MATH 519 Notes

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May 2, 2025

These notes were live-texed from a Differential Geometry class (MATH 519) taught by Professor R.L. Fernandes in Spring 2025 at University of Illinois. Any mistakes and inaccuracies would be my own.

Contents

1 Riemannian Geometry

1.1 Riemannian Metrics

Definition 1.1.1. A *Riemannian metric* on a manifold M is a family of inner products

$$\langle -, - \rangle_p : T_pM \times T_pM \to \mathbb{R}$$

which vary smoothly with the point p. We then say M is a *Riemannian manifold*.

Remark 1.1.2. Equivalently, we can think of it as a map

$$g : \mathfrak{X}(M) \to \mathfrak{X}(M) \to C^\infty(M)$$

of vectors field $\mathfrak{X}(M)$ on manifold M, defined by

$$g_p(x, y) = \langle x, y \rangle_p$$

and satisfying the properties that

a. it is C^∞-bilinear,

b. symmetric, i.e., $g(x, y) = g(y, x)$, and

c. positive-definite, i.e., $g_p(x, x) = 0$ if and only if $x = 0$.

Therefore, g is a symmetric tensor of type-$(2, 0)$. In local charts (U, x^i), the tensor has local coordinates given by

$$\begin{aligned} g|_U &= \sum_{i,j} g_{ij} dx^i \otimes dx^j \\ &= \sum_{i,j} g_{ij} dx^i dx^j \text{ by symmetry} \end{aligned}$$

for $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. Here by convention, we denote the symmetric product $dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$.

Exercise 1.1.3. If (V, y^i) is another chart (such that $U \cap V \neq \varnothing$), then we have $g = \sum\limits_{i,j} \bar{g}_{i,j} dy^i dy^j$. How are g_{ij}'s related to $\bar{g}_{ij}$?

Remark 1.1.4. The (symmetric) tensors of type-$(p, 0)$ behave like differential forms. Therefore, say, given a $C^\infty(M)$-multilinear tensor of type-$(p, 0)$ $T : \mathfrak{X}(M)^p \to C^\infty(M)$, and given $\Phi : N \to M$, then we have a pullback

$$\Phi^* T : \mathfrak{X}(N)^p \to C^\infty(N)$$

which is defined as per differential forms, i.e.,

$$(\Phi^* T)_x(x_1, \ldots, x_p) = T_{\Phi(x)}(d_x\Phi(x_1), \ldots, d_x\Phi(x_p)).$$

In local coordinates, we can represent $\Phi = (\Phi^1, \ldots, \Phi^p)$ and $T = \sum\limits_{i_1,\ldots,i_p} T_{i_1,\ldots,i_p} dx^{i_1} \otimes \cdots \otimes dx^{i_p}$, then we have

$$(\Phi^* T) = \sum_{i_1,\ldots,i_p} (T_{i_1,\ldots,i_p} \circ \Phi) d\Phi^{i_1} \otimes \cdots \otimes d\Phi^{i_p}.$$

Example 1.1.5. Consider $M = \mathbb{R}^3$ in coordinates (x, y, z), and $T = z^2 dx \otimes dy + 2xydz \otimes dx$ given by a 2-tensor. We have a map

$$\begin{aligned} \Phi : \mathbb{R}^2 &\to \mathbb{R}^3 \\ (u, v) &\mapsto (e^u v, v, uv), \end{aligned}$$

then to compute $\Phi^* T$, we assign $x = e^u$, $y = v$, and $z = uv$, then

$$\begin{aligned} \Phi^* T &= (uv)^2 de^u \otimes dv + 2e^u v d(uv) \otimes de^u \\ &= (uv)^2 e^u du \otimes dv + 2e^u v(vdu + udv) \otimes e^u du \\ &= 2ue^{2u} du \otimes du + u^2 v^2 e^u du \otimes dv + 2e^{2u} vudv \otimes du. \end{aligned}$$

Definition 1.1.6. A (smooth) map $\Phi : (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds is called an *isometric immersion* if it is an immersion and $\Phi^* g_2 = g_1$. In particular, if Φ is a (respectively, local) diffeomorphism, then we say Φ is a (respectively, local) isometry.

Example 1.1.7.

1. Consider $M = \mathbb{R}^n$ with local coordinates $(x^1, \dots, x^n)$, the inner product structure on the tangent space gives the *(standard) distance function* $g_0 = \sum_{i=1}^{n} (dx^i)^2$ as the metric.

2. If $N \subseteq \mathbb{R}^n$ is a submanifold, then the inclusion gives an induced Riemannian metric $g_N = i^* g_0$ where $i : N \hookrightarrow \mathbb{R}^n$ is the inclusion.

(⋆) Consider $N = \mathbb{S}^2_R \subseteq \mathbb{R}^3$ be the 2-sphere of radius R with local coordinates (θ, φ) and (x, y, z), respectively. We note that $x = R\cos(\theta)\sin(\varphi)$, $y = R\sin(\theta)\sin(\varphi)$, and $z = R\cos(\varphi)$. We should think of these expressions as defining the inclusion map i from the 2-sphere to $\mathbb{R}^3$, thereby inducing

$$\begin{aligned} g_{\mathbb{S}^2_R} &= i^* g_0 \\ &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= R^2(\sin^2(\varphi)(d\theta)^2 + (d\varphi)^2) \end{aligned}$$

3. Let G be a Lie group with Lie algebra $\mathfrak{g} = T_e G$. Picking an inner product $\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on the vector space $\mathfrak{g}$, we induce a Riemannian structure on the Lie group G, namely the left translation

$$g_h(x, y) = \langle d_h L_{h^{-1}}(x), d_h L_{h^{-1}}(y) \rangle$$

for $h \in G$ and left translation L.[1] This is a left-invariant metric, i.e., $g_y(u, v) = g_{L_x(y)}(d_y L_x u, d_y L_x v)$: every left translation $L_h : G \to G$ is an isometry. Moreover, one can show that the Lie algebra on the Lie group must be of compact type.

We end the lecture with two important results.

Theorem 1.1.8. Every manifold admits a Riemannian metric.

Proof 1. Use Whitney embedding theorem and pullback g_0. □

Proof 2. Use partition of unity: a $\mathbb{C}$-combination of inner products is still an inner product, so we get to glue the local inner product structures together as a global one. □

Remark 1.1.9. We see that the first proof is better than the second one, in the sense that it works in general for any analytic manifold, while the second one only works for Riemannian manifolds.

Theorem 1.1.10 (Nash Embedding)**.** Every Riemannian manifold (M, g) admits an isometric embedding $i : (M, g) \hookrightarrow (\mathbb{R}^n, g_0)$ for some n.

End of Lecture 1

1.2 Geodesics

Definition 1.2.1. Let (M, g) be a Riemannian manifold.

- For any $v \in T_x M$, we have $||v||^2 = g(v, v)$. In particular, if $v \in \mathbb{R}^m$, then this is its norm $|v|^2 = \sum_{i=1}^{m} v_i^2$.

[1] Correspondingly, there is a Riemannian structure given by the right translation.

- Given a path $\gamma : [a, b] \to M$ that is piecewise smooth, then its *length* is given by $L(\gamma) = \int_a^b ||\dot{\gamma}(t)||dt$, and its *energy* is given by $E(\gamma) = \frac{1}{2}\int_a^b ||\dot{\gamma}(t)||^2 dt$.
- The *distance function* $d : M \times M \to \mathbb{R}$ is given by

$$d(p, q) = \min\{L(\gamma) : \gamma : [a, b] \to M \text{ piecewise smooth curve} : \gamma(a) = p, \gamma(b) = q\}.$$

  In such cases, we usually assume M to be connected, or just deal with a connected component.

Proposition 1.2.2. The distance function is just the distance in the usual sense. That is, (M, d) is a metric space, and the topology it defined is the same as the one on M.

Proof. Let us first show that (M, d) is a metric space. Most of this is obvious, so the only part we need to show is that if $d(p, q) > 0$ whenever $p \neq q$. Fix a chart (U, φ) centered at $p \in M$, corresponding to $\varphi(U)$ on $\mathbb{R}^m$. Without loss of generality, we choose $q \notin U$. Choose $\varepsilon > 0$ such that $D_\varepsilon := \{v \in \mathbb{R}^m : |v| \leqslant \varepsilon\}$. If $\gamma(a) = p$ and $\gamma(b) = q$, then $L(\gamma) \geqslant L(\gamma \cap \varphi(D_\varepsilon))$. Therefore, it suffices to show that there exists $c > 0$ such that given a curve $\gamma' : [a, b] \to \varphi(D_\varepsilon)$ where $\gamma'(a) = p$ and $\gamma'(b) \in \varphi(\partial D_\varepsilon)$, then $L(\gamma') \geqslant \varepsilon$.

More specifically, let us write the chart as $\varphi = (x^1, \ldots, x^m)$ and $g = \sum_{i,j} g_{ij}(x)dx^i dx^j$. Let us define

$$\lambda(x) = \min\{g_{ij}(x)v^i v^j : |v| = 1, v \in \mathbb{R}^m\},$$

but since D_ε is compact, then we have $\lambda(x) \geqslant \lambda_0 > 0$ for all $x \in D_\varepsilon$, therefore

$$||v|| = \sum_{i,j} g_{ij}(x)v^i v^j = \sum_{i,j} g_{ij}(x)\frac{v^i}{|v|}\frac{v^j}{|v|}|v|^2 \geqslant \lambda_0|v|^2$$

which is true for any tangent vector in the disk D_ε. We compute that the length on the chart

$$\varphi \circ \gamma'(t) = (\gamma'^1(t), \ldots, \gamma'^m(t)),$$

where we find

$$\dot{\gamma}'(t) = \sum_{i=1}^m \dot{\gamma}'^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma'(t)}.$$

Therefore, we calculate

$$\begin{aligned} L(\gamma') &= \int_a^b ||\dot{\gamma}'(t)||dt \\ &= \int_a^b (g_{ij}\dot{\gamma}'^i(t)\dot{\gamma}'^j(t))^{\frac{1}{2}} \\ &\geqslant \int_a^b \lambda_0|\dot{\gamma}'(t)|dt \\ &\geqslant \lambda_0\varepsilon \\ &= c. \end{aligned}$$

To check that the topologies agree, we just need to check this on any chart. In particular, on a chart, we have

$$\lambda_0|v|^2 \leqslant g_{ij}(x)v^i v^j \leqslant \mu_0|v|^2$$

which sandwiches the distance between the two points in the ball. This means that for $x = \varphi(p)$ and $y = \varphi(q)$, then the distance

$$\lambda_0|x - y| \leqslant d(p, q) \leqslant \mu_0|x - y|$$

which means they define the same open set in any chart. $\square$

Remark 1.2.3. Length is invariant under parametrization. That is, given $\gamma : [a, b] \to M$ and $\tau : [c, d] \to [a, b]$ such that $\tau(c) = a$ and $\tau(d) = b$, then $L(\gamma \circ \tau) = L(\gamma)$. This is given by the chain rule: we have

$$\dot{\overline{(\gamma \circ \tau)}}(t) = \dot{\gamma}(\tau(t))\dot{\tau}(t),$$

so taking the length gives

$$\begin{aligned} L(\gamma \circ \tau) &= \int_c^d ||\dot{\overline{(\gamma \circ \tau)}}||dt \\ &= \int_c^d ||\dot{\gamma}(\tau(t))|| \cdot ||\dot{\tau}(t)||dt \\ &= \int_a^b ||\dot{\gamma}(s)||ds \\ &= L(\gamma) \end{aligned}$$

where we define $s = \tau(t)$.

Remark 1.2.4. Energy is not invariant due to the quadratic in its formula.

However, length and energy are related as follows.

Theorem 1.2.5 (Length-energy Inequality)**.** If $\gamma : [a, b] \to M$, then

$$L(\gamma)^2 \leqslant 2(b-a)E(\gamma)$$

with equality holds if and only if the length of the tangent $||\dot{\gamma}(t)||$ is constant.

