K-book Reading

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September 4, 2023

1 Projective Modules and Vector Bundles

The basic objects studied in algebraic K-theory are projective modules over a ring and vector bundles over schemes.

Rings are assumed to be non-trivial with multiplicative identity, and R-modules are assumed to be right modules with multiplications on the left.

1.1 Free modules, GL_n, and stably free modules

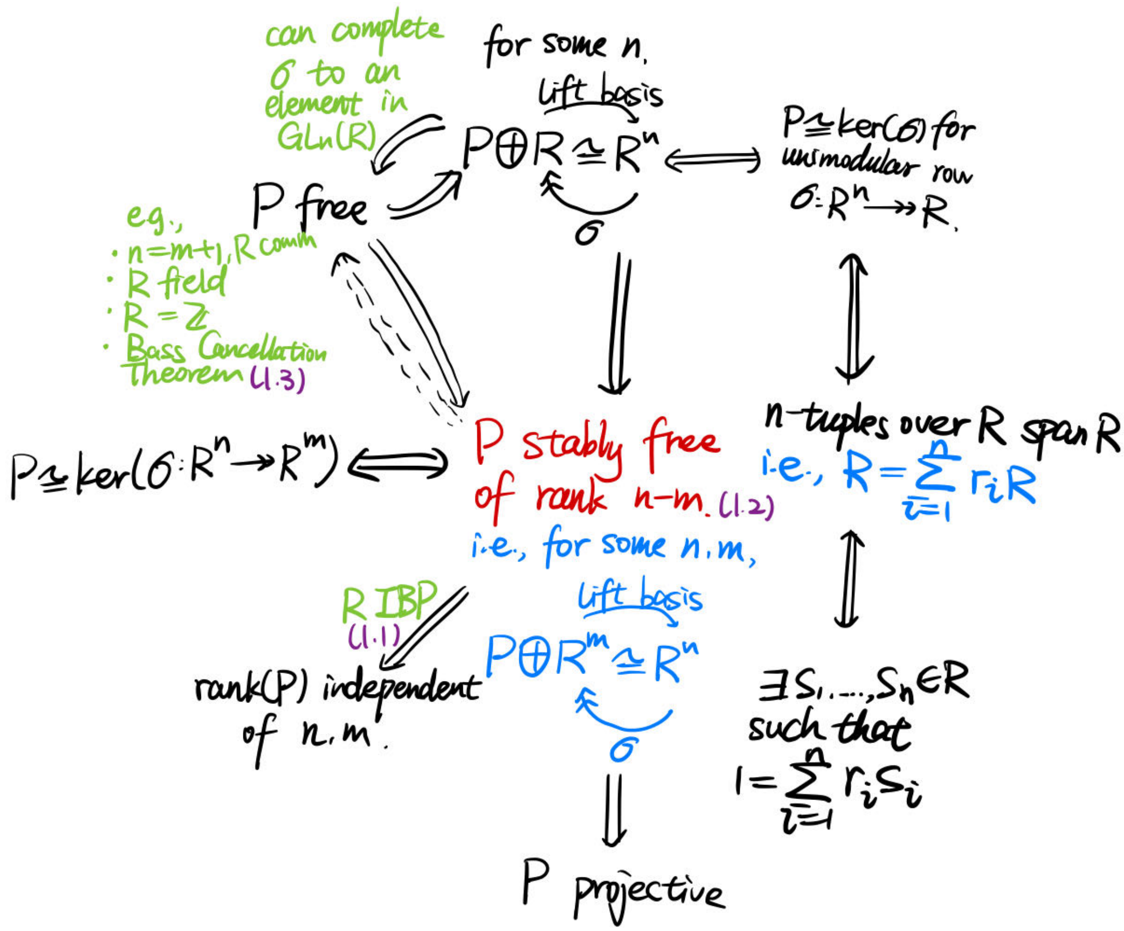

Figure 1: Chapter 1.1 Review

Exercise 1.1.1. If R is a semisimple ring, then R is a direct sum of a finite number of simple R-modules. Furthermore, every stably free module over R is free. In particular, semisimple rings satisfy IBP.

