Categorifying Spectra

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Categorification

Definition

Let A be an abelian group. **Categorifying an invariant** valued in A means finding a stable ∞-category $\mathcal{C}_A$ with $K_0(\mathcal{C}_A) \simeq A$ such that the given invariant lifts to functor valued in $\mathcal{C}_A$.

Example

Let $A = \mathbb{Q}$. [BGH$^+$19] proved a categorification of rationalization, i.e., for any stable ∞-category $\mathcal{C}$ and a set of primes $S \subseteq \mathbb{Z}$, one can construct a stable ∞-category $S^{-1}\mathcal{C}$ such that

$$K(S^{-1}\mathcal{C}) \simeq S^{-1}K(\mathcal{C}).$$

Main Results

Theorem (A)

Every spectrum is the K-theory of a stable ∞-category: for every spectrum M, there exists a small idempotent-complete stable ∞-category $\mathcal{C}_M$ such that

$$K(\mathcal{C}_M) \simeq M,$$

where K denotes the non-connective K-theory spectrum, and the assignment is functorial in M.

Corollary

Every abelian group is of the form $K_0(\mathcal{C})$ for some $\mathcal{C} \in \mathrm{Cat}_\infty^{perf}$.

Theorem (B)

The non-connective theorem of the heart is false in general.

Localizing Invariants

Definition

Consider the diagram

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

in $\mathrm{Cat}_{\infty}^{\mathrm{st}}$.
It is **Karoubi-Verdier** (KV) if f is fully faithful, $g \circ f$ is trivial, and the induced functor $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence up to idempotent completion.
A KV sequence is **Verdier** if g is essentially surjective, and the essential image of f is closed under retracts.

Localizing Invariants

Definition

Let $\mathcal{E}$ be a stable ∞-category, and let $E : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{E}$ be a functor.

- We say E is a **localizing invariant** if for any KV sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$, the sequence $E(\mathcal{A}) \to E(\mathcal{B}) \to E(\mathcal{C})$ is a fiber sequence.
- Suppose in addition that $\mathcal{E}$ is cocomplete. We say E is **finitary** if it preserves filtered colimits. There is a subcategory $\mathrm{Fun}^{\mathrm{loc,\,fin}}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathcal{E})$ of $\mathrm{Fun}(\mathrm{Cat}_\infty^{\mathrm{perf}}, \mathcal{E})$ of finitary localizing invariants.

Example

The non-connective K-theory functor $K : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$ is a finitary localizing invariant.

Comparison with [BGT13], [CDH$^+$20], and [Sau23]

<table>
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All notions are defined over $\mathrm{Cat}_\infty^{\mathrm{ex}}$ or equivalently $\mathrm{Cat}_\infty^{\mathrm{st}}$. [RSW24] follows the definitions in [BGT13], while [Sau23] mostly follows the definitions in [CDH+20]. The equivalences are proven in Proposition A.3.7 and Corollary A.1.10 of [CDH+20]. Localizing invariant had been defined differently among these sources.

- [BGT13] defines it over $E : \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{E}$ where $\mathcal{E}$ is stable presentable, and **assumes** E **to be finitary** in addition.
- [Sau23] defines a more general notion called Karoubi localizing over "Karoubi squares".
- [CDH+20] restricts the definition of [Sau23] to the context of Poincaré categories.

Stable K-theory

Let A be a ring. The Dennis trace map

$$\begin{array}{ccc} K(A) & \xrightarrow{\mathrm{tr}} & \mathrm{HH}(A) \\ & \searrow \quad \nearrow & \\ & K^S(A) & \end{array}$$

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Theorem (Dundas-McCarthy, [DM94])

For any simplicial ring R and simplicial R-bimodule M, there is a natural weak homotopy equivalence between $K^S(R, M)$ and $\mathrm{THH}(R; M)$.

Goal: establish an analogous result for non-connective K-theory.

Bimodules

Definition

Let $\mathcal{C}$ be a small stable ∞-category. A $\mathcal{C}$-bimodule T is an exact functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.