Proof. By Hölder's inequality, we have

$$\int_a^b |f(t)g(t)|dt \leqslant \left(\int_a^b |f|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b |g|^2 dt\right)^{\frac{1}{2}}$$

where equality holds if and only if there exists $\lambda, \mu \in \mathbb{R}$ such that $\lambda f^2 = \mu g^2$. Therefore, taking the length gives

$$\begin{aligned} L(\gamma)^2 &= \int_a^b ||\dot{\gamma}||dt \\ &\leqslant 2\left(\int_a^b 1dt\right) \cdot \frac{1}{2}\left(\int_a^b ||\dot{\gamma}||^2 dt\right) \\ &= 2(b-a)E(\gamma), \end{aligned}$$

and equality holds if and only if the length of the tangent is constant. □

Definition 1.2.6. A *geodesic* is a curve that minimizes energy.

Remark 1.2.7. To get around the fact that energy is not invariant under parametrization, we will define

$$P(p, q) = \{\gamma : [0, 1] \to M : \gamma(0) = p, \gamma(1) = q\},$$

then we can rewrite energy as

$$\begin{aligned} E : P(p, q) &\to \mathbb{R} \\ \gamma &\mapsto \frac{1}{2}\int_0^1 ||\dot{\gamma}(t)||^2 dt. \end{aligned}$$

Remark 1.2.8. If we try to minimize the length, since the length is invariant under parametrization, we can just restrict to the curves parametrized by the arc length, which means the norm of the derivative $||\dot{\gamma}(t)||$ is 1, but in that case this is equivalent as minimizing the energy function by Theorem 1.2.5. Conversely, we will show that the geodesic has "constant" norm of derivative, so again by Theorem 1.2.5, minimizing the length function and minimizing the energy function are the same thing.

Fix a curve $\gamma_0 : [0,1] \to M$, and take a piecewise smooth variation of γ_0. For simplicity, we may assume γ_0 is smooth, so that we only require a *smooth variation* of γ_0, which is a smooth curve $\gamma : (-\delta, \delta) \times [0,1]$ such that $\gamma(0,t) = \gamma_0(t)$, and we define $\gamma_\varepsilon(t) = \gamma(\varepsilon, t)$. If γ_0 minimizes E, then we will see that

$$0 = \frac{d}{d\varepsilon} E(\gamma_\varepsilon)\bigg|_{\varepsilon=0} = \frac{1}{2}\frac{d}{d\varepsilon}\int_0^1 ||\gamma_\varepsilon||^2 dt\bigg|_{\varepsilon=0}. \tag{1.2.9}$$

We will denote $||\gamma_\varepsilon||^2$ by $\mathcal{L}(\gamma_\varepsilon)$, given by the function

$$\begin{aligned} \mathcal{L} : TM &\to \mathbb{R} \\ v &\mapsto g(v,v) = ||v||^2 \end{aligned}$$

known as the *Lagrangian*. In particular, Equation (1.2.9) is equivalent to the Euler-Lagrange equations for $\mathcal{L}$.

To do this in global coordinates, we will require Čech spaces. Instead, we will do this in local coordinates (U, φ), where we define

$$\mathcal{L}(x, v) := g_{ij}(x) v^i v^j.$$

End of Lecture 2

We now summarize the setting. For a Riemannian manifold (X, g), where we take

$$X = \{\gamma : [0,1] \to M \mid \gamma(0) = p, \gamma(1) = q\}$$

and we have an energy function

$$\begin{aligned} E : X &\to \mathbb{R} \\ \gamma &\mapsto \frac{1}{2}\int_0^1 ||\dot{\gamma}(t)||^2 dt. \end{aligned}$$

We are now interested in the critical points of this function, which is the interest in studying calculus of variations. In a more general setting, consider a function $\mathcal{L} : TM \to \mathbb{R}$ on the tangent bundle of the manifold, where we should think of as $\mathcal{L}(v) = \frac{1}{2} g(v,v)$ in our case. Now consider the function

$$\begin{aligned} \mathcal{F} : X &\to \mathbb{R} \\ \gamma &\mapsto \frac{1}{2}\mathcal{L}(\gamma(t), \dot{\gamma}(t)) dt. \end{aligned}$$

Let us fix a chart $U \subseteq M$ so we can run a local argument (assuming $p, q \in U$), that is, assuming $\gamma_0 : [0,1] \to U$. Now take $\gamma(\varepsilon, t) = \gamma_\varepsilon : [0,1] \to U$ for

$$\gamma : (-\delta, \delta) \times [0,1] \to \mathbb{R}$$

for some small $\delta > 0$. Therefore $\gamma_0 \in X$ is a point, and γ_ε is now a curve on X, which defines a function $\mathcal{F}$ into $\mathbb{R}$. We are therefore interested in finding curves so that

$$\frac{d}{d\varepsilon}\mathcal{F}(\gamma_\varepsilon)\bigg|_{\varepsilon=0} = 0.$$

This is just asking

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\int_0^1 \mathcal{L}(\gamma_\varepsilon(t),\dot{\gamma}_\varepsilon(t))dt\bigg|_{\varepsilon=0} = \int_0^1\left(\sum_{i=1}^m \frac{\partial\mathcal{L}}{\partial x^i}(\gamma_\varepsilon(t))\frac{d\gamma_\varepsilon^i}{d\varepsilon} + \frac{\partial\mathcal{L}}{\partial v^i}(\gamma_\varepsilon(t),\dot{\gamma}_\varepsilon(t))\frac{d}{d\varepsilon}\dot{\gamma}_\varepsilon^i(t)\bigg|_{\varepsilon=0}\right)\\
&=\int_0^1\sum_{i=1}^m\left(\frac{\partial\mathcal{L}}{\partial x^i}(\gamma_0(t),\dot{\gamma}_0(t))\frac{d\gamma_\varepsilon^i}{d\varepsilon}\bigg|_{\varepsilon=0} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_0,\dot{\gamma}_0(t))\frac{d\gamma_\varepsilon^i}{d\varepsilon}\bigg|_{\varepsilon=0} + \frac{d}{dt}\left(\frac{d\mathcal{L}}{dv^i}(\gamma_0(t),\dot{\gamma}_0(t))\frac{d}{d\varepsilon}\gamma_\varepsilon^i(t)\bigg|_{\varepsilon=0}\right)\right)\\
&=\int_0^1\sum_{i=1}^m\left(\frac{\partial\mathcal{L}}{\partial x^i}(\gamma_0(t),\dot{\gamma}_0(t)) - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_0(t),\dot{\gamma}_0(t))\right)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}\bigg|_{\varepsilon=0}dt + \sum_{i=1}^m\left(\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_0(t),\dot{\gamma}_0(t))\ \frac{d}{d\varepsilon}\gamma_\varepsilon^i(t)\bigg|_{\varepsilon=0}\bigg|_{t=0}^{t=1}\right)\\
&=\int_0^1\sum_{i=1}^m\left(\frac{\partial\mathcal{L}}{\partial x^i}(\gamma_0(t),\dot{\gamma}_0(t)) - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_0(t),\dot{\gamma}_0(t))\right)\frac{d\gamma_\varepsilon(t)}{d\varepsilon}\bigg|_{\varepsilon=0}dt
\end{aligned}
$$

under enough smoothness conditions. But this expression is zero for any γ_ε, therefore this implies that

$$
\int_0^1\sum_{i=1}^m\left(\frac{\partial\mathcal{L}}{\partial x^i}(\gamma_0(t),\dot{\gamma}_0(t)) - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_0(t),\dot{\gamma}_0(t))\right)\frac{d\gamma_\varepsilon(t)}{d\varepsilon}\bigg|_{\varepsilon=0}dt = 0 \tag{1.2.10}
$$

for any $t\in[0,1]$ and $i=1,\dots,m$. This is known as the *Euler-Lagrange equation.*

With $\mathcal{L}(x,v)=\sum_{i,j} g_{ij}(x)v^iv^j$, we have the Euler-Lagrange Equation local charts (U,φ) given by

$$
\ddot{\gamma}_0^i(T) + \sum_{j,k=1}^m \Gamma_{jk}^i(\gamma_0(t))\dot{\gamma}^j(t)\dot{\gamma}^k(t) = 0 \tag{1.2.11}
$$

for $i=1,\dots,m$, where we define *Christoffel symbols*

$$
\Gamma_{jk}^i = \frac{1}{2}\sum_{\ell=1}^m g^{i\ell}\left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right)
$$

where (g^{ij}) is the inverse of (g_{ij}) given as a matrix. In particular, the Christoffel symbols are not tensors.

Definition 1.2.12 (Einstein Convention). A (p,q)-tensor $T\in\Gamma(\bigotimes^p T^*M\bigotimes^q TM)$ can be described as a C^∞-multilinear function

$$
T:(\mathfrak{X}^1(M))^p\times(\Omega^1(M))^q\to C^\infty(M)
$$

and in particular we can write

$$
T = T_{i_1,\dots,i_p}^{j_1,\dots,j_q}(x)dx^{i_1}\otimes\cdots dx^{i_p}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_q}}
$$

in local charts. This avoids writing over the summation $\sum_{\substack{i_1,\dots,i_p\\ j_1,\dots,j_q}}$ and we can just denote $g=g_{ij}dx^idx^j$.

Exercise 1.2.13. Show that any solution γ_0 of Equation (1.2.11) has $||\dot{\gamma}(t)||$ constant.

1.3 Connections

Definition 1.3.1 (Affine Connection). An affine connection on a manifold M is a $\mathbb{R}$-bilinear map

$$
\begin{aligned}
\nabla:\mathfrak{X}(M)\times\mathfrak{X}(M)&\to\mathfrak{X}(M)\\
(X,Y)&\mapsto\nabla_XY
\end{aligned}
$$

satisfying

i. it is C^∞-bilinear in the first entry: $\nabla_{fX} Yy = f\nabla_X Y$ for any $f \in C^\infty(M)$;

ii. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$ for any $f \in C^\infty(M)$.

Remark 1.3.2. There are other ways of defining connections. For instance, we can say it is a linear operator

$$d^\nabla : \Omega^*(M, TM) \to \Omega^{*+1}(M, TM)$$

satisfying $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge d^\nabla \eta$ for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M, TM)$.

Example 1.3.3.

1. Set $M = \mathbb{R}^n$ and $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$, then

$$\begin{aligned}\nabla_X Y &= X^i \nabla_{\frac{\partial}{\partial x^i}} \left(Y^j \frac{\partial}{\partial x^j} \right) \\ &= X^i Y^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} + X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j}.\end{aligned}$$

   We can now set

$$\Gamma^k_{ij} \frac{\partial}{\partial x^k} = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$$

   to be arbitrary functions so that we get a connection. For instance, we can set them to be zeros, which gives a canonical connection in $\mathbb{R}^n$, namely the *flat connection* or the trivial connection in $\mathbb{R}^n$ with $\Gamma^k_{ij} \equiv 0$. That is,

$$\nabla_X Y = X(Y^j) \frac{\partial}{\partial x^j}.$$

2. Let $M = G$ be a Lie group with a Lie algebra $\mathfrak{g} = T_e G$, and fix a basis $\{e_1, \dots, e_n\}$ for $\mathfrak{g}$, with left-invariant vector fields $\{E_1, \dots, E_n\} \subseteq \mathfrak{X}(M)$. This gives a basis, so any vector fields $X, Y \in \mathfrak{X}(M)$ can be written as $X = X^i E_i$ and $Y = Y^j E_j$. Now we get

$$\nabla_X Y = X^i Y^j \nabla_{E^i} E^j + X(Y^i) E_j$$

   just as in the previous example. If we set this to be arbitrary, we may get any connection. In particular, for $\nabla_{E_j} E_j = c[E_i, E_j]$ to be a Lie bracket multiplied by some fixed constant c.