Proof. by Artin-Wederburn Theorem, we know

$$R \cong M_{n_1}(D_1) \times \cdots M_{n_k}(D_k)$$

where $D_i \cong \mathrm{End}_R(L_i)$ for some minimal ideal L_i of R. Note that the matrix rings over division rings are simple. Now let M be a stably free R-module, then since R is semisimple, we know M is a semisimple module. As finite-dimensional matrix ring algebras, we know each simple component satisfies IBP, and therefore R satisfies IBP.

Suppose $R^m \oplus P$ is now free of rank n, i.e., $R^m \oplus P \cong R^n$, then we have a short exact sequence

$$0 \longrightarrow R^m \longrightarrow R^n \longrightarrow P \longrightarrow 0$$

and since R has the invariant basis property, we find a basis of R^n by extending the basis of R^m, therefore we have $P \cong R^n/R^m \cong R^{n-m}$, as desired.

not done □

Exercise 1.1.2. Consider the following conditions on a ring R:

(i) R satisfies IBP;

(ii) for all m and n, if $R^m \cong R^n \oplus P$, then $m \geqslant n$;

(iii) for all n, if $R^n \cong R^n \oplus P$, then $P = 0$.

If $R \neq 0$, show that $(iii) \Rightarrow (ii) \Rightarrow (i)$.

Proof. Let $R \neq 0$.

- $(iii) \Rightarrow (ii)$: Let R^n be the free R-module of rank n. Suppose R^n can also be generated by m elements, then there is an epimorphism $\pi : R^m \twoheadrightarrow R^n$. By construction, this extends to a short exact sequence

$$0 \longrightarrow P \xrightarrow{\ i\ } R^m \xrightarrow{\ \pi\ } R^n \longrightarrow 0$$

where $P \cong \ker(\pi)$. Since R^n is free, then the short exact sequence splits, hence $R^m \cong R^n \oplus P$. We claim that $m \geqslant n$, i.e., any generating set of a free module of rank n has at least m elements. By construction, we know R^n is now generated by m elements, so the m elements span R^n, but they are not necessarily linearly independent. Suppose, towards contradiction, that $m < n$, then we can extend this set to a set of n elements, and the new set still generates R^n, so now we have a short exact sequence

$$0 \longrightarrow Q \xrightarrow{\ i\ } R^n \xrightarrow{\ \pi'\ } R^n \longrightarrow 0$$

where π' is the augmented map from π. Note that this sequence splits again so $R^n \cong Q \oplus R^n$, but by assumption now $\ker(\varphi) = Q = 0$, so $\pi' : R^n \to R^n$ is an isomorphism. This is not possible since the domain R^n is extended from a surjection already, so the images of the added generators is already contained in the image of R^m, therefore the map would not be injective, contradiction. Hence, we know $m \geqslant n$.

- $(ii) \Rightarrow (i)$: Let A be a free R-module of rank m with respect to basis B_1 and of rank n with respect to basis B_2, it suffices to show that $m = n$. The statement of (ii) is equivalent to the following: the generating set of a free R-module of rank n has at least n elements. Therefore, by (ii), we know B_1 has at least m elements, and B_2 has at least n elements. If $m \neq n$, then say $m > n$, but now A as a free R-module of rank m cannot be generated by $n < m$ elements, contradiction, thus $m = n$.

□

Exercise 1.1.3. Show that (iii) and the following matrix conditions are equivalent:

(a) for all n, every surjection $R^n \to R^n$ is an isomorphism;

(b) for all n and for $f, g \in M_n(R)$, if $fg = 1_n$, then $gf = 1_n$ and $g \in \mathrm{GL}_n(R)$.

Then show that commutative rings satisfy (b), hence (iii).