In particular, T gives rise to a colimit-preserving functor

$$T : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}).$$

Example

Let R be a ring spectrum and M be an R-bimodule, then

$$M \otimes_R - : \mathrm{Perf}_R \to \mathrm{Ind}(\mathrm{Perf}_R)$$

is a Perf_R-bimodule.

Twisted Endomorphism

Definition

Let $\mathcal{C}$ be a small stable ∞-category, and $T : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ be a $\mathcal{C}$-bimodule. The ∞-category $\mathrm{End}(\mathcal{C}; T)$ of twisted endomorphisms is the lax equalizer

$$\mathcal{C} \underset{T}{\overset{よ}{\rightrightarrows}} \mathrm{Ind}(\mathcal{C})$$

That is, $\mathrm{End}(\mathcal{C}; T)$ is the pullback of the cospan

$$\begin{array}{ccc} & & \mathrm{Fun}(\Delta^1, \mathrm{Ind}(\mathcal{C})) \\ & & \big\downarrow{\scriptstyle (s,t)} \\ \mathcal{C} & \xrightarrow[(よ,T)]{} & \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C}) \end{array}$$

whose objects are pairs $(x, f : x \to Tx)$ where $x \in \mathcal{C}$, and the morphisms are the corresponding commutative squares.

Important Remark

Fix A to be a ring spectrum and M to be an A-bimodule. Consider $\mathcal{C} = \mathrm{Perf}_A$ and $T = \Sigma M \otimes_A -$.

$$\mathrm{Perf}_{A \oplus M} \to \mathrm{End}(\mathrm{Perf}_A; \Sigma M \otimes_A -)$$

is a fully faithful embedding whose essential image consists of nilpotent twisted endomorphisms. More explicitly, an element in the essential image is a pair $(P, P \to \Sigma M \otimes_A P)$ such that for $n \gg 0$, the composite

$$P \longrightarrow \Sigma M \otimes_A P \longrightarrow \Sigma^2 M^{\otimes_A 2} \otimes_A P \longrightarrow \cdots \longrightarrow \Sigma^n M^{\otimes_A n} \otimes_A P$$

is null.

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is null. In the case where A and M are connective, the said embedding is an equivalence.

Fiber of Retraction

Let $\mathcal{C}$ be a small idempotent-complete stable ∞-category, and T be a $\mathcal{C}$-bimodule. The inclusion

$$\begin{aligned} i : \mathcal{C} &\hookrightarrow \mathrm{End}(\mathcal{C}; T) \\ x &\mapsto (x, 0 : x \to Tx) \end{aligned}$$

admits a retraction

$$\begin{aligned} r : \mathrm{End}(\mathcal{C}; T) &\to \mathcal{C} \\ (x, x \to Tx) &\mapsto x. \end{aligned}$$

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For a stable ∞-category $\mathcal{E}$, consider the functor $E : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{E}$. Define $\tilde{E}(C; T) := \mathrm{cofib}(E(i))$ for $E(i) : E(\mathcal{C}) \to E(\mathrm{End}(\mathcal{C}; T))$, therefore it is a direct summand of $E(\mathrm{End}(\mathcal{C}; T))$, so equivalently, it is the fiber of the retraction.

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In the case where $\mathcal{C} = \mathrm{Sp}^\omega = \mathrm{Perf}_{\mathbb{S}}$ and $T = M \otimes -$, we abbreviate $\tilde{E}(\mathrm{End}(\mathrm{Sp}^\omega; M)) := \tilde{E}(\mathrm{End}(\mathrm{Sp}^\omega; M \otimes -))$.

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Theorem (Dundas-McCarthy)

There is a natural equivalence

$$M \simeq \underrightarrow{\mathrm{colim}}\, \Omega^n \tilde{K}(\mathrm{Sp}^\omega; \Sigma^n M),$$

where the forward-direction functor is defined by the Goodwillie derivative $P_1 F := \underrightarrow{\mathrm{colim}}\, \Omega^n F(\Sigma^n -)$.