We now know a connection always exists for an arbitrary manifold: the first example tells us that the connections exist locally, so it is just a question of how we glue connections together.

Proposition 1.3.4. Every manifold M has a connection. The space of connections is an affine space modeled on the vector space of $(2, 1)$-tensors.

Proof. Given a chart, we apply the first example in Example 1.3.3. So take a cover $C = \{(U_i, \varphi_i)\}$ of M by charts, and choose a connection ∇^i on each chart U_i. Now take a partition of unity $\{\rho_i\}$ subordinated to C, then we can define a global connection

$$\nabla_X Y = \sum_i \rho_i \nabla^i_{X|_{U_i}} Y|_{U_i}.$$

To prove the second statement, given two connections ∇^1 and ∇^2, we note that $T(x, y) := \nabla^1_X Y - \nabla^2_X Y$ is C^∞-linear, so this defines a C^∞-linear map

$$T : \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \to \mathfrak{X}^1(M)$$

which defines a $(2, 1)$-tensor. ☐

End of Lecture 3

Let ∇ be the connection. For a chart (U, x^i), we observed that

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

characterized the vector fields.

Remark 1.3.5. Γ_{ij}^k's are not components of a tensor field. Note that the assignment $X \mapsto \nabla_X Y$ is $C^\infty(M)$-linear for fixed Y, but $Y \mapsto \nabla_X Y$ is not $C^\infty(M)$-linear.

Definition 1.3.6. The *torsion* of ∇ is the map

$$\begin{aligned} T^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X, Y) &\mapsto \nabla_X Y - \nabla_Y X - [X, Y] \end{aligned}$$

Remark 1.3.7. This is a $(2, 1)$-tensor: we can define

$$\begin{aligned} \tilde{T}^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Omega^1(M) &\to C^\infty(M) \\ (X, Y, \alpha) &\mapsto \langle T^\nabla(X, Y), \alpha \rangle \end{aligned}$$

which is $C^\infty(M)$-linear in each entry. Indeed,

- $\alpha \mapsto \tilde{T}^\nabla(X, Y, \alpha)$ is $C^\infty(M)$-linear,
- $\tilde{T}^\nabla(X, Y, \alpha) = -\tilde{T}^\nabla(Y, X, \alpha)$,
- and

$$\begin{aligned} T^\nabla(fX, Y, \alpha) &= \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y] \\ &= f\nabla_X Y - f\nabla_Y X - Y(f)X - f[X, Y] + Y(f)X \\ &= fT^\nabla(X, Y). \end{aligned}$$

Therefore, in a local chart (U, x^i), we get

$$T^\nabla \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = T_{ij}^k \frac{\partial}{\partial x^k}$$

and therefore

$$T^\nabla = T_{ij}^k dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} = \frac{1}{2} T_{ij}^k dx^i \wedge dx^j \otimes \frac{\partial}{\partial x^k}$$

In particular, T_{ij}^k's are symmetric in i and j. In terms of Christoffel symbols, we write

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

Definition 1.3.8. A connection ∇ is called *symmetric* or *torsion-free* if the torsion T^∇ vanishes.

Remark 1.3.9. In a local chart (U, x^i), ∇ is torsion-free if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all i, j.

Example 1.3.10.

1. $\mathbb{R}^n$ with ∇ determined by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$$

   which is torsion-free.

2. For a Lie group G with ∇ determined by $\nabla_X Y = c[X, Y]$ for any $X, Y \in \mathfrak{X}_{\text{left-invariant}}(G)$. This connection has torsion:

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 2c[X, Y] - [X, Y] = (2c - 1)[X, Y].$$

   Therefore, ∇ is torsion-free if either

- $\mathfrak{g}$ is abelian, i.e., $[X, Y] = 0$ for all $X, Y \in \mathfrak{X}_{\text{left-invariant}}(G) \simeq \mathfrak{g}$, or
- $\mathfrak{g}$ is arbitrary but $c = \frac{1}{2}$.

Remark 1.3.11. Given a connection, we can differentiate any tensor fields along a vector field.

- For a 1-form $\alpha \in \Omega^1(M)$, we construct a new 1-form
$$\nabla_X \alpha \in \Omega^1(M)$$
given by $(\nabla_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y)$. This is equivalent to the property
$$X(\langle \alpha, Y\rangle) = \langle \nabla_X \alpha, Y\rangle + \langle \alpha, \nabla_X Y\rangle$$
for $\langle \alpha, Y\rangle = \alpha(Y)$.
- In general, given any (p, q)-tensor, we think of it as a map
$$T : (\mathfrak{X}(M))^p \times (\Omega^1(M))^q \to C^\infty(M),$$
we define a (p, q)-tensor
$$\begin{aligned}&\nabla_X T : (\mathfrak{X}(M))^p \times (\Omega^1(M))^q \to C^\infty(M)\\ &X(T(Y_1, \dots, Y_p, \alpha_1, \dots, \alpha_q)) = (\nabla_X T)(Y_1, \dots, Y_p, \alpha_1, \alpha_q) + \sum_i T(Y_1, \dots, \nabla_X Y_i, \dots, Y_p, \alpha_1, \dots, \alpha_q)\\ &\qquad + \sum_i T(Y_1, \dots, Y_p, \alpha_1, \dots, \nabla_X \alpha_i, \dots, \alpha_q)\end{aligned}$$
where we think of $T \in C^\infty(M)$ so we get to apply X on T since $\mathfrak{X}(M)$ is the set of derivations $X : C^\infty(M) \to C^\infty(M)$.
In the notation that
$$T = T^{j_1, \dots, j_q}_{i_1, \dots, i_p} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_p} \otimes X_{j_1} \otimes \cdots \otimes X_{j_q},$$
then we can also rewrite $\nabla_X T$ in this form as well.

Definition 1.3.12. A connection ∇ is *compatible* with a Riemannian metric g if $\nabla_X g = 0$ for all $X \in \mathfrak{X}(M)$. We also just write $\nabla g = 0$.

Remark 1.3.13. Explicitly, $\nabla g = 0$ is equivalent to the statement that $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for any $X, Y, Z \in \mathfrak{X}(M)$.

Exercise 1.3.14. Show that if ∇ is compatible with g and $\nabla_X Y = 0$, then $||Y||$ is constant along the orbit, i.e., the integral curves of the vector field X, that is, $X(||Y||^2) = 0$.

Theorem 1.3.15. Given a Riemannian manifold (M, g), there exists a unique torsion-free connection compatible with the Riemannian metric g.

Definition 1.3.16. The connection specified in Theorem 1.3.15 is called the *Levi-Civita connection* of (M, g).

Remark 1.3.17. Not all torsion-free connections the Levi-Civita connection of some Riemannian manifold.

Proof. Assuming ∇ satisfies $\nabla g = 0$ and $T^\nabla = 0$, we see that

- $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,
- $Y(g(Z, X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$, and
- $Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$,

therefore

$$\begin{aligned} X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) &= g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X) \\ &= g(2\nabla_X Y + [Y,X], Z) + g([X,Z],Y) + g([Y,Z],X), \end{aligned}$$

therefore

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}(&X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &- g([Y,X],Z) - g([X,Z],Y) - g([Y,Z],X)). \end{aligned}$$

One can then check that this is the torsion-free connection we need: in particular, show that X and Y in $\nabla_X Y$ satisfies the properties of a connection. $\square$

End of Lecture 4

Remark 1.3.18. On a local chart (U, x^i), we see that

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k},$$

by writing $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, and $Z = \frac{\partial}{\partial x^k}$, we have

$$\Gamma_{ij}^k g_{\ell k} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right).$$

Example 1.3.19.

1. Consider $\mathbb{R}^n$ with $g_0 = \sum_{i=1}^m (dx^i)^2$, the Levi-Civita connection is the flat connection given by $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$.

2. Let G be a Lie group, we have the torsion-free connection

$$\nabla_X Y = \frac{1}{2}[X,Y]$$

for any $X, Y \in \mathfrak{X}_{\text{left-invariant}}(G)$.

For any connection ∇, we know

$$\nabla_{fX} Y = f\nabla_X Y.$$

Therefore, suppose X_1 and X_2 agree at a point x, i.e., $X_1|_x = X_2|_x$, then

$$(\nabla_{X_1} Y)|_x = (\nabla_{X_2} Y)|_x.$$

Therefore, for any tangent vector $v \in T_x M$ and any tangent field Y defined in a neighborhood of x, $\nabla_v Y \in T_x M$ is a well-defined tangent vector of M at x.

Definition 1.3.20. Let $\gamma : [a,b] \to M$ be a path and $V : [a,b] \to TM$ be a vector field along γ, i.e., $V(t) \in T_{\gamma(t)} M$ for all $t \in [a,b]$, or

$$\begin{array}{ccc} & \overset{V}{\nearrow} & TM \\ & & \big\downarrow \pi \\ [a,b] & \xrightarrow[\gamma]{} & M \end{array}$$

then the *covariant derivative* of V along γ is the vector field $D_\gamma V$ along γ given by

$$(D_\gamma V)(t) = \nabla_{\dot\gamma(t)} \tilde{V}_t + \left.\frac{d}{dt}\tilde{V}_t\right|_{\gamma(t)},$$

where vector field $\tilde{V}_t \in \mathfrak{X}(M)$ is any time-dependent extension of V, i.e., it is smooth in both variables, such that $\tilde{V}_t(\gamma(t)) = \tilde{V}(\gamma(t), t) := V(t)$.

Remark 1.3.21. In general, one needs time-dependent extensions since curve may intersect itself. That is, if there is a self-intersecting curve, the tangent vector at the intersection point may change, depending on the time variable.

Remark 1.3.22. The definition is independent of the choice of extension. We just need to check this in a local chart (U, x^i). Consider $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$, and vector field $V(t) = V^i(t) \frac{\partial}{\partial x^i}\big|_{\gamma(t)}$, and let $\dot{\gamma}(t) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}\big|_{\gamma(t)}$. Given an extension

$$\tilde{V}_t = \tilde{V}^i(x,t)\frac{\partial}{\partial x^i},$$

with $\tilde{V}^i(\gamma(t),t) = V^i(t)$, we apply the formula and get

$$\begin{aligned}
(D_\gamma V)(t) &= \nabla_{\dot{\gamma}^i(t) \frac{\partial}{\partial x^i}|_{\gamma(t)}} \left(\tilde{V}^j(x,t)\frac{\partial}{\partial x^j}\right) + \frac{d}{dt}\left(\tilde{V}^i(x,t)\frac{\partial}{\partial x^i}\right)\Bigg|_{\gamma(t)} \\
&= \dot{\gamma}^i(t)\frac{\partial \tilde{V}^j(x,t)}{\partial x^i}\frac{\partial}{\partial x^j}\Bigg|_{\gamma(t)} + \dot{\gamma}^i(t)\tilde{V}^j(\gamma(t),t)\Gamma_{ij}^k(\gamma(t))\frac{\partial}{\partial x^k}\Bigg|_{\gamma(t)} + \frac{\partial \tilde{V}^j}{\partial t}(\gamma(t),t)\frac{\partial}{\partial x^j}\Bigg|_{\gamma(t)} \\
&= \frac{d}{dt}\left(\tilde{V}^j(\gamma(t),t)\right)\frac{\partial}{\partial x^j}\Bigg|_{\gamma(t)} + \dot{\gamma}^i(t)\tilde{V}^j(\gamma(t),t)\Gamma_{ij}^k(\gamma(t))\frac{\partial}{\partial x^k}\Bigg|_{\gamma(t)} \quad \text{by chain rule} \\
&= \left(\dot{\tilde{V}}^k(t) + \dot{\gamma}^i(t)V^j(t)\Gamma_{ij}^k(\gamma(t))\right)\frac{\partial}{\partial x^k}\Bigg|_{\gamma(t)} \qquad (1.3.23)
\end{aligned}$$

which is independent of the choice of extension.