Proof.
- $(iii) \Rightarrow (a)$: Let $\pi : R^n \to R^n$ be a surjection, then this extends to a short exact sequence

$$0 \longrightarrow \ker(\pi) \longrightarrow R^n \xrightarrow{\pi} R^n \longrightarrow 0$$

  In particular, R^n is free hence the short exact sequence splits, thus $R^n \cong \ker(\pi) \oplus R^n$, but by (iii) we have $\ker(\pi) = 0$, so this is an injection as well, thus we have a bijection of R-modules which is just an isomorphism.

- $(a) \Rightarrow (b)$: Let $f, g \in M_n(R)$ be such that $fg = 1_n$, f is a surjection and g is an injection. By assumption, $f \in M_n(R)$ is a surjection $f : R^n \to R^n$, so f is an isomorphism, therefore there exists a unique two-sided inverse h of f, but now $fg = fh = 1_n$ where f is a monomorphism of this category, so by left cancellation we have $g = h$, therefore g is the inverse of f, then the claim follows.

- Using the same argument, if we have $R^n \cong R^n \oplus P$, then this corresponds to an extension of the surjection $f : R^n \to R^n$ via the short exact sequence

$$0 \longrightarrow P \xrightarrow{g} R^n \xrightarrow{f} R^n \longrightarrow 0$$

  where $g \in M_n(R)$ is the inclusion and $P \cong \ker(f)$. Therefore, $fg = 1_n$, so by assumption we know $gf = 1_n$ and $g \in \mathrm{GL}_n(R)$, therefore this says f is an isomorphism, hence $\ker(f) = 0$. Therefore, $P = 0$.

□

Exercise 1.1.4. Show that right Noetherian rings satisfy condition (b) of Exercise 1.1.3, hence they satisfy (iii) and have the right invariant basis property.

Proof. We claim that (a) is true. Suppose R is Noetherian, then let $u : R^n \to R^n$ be a surjection, and suppose u is not an injection, then the ascending chain of $\ker(u^k)$'s cannot stabilize applying u always creates new elements into the kernel, so it contradicts the fact that R is Noetherian. Therefore, a surjection must be an isomorphism, therefore (b) is also true. □

Exercise 1.1.5.
(a) Show that (S_n) holds for all $n \geqslant \mathrm{sr}(R)$.

(b) If $\mathrm{sr}(R) = n$, show that all stably free projective modules of rank $\geqslant n$ are free. *Hint*: compare $(r_0, \dots, r_n)$, $(r_0, r_1', \dots, r_n')$, and $(1, r_1', \dots, r_n')$.

(c) Show that $\mathrm{sr}(R) = 1$ for every Artinian ring R. Conclude that all stably free projective R-modules are free over Artinian rings.

(d) Show that if I is an ideal of R, then $\mathrm{sr}(R) \geqslant \mathrm{sr}(R/I)$.

(e) If $\mathrm{sr}(R) = n$ for some n, show that R satisfies the IBP. *Hint*: consider an isomorphism $B : R^N \cong R^{N+n}$, and apply (S_n) to convert B into a matrix of the form $\binom{C}{0}$.

Proof. (a)

□

Exercise 1.1.6. Let D be a division ring which is not a field. Choose $\alpha, \beta \in D$ such that $\alpha\beta - \beta\alpha \neq 0$, and show that $\sigma = (x + \alpha, y + \beta)$ is a unimodular row over $R = D[x, y]$. Let $P = \ker(\sigma)$ be the associated rank 1 stably free module; $P \oplus R \cong R^2$. Prove that P is not a free $D[x, y]$-module, using these steps:

(i) If $P \cong R^n$, show that $n = 1$. Thus we may suppose that $P \cong R$ with $1 \in R$ corresponding to a vector $\begin{bmatrix} r \\ s \end{bmatrix}$ with $r, s \in R$.

(ii) Show that P contains a vector $\begin{bmatrix} f \\ g \end{bmatrix}$ with $f = c_1x + c_2y + c_3xy + c_4y^2$ and $g = d_1x + d_2y + d_3xy + d_4x^2$ $(c_i, d_i \in D)$.