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For a simplicial ring A, this recovers Dundas-McCarthy Theorem in the classical sense.

Proof Sketch

- Note that the functor $\varinjlim \Omega^n \tilde{K}(\mathrm{Sp}^\omega; \Sigma^n -) : \mathrm{Sp} \to \mathrm{Sp}$ is exact. Since both $K(-)$ and $\mathrm{End}(\mathcal{C}; -) : \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathrm{Ind}(\mathcal{C})) \to \mathrm{Cat}_\infty^{\mathrm{perf}}$ preserve filtered colimits, then so does $\varinjlim \Omega^n \tilde{K}(\mathrm{Sp}^\omega; \Sigma^n -)$.

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- To identify such a functor, it suffices to identify the image of $\mathbb{S}$, i.e., $\varinjlim \Omega^n \tilde{K}(\mathrm{Sp}^\omega; \Sigma^n \mathbb{S})$. But applying our important remark to the case where $A = \mathbb{S}$ and $M = \Sigma^n \mathbb{S}$, it then suffices to understand

$$\varinjlim \Omega^n \tilde{K}(\mathbb{S} \oplus \Sigma^{n-1} \mathbb{S}).$$

In the spirit of Dundas-McCarthy Theorem, this is just $\mathbb{S}$.

Return to Theorem (A)

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Every spectrum is the K-theory of a stable ∞-category: for every spectrum M, there exists a small idempotent-complete stable ∞-category $\mathcal{C}_M$ such that

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- We show that the suspension, loops, and certain cofibers of K-theory spectrum can be categorified, i.e., naturally lifted to constructions on the categorical level.
- Apply Dundas-McCarthy theorem and performs the above constructions at the categorical level.

Categorification

Definition

Let $\mathcal{C}$ be a small stable ∞-category, then we define $\mathrm{Calk}(\mathcal{C})$ to be the ω_1-small Calkin category of $\mathcal{C}$, that is, the idempotent completion of the Verdier quotient of the Yoneda embedding $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})^{\omega_1}$.

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Since $\mathrm{Ind}(\mathcal{C})^{\omega_1}$ admits an Eilenberg swindle, then in particular $K(\mathrm{Ind}(\mathcal{C})^{\omega_1}) \simeq 0$. In particular,

Proposition

the functor Calk *categorifies the suspension. That is, for any small idempotent complete stable ∞-category $\mathcal{C}$, $K(\mathrm{Calk}(\mathcal{C})) \simeq \Sigma K(\mathcal{C})$.*

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Theorem

Γ *also categorifies loops. That is, there exists a functor* $\Gamma : \mathrm{Cat}_\infty^{perf} \to \mathrm{Cat}_\infty^{perf}$ *such that there is a canonical equivalence*

$$K(\Gamma\mathcal{C}) \simeq \Omega K(\mathcal{C}).$$

Categorification

Let $F\mathcal{C} \subseteq \mathrm{Fun}(\mathbb{N}, \mathcal{C})$ be the full subcategory of filtered objects in $\mathcal{C}$ which stabilize after finitely many steps, and let $F^q\mathcal{C} \subseteq F\mathcal{C}$ be the full subcategory of filtered object which stabilize at 0. This defines a grading functor $\mathrm{gr} : F\mathcal{C} \to \bigoplus_{\mathbb{N}} \mathcal{C}$. Define pullback diagrams

$$\begin{array}{ccc} B\mathcal{C} & \longrightarrow & F\mathcal{C} \\ \downarrow & & \downarrow \\ F\mathcal{C} & \longrightarrow & \bigoplus_{\mathbb{N}} \mathcal{C} \end{array} \qquad \begin{array}{ccc} B^q\mathcal{C} & \longrightarrow & F^q\mathcal{C} \\ \downarrow & & \downarrow \\ F^q\mathcal{C} & \longrightarrow & \bigoplus_{\mathbb{N}} \mathcal{C} \end{array} \qquad \begin{array}{ccc} B^t\mathcal{C} & \longrightarrow & F^q\mathcal{C} \\ \downarrow & & \downarrow \\ F\mathcal{C} & \longrightarrow & \bigoplus_{\mathbb{N}} \mathcal{C} \end{array}$$

The diagonal functor gives rise to $\Delta : F \Rightarrow B$ and $\Delta^q : F^q \Rightarrow B^q$. Since $\mathrm{cofib}(K(\Delta^q)) \simeq \Omega K(\mathcal{C})$, it suffices to prove that

Categorification

the cofiber of K-theory maps can be categorified.