Definition 1.3.24. Given a path $\gamma : [a,b] \to M$,

1. a vector field V along γ is *parallel* if $D_\gamma V(t) = 0$ for any $t \in [a,b]$,
2. γ is a *geodesic* if $\dot{\gamma}(t)$ is parallel along γ, that is, $(D_\gamma \dot{\gamma})(t) = 0$ for any $t \in [a,b]$.

Remark 1.3.25. Equation (1.3.23) now explains what a parallel vector field and a geodesic would be.

- A vector field V along γ is parallel if it satisfies the equation

$$\dot{V}^k(t) + \Gamma_{ij}^k(\gamma(t))\dot{\gamma}^i(t)V^j(t) = 0 \qquad (1.3.26)$$

  for any $t \in [a,b]$ and $k = 1, \dots, m$.

- A geodesic satisfies the equation

$$\ddot{\gamma}^k(t) + \Gamma_{ij}^k(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0 \qquad (1.3.27)$$

  for any $t \in [a,b]$. Note that we can also rewrite this a system of first-order differential equations given by

$$\begin{cases} \dot{\gamma}^k &= v^k(t) \\ \dot{v}^k(t) &= -\Gamma_{ij}^k(\gamma(t))v^iv^j \end{cases} \qquad (1.3.28)$$

  interpreted from manifold to tangent bundle.

Proposition 1.3.29. Given a connection ∇ and path $\gamma : [a,b] \to M$, for any tangent vector $v_0 \in T_{\gamma(a)}M$, there exists a unique parallel vector field V along γ such that $V(a) = v_0$.

Remark 1.3.30. Because Equation (1.3.26) is a first-order linear ordinary differential equation, then given a tangent vector at the beginning of the path, we can "parallel transport" it along the path, and get a tangent vector at the end of the path, which then gives a vector field.

Definition 1.3.31. The *parallel transport* along a path $\gamma : [a,b] \to M$ is

$$\begin{aligned} \tau_\gamma : T_{\gamma(a)}M &\to T_{\gamma(b)}M \\ v_0 &\mapsto V(b) \end{aligned}$$

where $V(t)$ is given by Proposition 1.3.29.

Remark 1.3.32. We see that the parallel transport is a linear isomorphism, due to the following properties of covariant derivative $D_\gamma V$ of V along γ.

- D is linear: $D_\gamma(V_1 + V_2) = D_\gamma V_1 + D_\gamma V_2$.
- D satisfies the Leibniz rule: $D_\gamma(fV) = fD_\gamma(V) + \langle df, \dot{\gamma}\rangle V$.

As opposed to Equation (1.3.26), Equation (1.3.27) is a second-order non-linear ordinary differential equation, which means the solution tends to appear only in small intervals of time. This draws the following result.

Proposition 1.3.33. Given ∇ on M and $v_0 \in T_{x_0}(M)$, there exists a unique maximal geodesic $\gamma : I \to M$ such that $\gamma(0) = x_0$ and $\dot{\gamma}(0) = v_0$, where I is an open interval containing 0.

Example 1.3.34. Consider $\mathbb{R}^n$ with flat connection ∇, a path γ is a geodesic if and only if $\ddot{\gamma}^i(t) = 0$ for all i, therefore the geodesics are straight lines.

Definition 1.3.35. A connection is *complete* if geodesics exist for all time.

Definition 1.3.36. A *geodesic* in a Riemannian manifold (M, g) is a geodesic for Levi-Civita connection.

Remark 1.3.37. Arbitrary connections on compact manifolds will not be complete. However, we will see that this will happen for Riemannian manifolds.

The following definition is motivated by Equation (1.3.28).

Definition 1.3.38. On (M, ∇), the *spray* $X^\nabla \in \mathfrak{X}(TM)$ in local coordinates (x^i, v^j) is given by

$$X^\nabla(x, v) = X^\nabla\big|_{x,v} = v^i \frac{\partial}{\partial x^i} - \Gamma^k_{ij} v^i v^j \frac{\partial}{\partial v^k}.$$

The flow of the spray is called the *geodesic flow*.

Remark 1.3.39. Let $p : TM \to M$ be the projection and $m_t(v) = tv$ be the multiplication. The spray as a vector field is the unique one satisfying

- $d_v p(X^\nabla_v) = v$, and
- $(m_t)_*(X^\nabla) = \frac{1}{t}X^\nabla$ for all $t \in \mathbb{R}_+$.

End of Lecture 5

Exercise 1.3.40. Show that

a. X^∇ is independent of the choice of local charts;

b. X^∇ satisfies two properties:
   - Given the projection of tangent bundle $p : TM \to M$, we have $d_v p(X^\nabla) = v$;
   - Given the multiplication $m_t(v) = tv$, we have $(m_t)_* X^\nabla = \frac{1}{t}X^\nabla$ for all $t > 0$;

c. any vector field $X \in \mathfrak{X}(TM)$ that satisfies part a. and b. is the spray of a connection ∇.

Remark 1.3.41. Note that Equation (1.3.27) or X^∇ only depend on the symmetric part of Γ^k_{ij}:

$$\Gamma^k_{ij}(x) v^i v^j = \frac{1}{2}\left(\Gamma^k_{ij}(x) + \Gamma^k_{ji}(x)\right) v^i v^j.$$

(Here we implicitly assume there is a summation going on, as it usually happens in Einstein notation.) Therefore, geodesics do not give a complete characterization for the torsion.

Proposition 1.3.42.

i. Given any connection ∇, there exists a unique connection $\bar{\nabla}$ that has the same geodesic as ∇, but is torsion-free, i.e., $T^{\bar{\nabla}} = 0$.

ii. Two connections ∇_1 and ∇_2 with the same geodesics and torsions coincide.

Proof. Given ∇, we can define a *dual connection* ∇^* by

$$\nabla^*_X Y = \nabla_Y X + [X, Y].$$

Indeed,

$$\begin{aligned}
\nabla^*_{fX} Y &= \nabla_Y (fX) + [fX, Y] \\
&= f\nabla_Y X + Y(f)X + f[X, Y] - Y(f)X \\
&= f\nabla^*_X Y,
\end{aligned}$$

and

$$\begin{aligned}
\nabla^*_X (fY) &= \nabla_{fY} X + [X, fY] \\
&= f\nabla_Y X + f[X, Y] + X(f)Y \\
&= f\nabla^*_X Y + X(f)Y,
\end{aligned}$$

so taking combinations give a connection $\bar{\nabla} = \frac{1}{2}(\nabla_X Y + \nabla^*_X Y)$, such that

$$T^{\bar{\nabla}}(X, Y) = 0.$$

□

Remark 1.3.43. Let $v \in T_x M$ and $\gamma_v : [0, b) \to M$ be the geodesic with $\dot{\gamma}(0) = v$. Take $\lambda > 0$, we have a parametrization $t \mapsto \gamma_{\lambda v}(t) \equiv \gamma(t)$, and

$$\begin{cases} (D_\gamma \dot{\gamma})(t) & = \lambda^2 (D_{\gamma_v} \dot{\gamma}_v)(\lambda t) = 0 \\ \dot{\gamma}(0) & = \lambda v \end{cases}$$

therefore $\gamma : \left[0, \frac{b}{\lambda}\right) \to M$ is a geodesic with $\dot{\gamma}(0) = \lambda v$, and in particular $\gamma = \gamma_{\lambda v}$. Therefore, if we choose v sufficiently small, we can choose γ so that the geodesic exists for $t \in [0, 1]$.

Definition 1.3.44. The *exponential map* is defined as

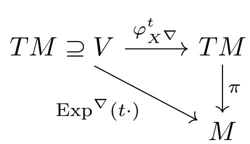
$$\begin{aligned}
\operatorname{Exp}^\nabla : V &\to M \\
v &\mapsto \gamma_v(1)
\end{aligned}$$

which exists in a neighborhood $0_M \subseteq V \subseteq TM$ containing the zero section 0_M. We denote Exp^M_x to be the map

$$\operatorname{Exp}^M_x : V \cap T_x M \to M$$

for $x \in M$.

Remark 1.3.45. The exponential map $\operatorname{Exp}^\nabla(t\cdot)$[2] cannot be a flow of a vector field, but if we take the flow of the geodesic spray, the diagram



commutes.

[2] We write "$t\cdot$" to represent an one-parameter group of diffeomorphisms.

Theorem 1.3.46. Given $x \in M$, there exists an open neighborhood $0_x \in V \subseteq T_xM$ of x and an open neighborhood $x \in U \subseteq M$ such that $\operatorname{Exp}_x^\nabla : V \to U$ is a diffeomorphism.

Proof. We need to check the differential of exponential map around 0_x is zero, then we have such a construction. That is, we need to check that

$$d_{0_x} \operatorname{Exp}_x^\nabla : T_{0_x}(T_xM) \simeq T_xM \to T_xM$$

is a linear isomorphism, which as we will see, is actually the identity map. Indeed,

$$\begin{aligned} d_{0_x} \operatorname{Exp}_x^\nabla(v) &= \frac{d}{dt} \operatorname{Exp}_x^\nabla(tv)\Big|_{t=0} \\ &= \frac{d}{dt}\gamma_{tv}(1)\Big|_{t=0} \\ &= \frac{d}{dt}\gamma_v(t)\Big|_{t=0} \\ &= v. \end{aligned}$$

□

Definition 1.3.47. The local coordinates in the chart given by Theorem 1.3.46 are called the *normal coordinates* centered at $x \in M$, i.e.,

$$U \xrightarrow{\left(\operatorname{Exp}_x^\nabla\right)^{-1}} V \subseteq T_xM \xrightarrow{\simeq} \mathbb{R}^m$$

where we choose a basis $\{e_1, \dots, e_m\}$ for T_xM to get the isomorphism.

Remark 1.3.48. In normal coordinates centered at $x \in M$,

- geodesics through x correspond to straight lines,
- geodesics through $y \neq x$ are not, in general, straight lines.

1.4 Geodesics in Riemannian Geometry

Recall that geodesics for (M, g) are just the geodesics for the Levi-Civita connection ∇.

Lemma 1.4.1. Geodesics have constant velocity.

Proof. Let $\gamma : [a, b] \to M$ be a geodesic, then

- the derivative

$$\begin{aligned} \frac{d}{dt}||\dot\gamma(t)||^2 &= \frac{d}{dt} g(\dot\gamma(t), \dot\gamma(t)) \\ &= g(D_\gamma\dot\gamma(t), \dot\gamma(t)) + g(\dot\gamma(t), D_\gamma\dot\gamma(t)) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Note $\nabla g = 0$, therefore

$$\begin{aligned} g(Y, \nabla_X Z) &= X(g(Y, Z)) - g(\nabla_X Y, Z) \\ &= \nabla_X g(Y, Z) \\ &= 0. \end{aligned}$$

□

Definition 1.4.2. For a Riemannian manifold (M, g), let ∇ be the Levi-Civita connection, then for $x \in U \subseteq M$ and $0_x \in V \subseteq T_xM$, we have a diagram

$$\begin{array}{ccccc} U & \xrightarrow{\left(\mathrm{Exp}_x^\nabla\right)^{-1}} & V & \hookrightarrow & \mathbb{R}^n \\ & & \big\downarrow & \swarrow_{\simeq} & \\ & & T_xM & & \end{array}$$

after choosing orthonormal basis $\{e_1, \ldots, e_n\}$. The coordinates given by this diagram is called the *metric normal coordinates*. Given such g_x, we build up a local chart (U, x^i).

Remark 1.4.3. In this chart, writing $g = g_{ij}(x)dx^i dx^j$ gives $g_{ij}(0) = \delta_{ij}$. At the origin, we have the Euclidean metric, but that is not true outside the origin. Instead, we get

$$\begin{aligned} g &= g_{ij}(x)dx^i dx^j \\ &= \sum_{i=1}^m (dx^i)^2 + O(2) \end{aligned}$$

is of second-order in x. Indeed, the geodesics through $x = 0$ are assigned as $t \mapsto vt$ for $v \in \mathbb{R}^n$, therefore $\Gamma_{ij}^k(0) = 0$ and $\frac{\partial g_{ij}}{\partial x^k}(0) = 0$.

