(\mathbb{C})$.

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