Proposition

Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor of small stable ∞-categories, then there exists a fully faithful exact functor $G : \mathcal{C} \to \mathcal{D}'$ and an exact functor $P : \mathcal{D}' \to \mathcal{D}$ such that $P \circ G \simeq F$ and P fits into a right-split Verdier sequence

$$\mathrm{Ind}(\mathcal{C})^{\omega_1} \longrightarrow \mathcal{D}' \xrightarrow{P} \mathcal{D}$$

In particular,

- *such factorization can be chosen functorially in F, and*
- *there exists a small stable ∞-category $\mathrm{Cone}(F)$ and an exact functor $\mathcal{D} \to \mathrm{Cone}(F)$ that induces an equivalence*

$$\mathrm{cofib}(K(F) : K(\mathcal{C}) \to K(\mathcal{D})) \simeq K(\mathrm{Cone}(F)).$$

Categorification

Theorem

The unit equivalence $\eta : x \xrightarrow{\simeq} \Omega\Sigma x$ *can be categorified as well. That is, there exists a natural functor* $\mathcal{C} \to \Gamma\,\mathrm{Calk}(\mathcal{C})$ *that induces the unit equivalence* $K(\mathcal{C}) \simeq \Omega\Sigma K(\mathcal{C})$, *under the aforementioned categorifications.*

Let $S\mathcal{C}$ be the Verdier quotient of $\mathrm{Ind}(\mathcal{C})^{\omega_1}$ by $\mathcal{C}$. There is a commutative diagram of KV sequences

$$\begin{array}{ccccc}
\mathcal{C} & \xrightarrow{\text{よ}} & \mathrm{Ind}(\mathcal{C})^{\omega_1} & \longrightarrow & S\mathcal{C} \\
{\scriptstyle \tau_0}\downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \bar{\tau}} \\
B^q S\mathcal{C} & \longrightarrow & B^t S\mathcal{C} & \longrightarrow & B^t S\mathcal{C}/B^q S\mathcal{C} \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \downarrow \\
B^q S\mathcal{C} & \longrightarrow & BS\mathcal{C} & \longrightarrow & BS\mathcal{C}/B^q S\mathcal{C} \\
{\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\
F^q S\mathcal{C} & \longrightarrow & FS\mathcal{C} & \longrightarrow & FS\mathcal{C}/F^q S\mathcal{C}
\end{array}$$

Categorification

Applying the previous proposition, we may show that

$$\mathcal{C} \xrightarrow{\tau_0} B^q S\mathcal{C} \longrightarrow \operatorname{Cone}(\Delta_{S\mathcal{C}})$$

induces an equivalence by applying K-theory functor. Since K-theory is invariant under idempotent completion, then the natural functor $\operatorname{Cone}(\Delta_{S\mathcal{C}}) \to \operatorname{Cone}(\Delta_{\operatorname{Calk}(\mathcal{C})})$ induces an equivalence after apply K-theory. Therefore, we define

$$\mathrm{id} \overset{\tau_0}{\Longrightarrow} B^q S(-) \Longrightarrow \operatorname{Cone}(\Delta_{S(-)}) \Longrightarrow \operatorname{Cone}(\Delta_{\operatorname{Calk}(-)}) \simeq \Gamma \operatorname{Calk}(-)$$