End of Lecture 6

Remark 1.4.4. If $\gamma : [a, b] \to M$ is a geodesic, then we know that $||\dot{\gamma}(t)||$ is constant. Now suppose we have

$$\begin{aligned} s : [a, b] &\to [0, L(\gamma)] \\ t &\mapsto \int_0^t ||\dot{\gamma}(t)||dt, \end{aligned}$$

then we get to write

$$s(t) = \frac{L(\gamma)}{b-a}(t-a)$$

is an affine function. Therefore, such repraparametrization $\gamma = \gamma(s)$ is still a geodesic.

In general, if we choose an arbitrary reparametrization $\tau : [0, d] \to [a, b]$, then $\gamma \circ \tau$ is not a geodesic. For instance, we can take $||\dot{\overline{\gamma \circ \tau}}|| = ||\dot{\gamma}(\tau(t))|| \cdot |\tau'|$ but this may not be constant.

We saw last time the notion of normal neighborhood U for (M, g) centered at $x_0 \in M$. This is given by

$$\mathbb{R}^n \simeq T_{x_0}M \supseteq V \xrightarrow{\mathrm{Exp}_{x_0}} U \subseteq M, \qquad \varphi : U \to \mathbb{R}^n$$

landing back in $\mathbb{R}^n$ after fixing an orthonormal basis for $T_{x_0}M$. We take up the following conventions.

- The *normal sphere* is denoted $S_\varepsilon(x_0) = \{x \in U : |\varphi(x)| = \varepsilon\}$.
- The *normal ball* is denoted $B_\varepsilon(x_0) = \{x \in U : |\varphi(x) < \varepsilon|\}$.

These notions don't depend on choices. In a normal chart, the metric is given by

$$\begin{aligned} g|_U &= g_{ij}(x)dx^i dx^j \\ &= g_{ij}(0)dx^i dx^j + \frac{\partial g_{ij}(0)}{\partial x^k} x^k dx^i dx^j + \cdots \\ &= \sum_{i=1}^n (dx^i)^2 + O(2). \end{aligned}$$

In *"spherical" normal coordinates*, we write

$$\begin{cases} x^1 & = r\sin(\varphi^1) \\ x^2 & = r\cos(\varphi^1)\sin(\varphi^2) \\ \vdots & \\ x^{n-1} & = r\cos(\varphi^1)\cos(\varphi^2)\cdots\cos(\varphi^{n-2})\sin(\varphi^{n-1}) \\ x^n & = r\cos(\varphi^1)\cos(\varphi^2)\cdots\cos(\varphi^{n-1}) \end{cases}$$

for $r \in [0,\infty)$, $\varphi^1,\dots,\varphi^{n-2} \in [0,\pi]$, and $\varphi^{n-1} \in [0,2\pi]$.

Proposition 1.4.5. In spherical normal coordinates,

$$g = (dr)^2 + g_{ij}(r,\varphi^1,\dots,\varphi^{n-1})d\varphi^i d\varphi^j$$

where $g_{ij}(0,\varphi^1,\dots,\varphi^{n-1}) = 0$.

Proof. We have

$$g = g_{rr}(dr)^2 + g_{ri}drd\varphi^i + g_{ij}d\varphi^i d\varphi^J$$

where $g_{rr}(r,\varphi) = g\left(\frac{\partial}{\partial r}\right)$. Recall that the assignment $\gamma : t \mapsto tv$ is geodesic, and $\frac{\partial}{\partial r}\big|_v$ are the derivatives $\dot{\gamma}(t)$, and in particular $\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0$, then by computation we get

$$\begin{aligned} \frac{\partial}{\partial r}g_{rr} &= \frac{\partial}{\partial r}\left(g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)\right) \\ &= 2g\left(\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) \\ &= 0. \end{aligned}$$

Therefore, g_{rr} is constant along the ray, thus $g_{rr}(0,\varphi) = 1$ since $g(0) = \sum_{i=1}^{m}(dx^i)^2$, which means $g_{rr}(r,\varphi) = 1$.

We now observe that the vector fields commute, i.e.,

$$\left[\frac{\partial}{\partial r},\frac{\partial}{\partial\varphi^i}\right] = 0,$$

and since $T^\nabla = 0$, thus

$$\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial\varphi} = \nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial r}.$$

By definition, we have

$$g_{ri} = g_r\left(\frac{\partial}{\partial r},\frac{\partial}{\partial\varphi^i}\right),$$

thus

$$\begin{aligned} \frac{\partial}{\partial r}g_{ri} &= g\left(\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r},\frac{\partial}{\partial\varphi^i}\right) + g\left(\frac{\partial}{\partial r},\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial\varphi^i}\right) \\ &= 0 + \frac{1}{2}\frac{\partial}{\partial\varphi^i}g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) \\ &= 0. \end{aligned}$$

This implies that g_{ri} is constant along the ray, hence

$$g_{ri}(r,\varphi) = g_{ri}(0,\varphi) = 0.$$

□

Corollary 1.4.6. If $\gamma : [0,1] \to M$ is any curve such that $\gamma(0) = x_0$ and $\gamma(1) \in S_x(x_0)$, then

$$L(\gamma) \geqslant \varepsilon$$

and equality holds if and only if γ is a reparametrization of a geodesic (but not necessarily one itself).

Remark 1.4.7. When points are close enough, i.e., contained in the normal neighborhood, the geodesics minimize the length.

Proof. We may assume that

- $\gamma(t) \neq x_0$ for all $t \in [0,1]$ by reparametrization, since this would not affect the length,
- the curve $\gamma(t) \subseteq U$ for some normal neighborhood U of x_0.

Using sphereical normal coordinates, we get

$$\begin{aligned} L(\gamma) &= \int_0^1 ||\dot{\gamma}(t)||dt \\ &= \int_0^1 \left(\dot{\gamma}^{r^2}(t) + g_{ij}(\dot{\gamma}^{\varphi_i}(t))\dot{\gamma}^{\varphi_j}(t)\right)^2 dt \\ &\geqslant \int_0^1 |\dot{\gamma}^r(t)|dt \\ &= \gamma^r(1) - \gamma^r(0) \\ &= \varepsilon. \end{aligned}$$

In particular, the equality holds if and only if

$$\begin{cases} g_{ij}\dot{\gamma}^{\varphi_i}\dot{\gamma}^{\varphi_j} & = 0 \\ \dot{\gamma}^r(t) & > 0 \end{cases}$$

which gives

$$\begin{cases} \dot{\gamma}^{\varphi_i}(t) & = 0 \\ \dot{\gamma}^r(t) & > 0 \end{cases}$$

In this case, $\gamma^{\varphi_i}(t) = \varphi^i(0)$ is constant, then we have

$$\gamma(t) = \exp\left((\gamma^r(t), \varphi^1(0), \dots, \varphi^{n-1}(0))\right)$$

given by the exponential map acting on a reparametrization of $t \mapsto \dot{\gamma}(0)t$. □

What can we say when the sphere is huge?

Theorem 1.4.8. Suppose $\gamma : [a,b] \to M$ is a smooth curve such that $\gamma(0) = x$ and $\gamma(b) = y$, and for every piecewise smooth curve $\eta : [c,d] \to M$ with $\eta(c) = x$ and $\eta(d) = y$, one has

$$L(\eta) \geqslant L(\gamma),$$

then γ is a reparametrized geodesic.

Proof. If γ is contained in a normal neighborhood, we may apply Corollary 1.4.6. Otherwise, the intersection of γ with any normal neighborhood U satisfies the assumption of Theorem 1.4.8 for any values of parameter t such that for any $t \in [c,d]$, we have $\gamma(t) \in U$. We may then apply the local case again. □

Question. Here is a rather open question. In a Riemannian manifold (M, g) with $(x, y) \in M$ fixed, are there geodesics connecting x and y? If yes, how many? In this case geodesics mean either unparametrized geodesics or ones up to reparametrization.

End of Lecture 7

Remark 1.4.9. The proof of Theorem 1.4.8 last time actually requires more than just having normal neighborhoods. What we need is a notion of totally normal neighborhoods.

Definition 1.4.10. A *totally normal neighborhood* $U \subseteq M$ is one such that for any $x \in U$, $U \subseteq B_\varepsilon(x)$ for some $\varepsilon > 0$.

Proposition 1.4.11. Totally normal neighborhoods always exists.

Proof. We define the geodesic flow

$$\begin{aligned} \Phi : \mathbb{R} \times TM \supseteq D &\to \mathbb{R} \times TM \\ (t, v) &\mapsto (t, \varphi^t_{X^\nabla}(v)) \end{aligned}$$

For any $x_0 \in M$, we have $(0, 0_{x_0}) \in D$, and Φ is a diffeomorphism on some open V containing this point. Therefore, there exists $0_{x_0} \in \bar{V} \subseteq T_{x_0}M$ and $\varepsilon > 0$ such that $[0, \varepsilon] \times \bar{v} \subseteq V$, which means that $U = \exp_{x_0}(V)$ is a totally normal neighborhood. $\square$

Corollary 1.4.12. Geodesics contained in a totally normal neighborhood are length-minimizing.

Example 1.4.13. Given two points, is there a geodesic connecting them? How many are there precisely?

1. Suppose $M = \mathbb{R}^n$ with standard Euclidean metric g_0. The geodesics are the straight lines, therefore any two points are connected by a unique geodesic.

2. Suppose $M = \mathbb{R}^n \backslash \{0\}$ with induced metric $g = g_0|_M$. Note that the points x and $-x$ are not connected by a geodesic.

3. Suppose $M = \mathbb{S}^n \subseteq \mathbb{R}^{n+1} \backslash \{0\}$ with induced metric $g = g_0|_{\mathbb{S}^n}$. In this case, the geodesics are maximal circles. To see why,
   - take $v \in T_x\mathbb{S}^n$ and let $\gamma_v(t)$ be the geodesic with $\dot{\gamma}_v(0) = v$;
   - isometries take geodesics to geodesics;
   - let H be the 2-plane containing x and v, then set $r : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ to be the reflection on H, which is an isometry on $\mathbb{R}^{n+1}$ with g_0;
   - in particular, $r = r|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{S}^n$ is still an isometry, and in particular $\gamma(t) = r \circ \gamma_v(t)$ is a geodesic;
   - but note that γ satisfies

     $$\begin{cases} \gamma(0) &= x \\ \dot{\gamma}(0) &= dr \circ \dot{\gamma}_v(0) = v \end{cases}$$

     which means that $\gamma = \gamma_v$.

   Therefore, for any $x, y \in \mathbb{S}^n$ that are not antipodal, i.e., $y \neq -x$, there are two geodesics containing x and y. In the case where $y = -x$, there are infinitely many geodesics connecting x and y.

4. Suppose $M = \mathbb{T}^n$ and let $g = (d\theta^1)^2 + \cdots + (d\theta^m)^2$, then any two points are connected by infinitely many geodesics.

Definition 1.4.14. A geodesically-complete Riemann manifold is a Riemannian manifold (M, g) such that every maximal geodesic $\gamma : I \to M$ has $I = \mathbb{R}$.

Theorem 1.4.15 (Hopf-Rinow)**.** Given a Riemannian manifold (M, g), the following are equivalent:

i. (M, g) is geodesic-complete;

ii. (M, d) is a complete metric space, where d is the metric induced by length;

iii. there exists a point $p \in M$ such that the exponential map $\exp_x$ has domain the entire tangent space $T_x M$.

Moreover, if any of the conditions above holds, then for any $x, y \in M$, there exists a geodesic connecting x and y, with $d(x, y) = L(\gamma)$.

Remark 1.4.16. By completeness, the last condition is actually true for any point $p \in M$.

Corollary 1.4.17. Every compact Riemannian manifold is geodesically-complete.