(iii) Show that P cannot contain any vector $\begin{bmatrix} f \\ g \end{bmatrix}$ with f and g linear polynomials in x and y. Conclude that the vector in (i) must be quadratic and may be taken to be of the form given in (ii).

(iv) Show that P contains a vector $\begin{bmatrix} f \\ g \end{bmatrix}$ with $f = \gamma_0 + \gamma_1 y + y^2$, $g = \delta_0 + \delta_1 x - \alpha y - xy$, and $\gamma_0 = \beta u^{-1}\beta u \neq 0$. This contradicts (iii), so we cannot have $P \cong R$.

Proof. First off, we have

$$\begin{aligned}\frac{1}{\alpha\beta - \beta\alpha}\left((x+\alpha)(\beta+y) + (y+\beta)(-\alpha-x)\right) &= \frac{1}{\alpha\beta-\beta\alpha}\left(x\beta + \alpha\beta + xy + \alpha y - y\alpha - \beta\alpha - yx - \beta x\right) \\ &= \frac{1}{\alpha\beta-\beta\alpha} \times (\alpha\beta - \beta\alpha) \\ &= 1.\end{aligned}$$

(i) We know division rings satisfy IBP, so given $P \cong R^n$, we have $R^n \oplus R \cong R^2$, then rank gives $n+1 = 2$, so $n = 1$.

(ii) By the calculation above, we know $\begin{bmatrix} A(y+\beta) \\ B(x+\alpha) \end{bmatrix}$ is an element in P and write it as $\begin{bmatrix} m \\ n \end{bmatrix}$. Now multiplying this element by $(x+y)$, we obtain a corresponding element

$$\begin{aligned}\begin{bmatrix} m \\ n \end{bmatrix}(x+y) &= \begin{bmatrix} m(x+y) \\ n(x+y) \end{bmatrix} \\ &= \begin{bmatrix} A(y+\beta)(x+y) \\ B(x+\alpha)(x+y) \end{bmatrix} \\ &= \begin{bmatrix} (A\beta)x + (A\beta)y + Axy + Ay^2 \\ (B\alpha)x + (B\alpha)y + Bxy + By^2 \end{bmatrix}\end{aligned}$$

which we will define by $\begin{bmatrix} f \\ g \end{bmatrix}$.

(iii) Since this element is in the kernel, then multiplying with σ should give zero. This would not be possible if f contains a linear term in x and g contains a linear term in y, there are no other corresponding basis elements in x^2 and y^2 by then. It now suffices to show that the vector $\begin{bmatrix} r \\ s \end{bmatrix}$ cannot be linear. If it were linear, then we obtain $\begin{bmatrix} f \\ g \end{bmatrix}$ by multiplication with respect to some linear polynomial in y, according to f, but that means g still contains linear terms in y, contradiction. Therefore, the basis vector $\begin{bmatrix} r \\ s \end{bmatrix}$ must be quadratic, so we take it to be $\begin{bmatrix} f \\ g \end{bmatrix}$ as in (ii).

(iv)