Lemma that Describes the Behavior of Sequential Limits

Consider a sequential diagram $x_0 \xrightarrow{\alpha_0} x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots$ in a stable ∞-category. Suppose that each α_n admits a factorization

$$x_n \xrightarrow{\varphi_n} x_n' \xrightarrow{\psi_n} x_n'' \xrightarrow{\beta_n} x_{n+1}$$

where φ_n and ψ_n are equivalences, then $\varinjlim x_n$ is equivalent to the cofiber of

$$\bigoplus_{n\in\mathbb{N}} (x_n \oplus x_n'') \to \bigoplus_{n\in\mathbb{N}} (x_n \oplus x_n'),$$

represented by the diagram

$$\begin{tikzcd}
x_0 \arrow[d, "\mathrm{id}"'] \arrow[drr, "-\varphi_0"] & \oplus & x_0'' \arrow[d, "\psi_0^{-1}"] \arrow[drr, "-\beta_0"] & \oplus & x_1 \arrow[d, "\mathrm{id}"] \arrow[drr, "-\varphi_1"] & \oplus & x_1'' \arrow[d, "\psi_1^{-1}"] \arrow[drr, "-\beta_1"] & \oplus & x_2 \arrow[d, "\mathrm{id}"] \arrow[drr, "-\varphi_2"] & \oplus & \cdots \\
x_0 & \oplus & x_0'' & \oplus & x_1 & \oplus & x_1'' & \oplus & x_2 & \oplus & \cdots
\end{tikzcd}$$

Proof

Define $\gamma_n = \psi_n \varphi_n : x_n \to x''_n$, then $\bigoplus_{n\in\mathbb{N}} (x_n \oplus x''_n) \to \bigoplus_{n\in\mathbb{N}} (x_n \oplus x'_n)$ is just represented by the diagram

$$\begin{array}{ccccccccccccc}
x_0 & \oplus & x''_0 & \oplus & x_1 & \oplus & x''_1 & \oplus & x_2 & \oplus & \cdots \\
\downarrow \text{id} & \searrow^{-\gamma_0} & \downarrow \text{id} & \searrow^{-\beta_0} & \downarrow \text{id} & \searrow^{-\gamma_1} & \downarrow \text{id} & \searrow^{-\beta_1} & \downarrow \text{id} & \searrow^{-\gamma_2} & \\
x_0 & \oplus & x''_0 & \oplus & x_1 & \oplus & x''_1 & \oplus & x_2 & \oplus & \cdots
\end{array}$$

The cofiber is then the colimit of the diagram

$$x_0 \xrightarrow{\gamma_0} x''_0 \xrightarrow{\beta_0} x_1 \xrightarrow{\gamma_1} x''_1 \xrightarrow{\beta_1} x_2 \xrightarrow{\gamma_2} \cdots$$

which is just $\varinjlim x_n$.

Proof Sketch of Theorem (A)

Let M be an arbitrary spectrum, then by Dundas-McCarthy Theorem, we identify M to be the colimit

$$\varinjlim(\tilde{K}(\mathrm{Sp}^{\omega}; M) \to \Omega\tilde{K}(\mathrm{Sp}^{\omega}; \Sigma M) \to \Omega^2\tilde{K}(\mathrm{Sp}^{\omega}; \Sigma^2 M) \to \cdots).$$

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The functor we want to study is $\tilde{K}(\mathrm{Sp}^\omega; \Sigma^n M)$, which is the cofiber of

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with commutative square

$$\begin{array}{ccc} \mathrm{End}(\mathrm{Sp}^\omega; \Sigma^n M) & \xrightarrow{p_n} & \mathrm{End}(\mathrm{Sp}^\omega; 0) \\ \downarrow & & \downarrow \\ \mathrm{End}(\mathrm{Sp}^\omega; 0) & \xrightarrow[i_n]{} & \mathrm{End}(\mathrm{Sp}^\omega; \Sigma^{n+1} M) \end{array}$$

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Rest of the proof: waving your hands frequently to make enough identifications on a categorical level.