Proof. Any compact metric space is complete. □

Corollary 1.4.18. A closed embedded submanifold N of a geodesically-complete Riemannian manifold (M, g) can be upgraded to a geodesically-complete Riemannian manifold $(N, g|_N)$.

Proof. If $\gamma : [a, b] \to M$ is a smooth curve with $\gamma(t) \in N$ for all $t \in [a, b]$, then since N is embedded, γ must also be a smooth curve in N. Therefore, for any $x, y \in N$, $d_M(x, y) \leqslant d_N(x, y)$, so a Cauchy sequence in N must be a Cauchy sequence in M. In particular, it has a subsequence that converges in M. Since N is closed and embedded, it convergences in N. Therefore, (N, d_N) is a complete metric space as well. □

Remark 1.4.19. We still keep the implicit assumption that Riemannian manifolds are connected: otherwise we may not have paths connecting two points.

We will now prove Theorem 1.4.15.

- The proof of i. implying iii. is obvious: this follows from the definition of the exponential map.
- To prove iii. implies ii., we require the following lemma.

  Lemma 1.4.20. If condition iii. of Theorem 1.4.15 holds, then for each $x, y \in M$, there exists a geodesic γ connecting x and y such that $L(\gamma) = d(x, y)$.

  To prove ii., we will show that having $K \subseteq M$ bounded and closed implying K is compact. Since K is bounded, then we have $K \subseteq B_R(x) = \{y \in M : d(x, y) \leqslant R\}$ for some R. By Lemma 1.4.20, for any $y \in K$, there exists a geodesic $\gamma : [a, b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$, with $L(\gamma) = d(x, y)$. Therefore, $K \subseteq \exp_x(\{v \in T_x M : ||v|| \leqslant R\})$, because the domain of $\exp_x$ is the entire tangent space. Note that $\{v \in T_x M : ||v|| \leqslant R\}$ is compact, and since $\exp_x$ is continuous, then $\exp_x(\{v \in T_x M : ||v|| \leqslant R\})$ is compact as well. Being a closed subset of a compact set, we note that K is compact as well.

- To prove that ii. implies i., let $\gamma : [a, b) \to M$ be a geodesic. Assume for now that $b < \infty$, then we have an increasing sequence $\{t_n\}_{n \geqslant 1}$ converging to b. The geodesic γ has the property $||\dot{\gamma}(t)|| = c$, so by reparametrization $s = \frac{t}{c}$, we assume $||\dot{\gamma}(s)|| = 1$. Therefore, we have

$$\begin{aligned} d(\gamma(t_n), \gamma(t_{n+1})) &\leqslant L(\gamma|_{[t_n, t_{n+1}]}) \\ &= \int_{t_n}^{t_{n+1}} ||\dot{\gamma}(t)|| dt \\ &= t_{n-1} - t_n \\ &\xrightarrow[n \to \infty]{} 0 \end{aligned}$$

  Since the sphere of radius 1 is compact, and $||\dot{\gamma}(t_m)|| = 1$, then there exists a converging subsequence $\dot{\gamma}(t_{n_k}) \to v$. In particular, $(\gamma(t)_{n_k}, \dot{\gamma}(t_{n_k}))$ converges, therefore $(\gamma(t), \dot{\gamma}(t))$ is an integral curve X^∇ that is bounded as $t \to b$, hence $(\gamma(t), \dot{\gamma}(t))$ exists in the interval $[a, b + \varepsilon)$ for some ε. But that means (a, b) is not maximal, contradiction. Therefore, $b = \infty$. Similar proof shows that $a = -\infty$.

End of Lecture 8

We omitted the proof of Lemma 1.4.20 in class, but we record it here for completeness.

Supplement. *Let $\rho = d(x, y)$. Choose $0 < \varepsilon < \rho$ such that $S_\varepsilon(x)$ is a normal sphere. This is compact, therefore there exists $x_0 \in S_\varepsilon(x)$ such that*

$$d(x_0, y) = \min\{d(z, y) : z \in S_\varepsilon(x_0)\}.$$

By the definition of a normal sphere, there exists some $v \in T_xM$ such that $||v|| = 1$ and $\exp_x(\varepsilon v) = x_0$. We claim that $y = \exp_x(\rho v)$, therefore $\gamma(t) = \exp_x(tv)$ is the desired geodesic. To prove this, let

$$A = \{t \in [0, \rho] : d(\exp_x(tv), y) = \rho - t\} \subseteq \mathbb{R}.$$

Since the assignment $f : t \mapsto d(\exp_x(tv), y) + t$ is continuous, then $A = f^{-1}(\rho)$ is closed, bounded, and non-empty since $0 \in A$. Therefore, A has a maximum. If we can show that any $t_0 \in [0, \rho)$ is not a maximum, then ρ must be a maximum. If that is the case, then $\rho \in A$, thus $d(\exp_x(\rho v), y) = 0$, hence $\exp_x(\rho v) = y$.

We may assume, towards contradiction, that $t_0 = \max(A) \in [0, \rho)$, and set $y_0 = \exp(t_0 v)$, then choose $\delta \in (0, \rho - t_0)$ such that $S_\delta(y_0)$ is a normal sphere. Let z_0 be such that $d(z_0, y) = \min\{d(z, y) : z \in S_\delta(y_0)\}$. It suffices to show that $z_0 = \exp_x((t_0 + \delta)v)$, and $d(z_0, y) = \rho - (t_0 + \delta)$, then $t_0 \neq \max(A)$. We note that

$$\begin{aligned} \rho - t_0 &= d(y_0, y) \\ &= \delta + \min_{z \in S_\delta(y_0)} d(z, y) \\ &= \delta + d(z_0, y), \end{aligned}$$

therefore

$$\begin{aligned} d(x, z_0) &\geqslant d(x, y) - d(z_0, y) \\ &= \rho - d(z_0, y) \\ &= t_0 + \delta. \end{aligned}$$

Set $z_0 = \exp_{y_0}(\delta w)$ for some w, then the curve

$$\exp_{x, 0<t<t_0}(tv) \cup \exp_{x, 0<t<\delta}(tw)$$

has length $t_0 + \delta$. In particular, this must be a reparametrized geodesic, therefore $\gamma(t) = \exp_x(tv)$ is a geodesic through z_0, so $\exp_x((t_0 + \delta)v) = z_0$, as desired.

Remark 1.4.21.

1. How do we count geodesics? We can show that in a complete Riemannian manifold (M, g), in every path-homotopy class, there exists a geodesic that minimizes the length among all curves in the class.
2. Every manifold admits a geodesically-complete metric. In fact, given any metric g, there exists a geodesically-complete metric $g' = fg$ for some non-negative function $0 < f \in C^\infty$. We say that g' is *conformal* to g, and f is a *conformal factor*. That is, in every conformal class, there exists a geodesically-complete metric.
3. Instead of curves, what about other submanifolds, e.g., minimizing surfaces? This is more of the current research.

1.5 Curvature

We saw that on normal charts, a metric g starts with a Euclidean metric with error term has of order at least 2. How do we measure this higher-order term, i.e., the failure of g being locally equivalent to Euclidean metric? More generally, how can we measure the failure of the connection ∇ being locally equivalent to the standard connection with $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$?

Definition 1.5.1. The *curvature* of a connection ∇ is

$$R^\nabla(X, Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Remark 1.5.2.

- This is a $(3,1)$-tensor, i.e., 3-covariant, 1-contravariant tensor, given by

$$\begin{aligned}\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\times\Omega^1(M)&\to C^\infty(M)\\ (X,Y,Z,\alpha)&\mapsto\left\langle R^\nabla(X,Y)Z,\alpha\right\rangle\end{aligned}$$

  and is $C^\infty(M)$-linear in each entry.

- There is an assignment

$$\begin{aligned}TM&\to TM\\ (X,Y)&\mapsto R^\nabla(X,Y)\end{aligned}$$

  such that $R^\nabla(X,Y)=-R^\nabla(Y,X)$ for $R^\nabla\in\Omega^2(M,\operatorname{End}(TM))$.

- For Euclidean connection ∇, $R^\nabla\equiv 0$.

Theorem 1.5.3 (Bianchi's Identity)**.** If $T^\nabla=0$, then the cyclic permutations of (X,Y,Z)

$$R^\nabla(X,Y)Z+\operatorname{cycPerm}(X,Y,Z):=R^\nabla(X,Y)Z+R^\nabla(Y,Z)X+R^\nabla(Z,X)Y=0.$$

Proof. Note that $T^\nabla=0$ if and only if $\nabla_XY-\nabla_YX=[X,Y]$, therefore we may compute

$$R^\nabla(X,Y)Z+\operatorname{cycPerm}(X,Y,Z)=[X,[Y,Z]]+\operatorname{cycPerm}(X,Y,Z)=0$$

by Jacobi identity. □

In a local chart (U,x^i), we have

$$R^\nabla\big|_U=R^\ell_{ijk}(x)dx^i\otimes dx^j\otimes dx^k\otimes\frac{\partial}{\partial x^\ell}$$

where $R^\ell_{ijk}=\left\langle R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k},dx^\ell\right\rangle$ with the property that

- $R^\ell_{ijk}=-R^\ell_{jik}$, and
- $R^\ell_{ijk}+R^\ell_{jki}+R^\ell_{kij}=0$.

We now give a geometric interpretation of the curvature, using covariant derivative along paths. Denote

$$\gamma:[0,1]\times[0,1]\to M$$

parametrized by (t,ε), then we get two families of disjoint curves intersecting each other that parametrizes the surface, given by

$$\begin{aligned}\gamma_\varepsilon:[0,1]&\to M\\ t&\mapsto\gamma_\varepsilon(t)\end{aligned}$$

and

$$\begin{aligned}\gamma_t:[0,1]&\to M\\ \varepsilon&\mapsto\gamma_t(\varepsilon)\end{aligned}$$

This allows us to define a vector field along γ

$$c:[0,1]\times[0,1]\to TM$$

where $c(t,\varepsilon)\in T_{\gamma(t,\varepsilon)}M$.

Proposition 1.5.4. For a connection ∇, the covariant derivative

$$T^{\nabla}(\dot{\gamma}_{\varepsilon}, \dot{\gamma}_t) = D_{\gamma_{\varepsilon}}\dot{\gamma}_t - D_{\gamma_t}\dot{\gamma}_{\varepsilon},$$

and

$$R^{\nabla}(\dot{\gamma}_{\varepsilon}, \dot{\gamma}_t)c = D_{\gamma_{\varepsilon}}D_{\gamma_t}c - D_{\gamma_t}D_{\gamma_{\varepsilon}}c.$$

Remark 1.5.5. Here $\dot{\gamma}_t$ is a vector field along γ_{ε} so we get to derive it, and the other way around.

We will postpone the proof of Proposition 1.5.4 because it will be a mess: taking a time-dependent parametrization will introduce a third variable, c.f., [Spi70]. Instead, after learning about pullback connections of vector bundles, we will come back to this: see Proposition 2.5.15.

Corollary 1.5.6. If ∇ is flat, i.e., $R^{\nabla} \equiv 0$, then parallel transport is invariant under path-homotopy. That is, if $\gamma_0 \sim \gamma_1$ is a path-homotopy, then $\tau_{x_0} = \tau_{x_1}$.