□

1.2 Projective modules

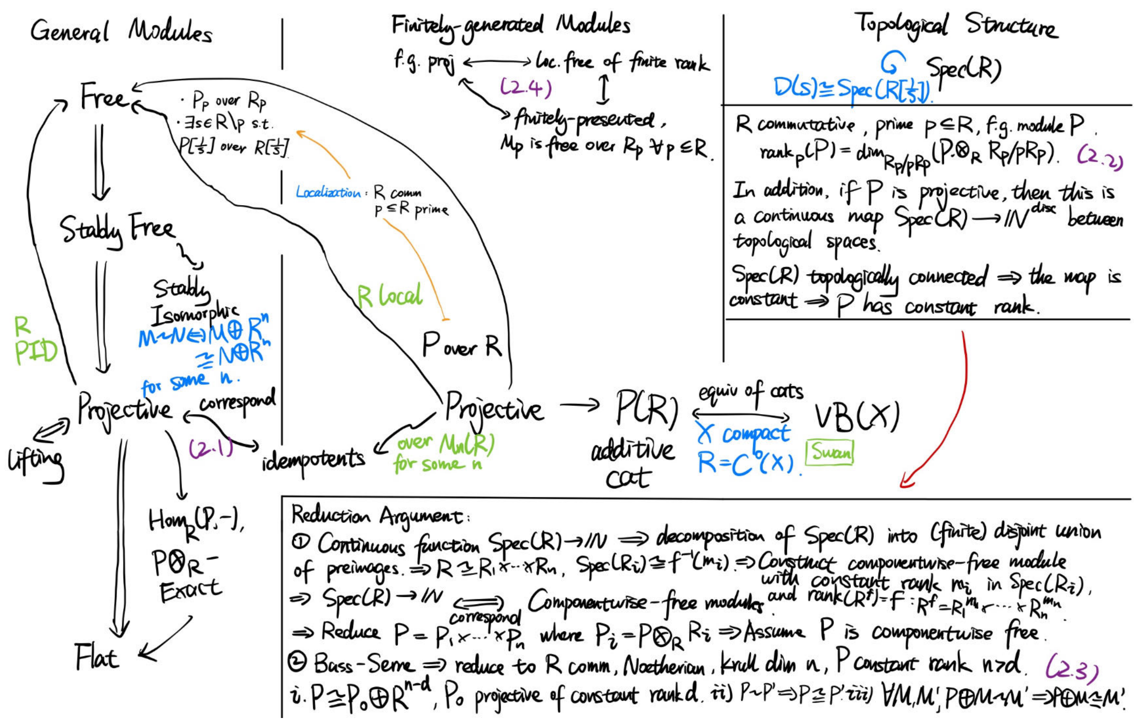

Figure 2: Chapter 1.1 Review

The rest of the section discusses patching free modules to obtain a projective module structure.

Remark 1.2.1 (Milnor Patching)**.** Let I be an ideal in R, and $f : R \to S$ be a ring homomorphism that sends I to an ideal of S that we identify by I as well. Therefore, R is the pullback of S and R/I, as

$$R = \{(\bar{r}, s) \in (R/I) \times S : \bar{f}(\bar{r}) = s \pmod I\},$$

so we obtain a Milnor square

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow[\bar{f}]{} & S/I \end{array}$$

In particular, if I is the conductor ideal, then we obtain a conductor square.

Given a Milnor square above, we construct $M = (M_1, g, M_2)$ as the R-module obtained by patching M_1 and M_2 together along g as follows: for S-module M_1, R/I-module M_2, and S/I-module isomorphism $g : M_2 \otimes_{R/I} S/I \cong M_1/IM_1$, then M becomes the kernel of

$$\begin{aligned} M_1 \times M_2 &\to M_1/IM_1 \\ (m_1, m_2) &\mapsto \bar{m}_1 - g(\bar{f}(m_2)). \end{aligned}$$

Theorem 1.2.2 (Milnor Patching)**.** Given a Milnor square above,

1. if P is obtained by patching together a finitely-generated projective S-module P_1 and a finitely-generated projective R/I-module P_2, then P is a finitely-generated projective R-module;

2. $P \otimes_R S \cong P_1$, and $P/IP \cong P_2$;

3. every finitely-generated projective R-module arises in this way;
4. if P is obtained by patching free modules along $g \in \mathrm{GL}_n(S/I)$ and Q is obtained by patching free modules along g^{-1}, then $P \oplus Q \cong R^{2n}$.

We would now talk about the case in infinitely-generated projective modules.

Example 1.2.3 (Eilenberg Swindle)**.** Let R^∞ be an infinitely-generated free module, if $P \oplus Q = R^n$, then we have $P \oplus R^\infty \cong R^\infty$ and $R^\infty \cong R^\infty \oplus R^\infty$. Moreover, if $P \oplus R^\infty \cong R^\infty$, then $R \cong R^\infty$.