How to Wave Your Hands Correctly

Now the structure maps of the sequential limit we had can be written down as a composition

$$\begin{array}{ccc} \tilde{K}(\mathrm{Sp}^\omega;\Sigma^n M) & \xrightarrow{\star} & \Omega\,\mathrm{cofib}(K(i_n)) = \Omega\tilde{K}(\mathrm{Sp}^\omega;\Sigma^{n+1}M) \\ {\scriptstyle\eta_n}\downarrow & & \uparrow{\scriptstyle f_n} \\ \Omega\Sigma\tilde{K}(\mathrm{Sp}^\omega;\Sigma^n M) & \xrightarrow[v_n]{} & \Omega\,\mathrm{cofib}(K(p_n)) \end{array}$$

where

- η_n is the unit map $\mathrm{id} \simeq \Omega\Sigma$,
- v_n is the inverse of the map induced on the horizontal cofibers of the pullback diagram

$$\begin{array}{ccc} K(\mathrm{Sp}^\omega;\Sigma^n M) & \xrightarrow{K(p_n)} & K(\mathrm{Sp}^\omega;0) \\ \downarrow & & \downarrow \\ \tilde{K}(\mathrm{Sp}^\omega;\Sigma^n M) & \longrightarrow & 0 \end{array}$$

- and f_n is the canonical map induced on cofibers.

How to Wave Your Hands Correctly

By lifting enough times, we construct φ_n, ψ_n, and β_n are required by the lemma before, then there exists a functor F represented by the diagram

$$
\begin{array}{ccccccccccc}
\widetilde{\mathrm{End}}(\mathrm{Sp}^\omega; M) & \oplus & \Gamma\,\mathrm{Cone}(p_0) & \oplus & \widetilde{\mathrm{End}}(\mathrm{Sp}^\omega; \Sigma M) & \oplus & \mathrm{Cone}(p_1) & \oplus & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & \searrow^{\Sigma\circ\varphi_0} & \downarrow{\scriptstyle \Gamma\psi_0} & \searrow^{\Sigma\circ\Gamma\beta_0} & \downarrow{\scriptstyle \mathrm{id}} & \searrow^{\Sigma\circ\varphi_1} & \downarrow{\scriptstyle \Gamma\psi_1} & \searrow^{\Sigma\circ\Gamma\beta_1} & \\
\widetilde{\mathrm{End}}(\mathrm{Sp}^\omega; M) & \oplus & \Gamma\,\mathrm{Calk}(\widetilde{\mathrm{End}}(\mathrm{Sp}^\omega; M)) & \oplus & \widetilde{\mathrm{End}}(\mathrm{Sp}^\omega; \Sigma M) & \oplus & \Gamma\,\mathrm{Calk}(\widetilde{\mathrm{End}}(\mathrm{Sp}^\omega; \Sigma M)) & \oplus & \cdots
\end{array}
$$

In particular, the lemma says that $K(\mathrm{Cone}(F)) \simeq M$. Finally, the choices we made shows that the construction of $\mathrm{Cone}(F)$ refines to a functor $\mathcal{C}_{(-)} : \mathrm{Sp} \to \mathrm{Cat}_\infty^{\mathrm{perf}}$ such that $K \circ \mathcal{C}_{(-)} \simeq \mathrm{id}$.

Theorem of the Heart

Theorem ([Bar15])

Let E be a stable ∞-category with a bounded t-structure, then the inclusion $E^{\heartsuit} \hookrightarrow E$ induces a weak equivalence

$$K(E^{\heartsuit}) \simeq K(E).$$

Here $K(-)$ is interpreted as the (Waldhausen) K-theory of an $(\infty, 1)$-category. However, the proof in [Bar15] makes use of the ∞-exact category structure which gives rise to a duality.

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This is an analogue of Neeman's Theorem of the Heart for the algebraic K-theory of $\triangle$-categories, which expresses an equivalence between the algebraic K-theory of a $\triangle$-category $\mathcal{T}$ equipped with a bounded t-structure and the Quillen K-theory of its heart $\mathcal{T}^{\heartsuit}$. ([Nee98])

General Conjectures

The following conjectures were recorded in [AGH19].