Proof. Let us take $\gamma : [0, 1] \times [0, 1] \to M$ be a path-homotopy between γ_0 and γ_1. We will assume that γ is C^{∞}: any C^0 path-homotopy can then by approximated by the smooth homotopies. We now have

$$\begin{cases} \gamma_0(t) = \gamma(t, 0), & \gamma(0, \varepsilon) = x_0 \\ \gamma_1(t) = \gamma(t, 1), & \gamma(1, \varepsilon) = x_0 \end{cases}.$$

Given a tangent vector $v_0 \in T_{x_0}M$, we define

$$\begin{aligned} c : [0, 1] \times [0, 1] &\to TM \\ (t, \varepsilon) &\mapsto \tau^t_{\gamma_{\varepsilon}}(v_0), \end{aligned}$$

which is equivalent to saying $D_{\gamma_{\varepsilon}}c = 0$. Since $c(0, \varepsilon) = v_0$, then

$$D_{\gamma_0(\varepsilon)=\gamma_{t=0}}c = 0,$$

and we want to show that $c(1, \varepsilon)$ is constant, i.e., $D_{\gamma_{t=1}}c = 0$. Because $R^{\nabla} = 0$, then

$$D_{\gamma_{\varepsilon}}D_{\gamma_t}c = D_{\gamma_t}D_{\gamma_{\varepsilon}}c = 0.$$

In particular, $D_{\gamma_{t=1}}c = 0$. □

Example 1.5.7. We can give an example where Corollary 1.5.6 fails if we remove the assumption of path-homotopy invariance. Take $M = \mathbb{T}^2$ and connection ∇ with

$$\begin{cases} \nabla_{\frac{\partial}{\partial x^1}}dx^1 = \nabla_{\frac{\partial}{\partial x^2}}dx^1 = 0 \\ \nabla_{\frac{\partial}{\partial x^1}}dx^2 = dx^2, \nabla_{\frac{\partial}{\partial x^2}}dx^2 = dx^2 = dx^1. \end{cases}$$

and consider the monodromy $\ell^{\nabla} : \pi_1(M, x_0) \to \mathrm{GL}(T_{x_0}M)$.

End of Lecture 9

Note that the curvature for connection ∇ is defined by

$$R^{\nabla}(X, X) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Definition 1.5.8. For a Riemannian manifold (M, g), the *Riemannian curvature tensor* is the 4-covariant tensor

$$R(x, y, z, w) = g(R^{\nabla}(X, Y)(Z), W).$$

In local coordinates (U, x^i), we can write

$$R^{\nabla} = R^{\ell}_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^\ell}$$

for

$$R^{\ell}_{ijk} = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right\rangle$$

and

$$R = R_{ijk\ell} dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell$$

for

$$R_{ijk\ell} = R\left(\partial_{x^i}, \partial_{x^j}, \partial_{x^k}, \partial_{x^\ell}\right) = g(R(\partial_{x^i}, \partial_{x^j})\partial_{x^k}, \partial_{x^\ell}) = g_{\ell m} R^m_{ijk}$$

where we write $\partial_{x^j} = \frac{\partial}{\partial x^j}$.

We have the following symmetries of R.

Proposition 1.5.9.

i. Bianchi's Identity: $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$.

ii. $R(X, Y, Z, W) = -R(Y, X, Z, W)$.

iii. $R(X, Y, Z, W) = -R(X, Y, W, Z)$.

iv. $R(X, Y, Z, W) = R(Z, W, X, Y)$.

Proof. We have already seen that the first two are true. We will prove iii. and iv.

iii. It is enough to show that $R(X, Y, Z, Z) = 0$ by polarity: if this holds, then

$$\begin{aligned} 0 &= R(X, Y, Z + W, Z + W) \\ &= R(X, Y, Z, Z) + R(X, Y, Z, W) + R(X, Y, W, Z) + R(X, Y, W, W) \\ &= 0 + R(X, Y, Z, W) + R(X, Y, W, Z) + 0 \\ &= R(X, Y, Z, W) + R(X, Y, W, Z). \end{aligned}$$

Since $\nabla g = 0$, then

$$\begin{cases} X(g(\nabla_Y Z, Z)) &= g(\nabla_X \nabla_Y Z, Z) + g(\nabla_Y Z, \nabla_X Z) \\ [X, Y](g(Z, Z)) &= 2g(\nabla_{[X,Y]} Z, Z). \end{cases}$$

Therefore,

$$\begin{aligned} R(X, Y, Z, Z) &= g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, Z) \\ &= X(g(\nabla_Y Z, Z)) - Y(g(\nabla_X Z, Z)) - \frac{1}{2}[X, Y]g(Z, Z) \\ &= \frac{1}{2}X(Y(g(Z, Z))) - \frac{1}{2}Y(X(g(Z, Z))) - \frac{1}{2}[X, Y](g(Z, Z)) \\ &= 0. \end{aligned}$$

iv. Apply Bianchi's identity (with appropriate signs) four times to ii. and iii.

□

Remark 1.5.10. We also have a point of view that characterize the curvature as an operator. By ii. and iii. of Proposition 1.5.9, we have

$$\tilde{\rho} : \Lambda^2 T_p M \otimes \Lambda^2 T_p M \to \mathbb{R}$$

$$(X \wedge Y, Z \wedge W) \mapsto R(X, Y, Z, W)$$

By iv., $\tilde{\rho}$ is a symmetric bilinear form on the vector space $\Lambda^2 T_p M$. Now g_p induces inner product on $\Lambda^2 T_p M$:

$$g_p(X \wedge Y, Z \wedge W) := \det \begin{pmatrix} g_p(X, Z) & g_p(X, W) \\ g_p(Y, Z) & g_p(Y, W) \end{pmatrix}$$

Therefore, we have the *curvature operator* $\rho : \Lambda^2 TM \to \Lambda^2 TM$ defined by

$$\Lambda^2 T_p M \xrightarrow{\tilde{\rho}} (\Lambda^2 T_p M)^* \xrightarrow{g_p} \Lambda^2 T_p M$$

where we use the metric to identify the dual with the vector space.

A different incarnation of curvature would be the sectional curvature.

Definition 1.5.11. The *sectional curvature* of a 2-plane generated by $v, w, \in T_p M$ is given by

$$K_p(v \wedge w) := \frac{R(v, w, w, v)}{||v \wedge w||^2} = \frac{R(v, w, w, v)}{g(v, v) g(w, w) - g(v, w)^2}.$$

Remark 1.5.12. This is not a linear map in v and w, but it associates a 2-plane with a real number. The 2-planes give a Grassmannian in the tangent bundle, therefore we can think of this as $K : \mathrm{Gr}_2(TM) \to \mathbb{R}$.

Proposition 1.5.13. The sectional curvature completely determines the Riemannian curvature tensor.

Proof. This is proven from the following observations.

- If R_1 and R_2 are tensors satisfying all the symmetries in Proposition 1.5.9, then so does their difference $R_1 - R_2$.
- If R satisfies Proposition 1.5.9 and
$$R(X, Y, Y, X) = 0$$
for all vector fields X, Y, then $R = 0$.

The first observation is obvious. We will prove the second observation using polarity. We have

$$\begin{aligned} 0 &= R(X + Z, Y, Y, X + Z) \\ &= R(X, Y, Y, X) + R(X, Y, Y, Z) + R(Z, Y, Y, X) + R(Z, Y, Y, Z) \\ &= 0 + R(X, Y, Y, Z) + R(Z, Y, Y, X) + 0 \\ &= R(X, Y, Y, Z) + R(Z, Y, Y, X) \\ &= 2R(X, Y, Y, Z), \end{aligned}$$

thus $R(X, Y, Y, Z) = 0$. We then take

$$\begin{aligned} 0 &= R(X, Y + Z, Y + Z, W) \\ &= R(X, Y, Y, W) + R(X, Y, Z, W) + R(X, Z, Y, W) + R(X, Z, Z, W) \\ &= 0 + R(X, Y, Z, W) + R(X, Z, Y, W) + 0 \\ &= R(X, Y, Z, W) + R(X, Z, Y, W), \end{aligned}$$

hence $R(X, Y, Z, W) = R(Z, X, Y, W)$. By Bianchi's identity,

$$R(X, Y, Z, W) = R(Y, Z, X, W) = R(Z, X, Y, W) = 0.$$

□

Remark 1.5.14. If $\dim(M) = 2$, then there is only one 2-plane, hence the Grassmannian is canonically $\mathrm{Gr}_2(TM) \simeq M$, and the sectional curvature becomes a function $K : M \to \mathbb{R}$, known as the *Gaussian curvature*. Again, this function completely determines the Riemannian curvature tensor in the case of $\dim(M) = 2$, then

$$R(X, Y, Z, W) = -K(g(X, Z) g(Y, W) - g(X, W) g(Y, Z)) \tag{1.5.15}$$

since we compute the Riemannian curvature tensor of $g(X, Z) g(Y, W) - g(X, W) g(Y, Z)$ to be -1. This ensures the 2-sphere has a positive sectional curvature.

Definition 1.5.16. A Riemannian manifold (M, g) is *isotropic* at a point $p \in M$ if the sectional curvature K_p at the point p is constant. We say (M, g) has *constant curvature* if it is isotropic and sectional curvature does not depend on p.

Remark 1.5.17. For any isotropic (M, g), the Riemannian curvature tensor is given by Equation (1.5.15).

Exercise 1.5.18. If $\dim(M) \geqslant 3$, then (M, g) being isotropic (at every point) implies constant curvature.

Example 1.5.19.

1. $\mathbb{R}^n$ with the flat metric $g_0 = \sum_{i=1}^{n}(dx^i)^2$ has $R \equiv 0$, therefore it has constant curvature 0.

2. Consider the n-sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\} \hookrightarrow \mathbb{R}^{n+1}$ with $g = g_0|_{\mathbb{S}^n}$. The orthonormal group $\mathrm{SO}(n+1)$ acts on $\mathbb{R}^{n+1}$ by isometries, but the action also preserves the sphere: $\mathrm{SO}(n+1)$ acts on $\mathbb{S}^n$ by isometries. For instance, fix the north pole $p = (0, \ldots, 0, 1) \in \mathbb{S}^n$, then the isometry group or stabilizer group is given by

$$\mathrm{SO}(n+1)_p = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathrm{SO}(n) \right\}.$$

Therefore, $\mathrm{SO}(n+1)_p \simeq \mathrm{SO}(n)$ gives an action on $T_p\mathbb{S}^n$ preserving the inner product g_p at the point p. As a vector space, $T_p\mathbb{S}^n$ is just $\mathbb{R}^n$, then the action acts transitively on 2-planes of $T_p\mathbb{S}^n$. In particular, K_p is constant (which can be calculated to be 1) at every point $p \in M$, which means $\mathbb{S}^n$ is of constant curvature 1.

More generally, taking a sphere of radius R gives radius $\frac{1}{R}$ for $\mathbb{S}^n_R$.

We see that a submanifold with restricted metric can have different curvature from the manifold.

End of Lecture 10

Remark 1.5.20. If $\dim(M) \geqslant 3$, then $K(P)$ is the Gaussian curvature of $\exp(P)$, where P is the span by vectors.

Example 1.5.21.

1. Consider the hyperbolic space $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : (x, x) = -1\}$ with Minkowski bilinear form $(v, w) = -v^0w^0 + \sum_{i=1}^{n} v^iw^i$. This is not inner product: it is not positive-definite. To get a bilinear form, we note there is an inclusion $i : \mathbb{H}^n \hookrightarrow \mathbb{R}^{n+1}$, where on $\mathbb{R}^{n+1}$ we take the metric

$$g = -(dx^0)^2 + \sum_{i=1}^{m}(dx^i)^2$$

as a symmetric bilinear tensor, then we pullback along i to get a Riemannian metric $(\mathbb{H}^n, i^*g)$.

Exercise 1.5.22. Check that $T_x\mathbb{H}^n = \{v \in \mathbb{R}^{n+1} : (x, v) = 0\}$, and $g|_{T_x\mathbb{H}^n}$ is actually positive-definite, which gives a Riemannian metric i^*g.

The linear transformations $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, such that $(Ax, Ay) = (x, y)$ for all x, y with $\det(A) = 1$, is denoted by $\mathrm{SO}(n, 1)$. We denote $G := \mathrm{SO}(n, 1)^\circ$ to be its connected component of the identity. This admits a (transitive) $\mathrm{SO}(n, 1)^\circ$-action on $\mathbb{H}^n$ by isometries of the hyperbolic space. Fixing a point $x \in \mathbb{H}^n$, we note that G_x is the isotropy group acting on $(T_x\mathbb{H}^n, g_x)$ by isometries. This action is still transitive on 2-planes, therefore it has constant sectional curvature.