Conjecture (A)

If $\mathcal{A}$ is a small abelian category, then $K_{-n}(\mathcal{A}) = 0$ for $n \geq 1$.

Conjecture (B)

If E is a small stable ∞-category with a bounded t-structure, then $K_{-n}(E) = 0$ for $n \geq 1$.

Conjecture (C)

If E is a small stable ∞-category with a bounded t-structure, then the natural map $K(E^{\heartsuit}) \to K(E)$ is an equivalence of non-connective K-theory spectra.

Remark

Conjecture (B) holds if and only if Conjecture (A) and Conjecture (C) hold.

Conjecture (C): Non-connective Theorem of the Heart

Theorem ([AGH19])

Let E be a small stable ∞-category with a bounded t-structure such that $E^\heartsuit$ is Noetherian, then the natural map

$$K(E^\heartsuit) \xrightarrow{\simeq} K(E)$$

of non-connective K-theory spectra is an equivalence.

Here the non-connective K-theory of the heart $K(E^\heartsuit) := K(\mathcal{D}^b(E^\heartsuit))$ is defined as that of the bounded derived category, which is a small idempotent-complete stable ∞-category.

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Theorem ([RSW24])

Conjecture (C) is false if we drop the Noetherian assumption of the heart.

Strategy

- Pick a spectrum M that is not $K(\mathbb{Z})$-local, e.g., the Morava K-theory spectrum $K(n)$ for $n \geq 2$. ([Mit90])
- By Theorem (A), pick $\mathcal{C} = \mathcal{C}_M$. This is a category "whose K-theory has sufficiently non-trivial chromatic behavior."
- Let $\hat{\mathcal{C}} = \mathrm{Fun}_\times(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ be the ∞-category of additive presheaves on $\mathcal{C}$, and let $\mathcal{C}^{\mathrm{fin}} \subseteq \hat{\mathcal{C}}$ be the smallest idempotent complete stable subcategory containing the image of $\text{よ} : \mathcal{C} \hookrightarrow \hat{\mathcal{C}}$.

Strategy

- The functor $よ : \mathcal{C} \to \mathcal{C}^{\text{fin}}$ is an initial additive functor into a small stable ∞-category. ([ES22]) This gives rise to an adjunction

$$\mathrm{Cat}_\infty^{\text{padd}} \underset{U}{\overset{(-)^{\text{fin}}}{\rightleftarrows}} \mathrm{Cat}_\infty^{\text{perf}}$$

  with counit $L : \mathcal{C}^{\text{fin}} \to \mathcal{C}$ for our choice of $\mathcal{C} \in \mathrm{Cat}_\infty^{\text{perf}}$.
- It turns out that L is also a Verdier localization map, so it gives rise to an exact sequence

$$\mathrm{Ac}(\mathcal{C}) \longrightarrow \mathcal{C}^{\text{fin}} \longrightarrow \mathcal{C}$$

  in $\mathrm{Cat}_\infty^{\text{perf}}$.

Strategy

- By [Kle20], the kernel $\mathrm{Ac}(\mathcal{C})$ is a stable ∞-category generated by cofibers of the natural maps $よ(b)/よ(a) \to よ(b/a)$ for morphisms $a \to b$ in $\mathcal{C}$. It is observed that $\mathrm{Ac}(\mathcal{C})$ admits a natural bounded t-structure, c.f., Theorem 5.1, or [Nee21].

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- This is a contradiction: if Conjecture (C) holds, then $K(\mathrm{Ac}(\mathcal{C}))$ is $K(\mathbb{Z})$-local as $\mathcal{D}^b(\mathrm{Ac}(\mathcal{C})) \simeq \mathcal{D}^b(\mathrm{Ac}(\mathcal{C})^{\heartsuit})$ is $\mathbb{Z}$-linear, i.e., with simple chromatic behavior. (Proposition 4.15, Corollary 4.16)

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