Remark 1.5.23. Applying the stereographic projection, we get a disk

$$D = \{x \in \mathbb{R}^n : |x| < 1\}$$

with a metric

$$g_D = \frac{4((dx^1)^2 + \cdots + (dx^n)^2)}{(1 - ||x||^2)^2}.$$

The isometry group is still the group G denoted above, just with a different notion of action on the disk model.

2. Suppose G is a Lie group with bi-invariant metric, i.e., a metric that is both left-translation invariant and right-translation invariant, then we have a Levi-Civita connection

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

for left-invariant vectors fields $X, Y \in \mathfrak{X}_{\text{left-invariant}}(G) \simeq \mathfrak{g}$. One can compute $R^\nabla(X, Y)Z = \frac{1}{4}[Z, [X, Y]]$ in terms of left-invariant vector fields. Since the metric is bi-invariant, then

$$g([Z, X], Y) + g(X, [Z, Y]) = 0$$

and therefore the Riemannian curvature tensor is

$$R(X, Y, Z, W) = g(R^\nabla(X, Y)Z, W) = -\frac{1}{4}g([X, Y], [Z, W]).$$

We may then calculate the sectional curvature to be non-negative since

$$R(X, Y, Y, X) = \frac{1}{4}||[X, Y]||^2.$$

The sectional curvature may not be constant. For abelian Lie group, this is indeed constant.

Definition 1.5.24. We define the *Ricci curvature* $\mathrm{Ric}(Y, Z)$ to be the trace of Riemannian curvature using the metric g. That is, if we consider the assignment

$$\begin{aligned} TM &\to TM \\ X &\mapsto R^\nabla(X, Y)Z \end{aligned}$$

this is a linear transformation. In particular, every linear map has a trace, therefore the precise definition would be

$$\mathrm{Ric}(Y, Z) = \mathrm{tr}(X \mapsto R^\nabla(X, Y)Z).$$

This is a symmetric covariant tensor.

Remark 1.5.25. Being a symmetric covariant tensor,

1. Ric is completely determined by the quadratic $Q(x) = \mathrm{Ric}(X, X)$ by the polarity argument;
2. in dimension 2 (or in the isotropic case: $\mathrm{Ric}(X, Y) = K(n-1)g(X, Y)$ for $n = \dim(M)$), we have

$$\mathrm{Ric}(X, Y) = Kg(X, Y).$$

following from Equation (1.5.15);

3. one can show that $Q(x) = \mathrm{Ric}(X, X)$ is the average of sectional curvature $K(P)$ for $X \subseteq P \subseteq T_x M$. Therefore, Ricci curvature does not determine the curvature tensor or the sectional curvature in general (i.e., for $\dim(M) \geqslant 3$, but this in fact still holds for $\dim(M) = 3$).
4. By definition, the Ricci curvature only depends on the connection, so it is defined more generally than the Riemannian manifolds.

In the remark above, we notice that the Ricci curvature is proportional to the metric.

Definition 1.5.26. An *Einstein metric* g is one such that

$$\mathrm{Ric} = cg$$

where c is the cosmological constant.

Remark 1.5.27. If M has constant sectional curvature, then $c = K(n-1)$ where $n = \dim(M)$. Therefore, constant sectional curvature implies Einstein metric, but not the other way around, e.g., *Fubini-Study metric* in $\mathbb{CP}^n$, c.f., Example 1.6.4 and Remark 1.6.5.

Given a bilinear form $\mathrm{Ric} : T_xM \times T_xM \to \mathbb{R}$, we get a mapping

$$\begin{aligned} T_xM &\to T_x^*M \\ v &\mapsto \mathrm{Ric}_x(v, -). \end{aligned}$$

This defines a mapping L_x via

$$T_xM \longrightarrow T_x^*M \xrightarrow{g} T_xM$$

Definition 1.5.28. The *scalar curvature* of (M, g) is defined by to be

$$\begin{aligned} S : M &\to \mathbb{R} \\ x &\mapsto \mathrm{tr}_g(\mathrm{Ric}_x) = \mathrm{tr}(L_x). \end{aligned}$$

In terms of local coordinates, we may express

$$\mathrm{Ric} = R_{ij} dx^i \otimes dx^j$$

where

$$R_{ij}(x) = R^\ell_{\ell ij}(x) = g^{\ell m} R_{\ell mij},$$

and we take the convention

$$dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i),$$

therefore $g^{\ell m}$ is just the inverse of $g = g_{ij} dx^i dx^j$. With this, we can now write

$$S(x) = g^{ij} R_{ij}$$

for the scalar curvature.

End of Lecture 11

Remark 1.5.29. Let us recall the following notions of curvature we have seen so far.

- The Riemann curvature tensor R, corresponding to the sectional curvature K.
- The Ricci tensor Ric, given by $\mathrm{Ric}(Y, Z) = \mathrm{tr}(X \mapsto R^\nabla(X, Y)Z)$.
- The scalar tensor $S = \mathrm{Tr}_g \mathrm{Ric}$.

The natural question being, why these tensors specifically? This is because the curvature tensor R has symmetries $S^2(\Lambda^2 V)$ that encode all of them except Bianchi's identity, and to encode this identity, we have a map

$$\begin{aligned} L : S^2(\Lambda^2 V) &\to \Lambda^4 V \\ L(R)(X, Y, Z, W) &= R(X, Y, Z, W) + \mathrm{cycPerm}(X, Y, Z) \end{aligned}$$

This motivates us to examine $\ker(L)$. The $\mathrm{O}(m)$-action on $V = (T_xM, g_x)$ gives an action on $S^2(\Lambda^2 V)$, which in turn lifts into an action on curvature tensor $R \in \ker(L)$. We may then decompose the $\mathrm{O}(m)$-action on $\ker(L)$ into irreducible subspaces

$$\ker(L) = V_0 \oplus V_1 \oplus V_2.$$

The corresponding decomposition of R is the following:

$$\begin{aligned} R(X, Y, Z) = &-\frac{S}{m(m-1)}\left(g(X, Y)g(Y, W) - g(X, W)g(Y, Z)\right) \\ &- \frac{1}{m-2}\left(\mathrm{Ric}_0(X, Z)g(Y, W) + \mathrm{Ric}_0(Y, W)g(X, Z) - \mathrm{Ric}_0(X, W)g(Y, Z) - \mathrm{Ric}_0(Y, Z)g(X, W)\right) \\ &+ W(X, Y, Z, W) \end{aligned}$$

where $\mathrm{Ric}_0(X, Y) = \mathrm{Ric}(X, Y) - \frac{1}{m} S g(X, Y)$ is the *traceless Ricci tensor*. We see that the three components corresponding to V_0, V_1, and V_2, and they are called the scalar curvature component, traceless Ricci component, and the Weyl tensor.

Remark 1.5.30.

- When the scalar curvature component is 0, R is called scalar flat.
- When the traceless Ricci component is 0, R is called Einstein.
- When both the traceless Ricci component and the Weyl tensor is 0, R is said to be isotropic.

1.6 Quotients and Isometry Groups

Given a Riemannian manifold (M, g) and a surjective submersion $\Phi : M \to N$, then M is like a quotient of N by some smooth equivalence relation. How do we build up an induced metric on N using the quotient and g? The general answer would be no, but we would like to understand when we can get one.

Suppose q is a point in N, with tangent space T_qN, then via $d_p\Phi$, it corresponds to $\Phi^{-1}(q)$ upstairs, where p is a point upon it. Other than the tangent space T_pM, we can look at the orthogonal complement $H_p = (\ker(d_p\Phi))^\perp$ as well, which gives an isomorphism

$$d_p\Phi : H_p = (\ker(d_p\Phi))^\perp \simeq T_qN,$$

so we would like to build up the metric on T_qN. The issue being, there are multiple points in the fiber. This motivates the following definition.

Definition 1.6.1. A Riemannian submersion is a submersion

$$\Phi : (M, g) \to (N, \bar{g})$$

such that

$$\bar{g}_{\Phi(p)}(d_p\Phi(v), d_p\Phi(w)) = g_p(v, w)$$

for all $v, w \in (\ker(d_p\Phi))^\perp$. This definition does not depend on $\bar{g}$: it is completely determined by the structure on g.

Remark 1.6.2. This is not just a pullback. This definition is talking about the inverse of the metric, i.e., given on the cotangent space. That is, given the corresponding injective map

$$(d_p\Phi)^* : T^*_{\Phi(p)}N \to T^*_pN$$

with metric g_p^{-1} downstairs, and the theorem says the corresponding metric matches.

Here is one way of getting a Riemannian submersion.

Theorem 1.6.3. Let G be a Lie group that acts on (M, g) properly and freely, and by isometries. By the assumption, the orbit space is a manifold, then the map to the orbit space $\pi : M \to M/G$ is a Riemannian submersion for a unique Riemannian metric $\bar{g}$ on M/G.

Proof. Set $N = M/G$, then the fibers of $q \in M/G$ are exactly given by the orbit of the action, i.e.,

$$\pi^{-1}(q) = O.$$

Fix $k \in G$, then

$$\begin{aligned} \Psi_k : M &\to M \\ x &\mapsto k \cdot x \end{aligned}$$

In particular, it sends the orbit into itself, i.e., $\Psi_k(O) \subseteq O$, and moreover, $d_p\Psi_k : T_pM \to T_{k\cdot p}M$ is an isometry. Note that the tangent space $T_pO = \ker(d_p\pi)$ by definition of the orbit space, then $d_p\Psi_k(T_pO) = T_{k\cdot p}O$, thus it must map the orthogonal to the orthogonal:

$$d_p\Psi_k((T_pO)^\perp) = (T_{k\cdot p}O)^\perp,$$

and in particular it maps the metric restricted to the orthogonal to the metric restricted to the orthogonal, i.e., preserves restriction $g|_{TO}$ for every $k \in G$. Therefore, π is a Riemannian submersion. ☐

Example 1.6.4.

1. Let G be a Lie group with a right-invariant Riemannian metric[3], and let $H \subseteq G$ be a closed subgroup. Now H acts on G via right translations $h \cdot k = kh^{-1}$. Therefore, this action is given by isometries, because G is right-invariant. By Theorem 1.6.3, there exists a unique metric $\bar{g}$ on G/H such that $\pi : (G, g) \to (G/H, \bar{g})$ is a Riemannian submersion.

2. We now apply this to construct a metric on $M = \mathbb{CP}^n$. This is the quotient $\mathbb{C}^{n+1}/\mathbb{C}^*$, given by lines on $\mathbb{C}^{n+1}$ with identification by non-zero multiplication. Since $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$, we equip it with a Euclidean metric g_0. The issue being, $\mathbb{C}^* \simeq \mathbb{R}_+ \times \mathbb{S}^1$ is given by dilations and rotations, but the dilations on $\mathbb{R}^{2n+2}$ will not be isometries, i.e., does not preserve the inner products, thus g_0 is not dilation invariant. Instead, we think of $\mathbb{CP}^n$ as a sphere, then it is only given by a $\mathbb{S}^1$-action, i.e., $\mathbb{CP}^n \simeq \mathbb{S}^{2n+1}/\mathbb{S}^1$. In this case, $(\mathbb{S}^{2n+1}, g_{\mathbb{S}^{2n+1}} := g_0|_{\mathbb{S}^{2n+1}})$ has an $\mathbb{S}^1$-action by isometries, which gives the *Fubini-Study metric* $(\mathbb{CP}^n, g_{\mathbb{CP}^n})$.

Remark 1.6.5. Here are some properties of the Fubini-Study metric.

1. This is an Einstein metric: $\mathrm{Ric} = (2n+1)g_{\mathbb{CP}^n}$. This implies having constant scalar curvature S.

2. However, the sectional curvature is not constant (when $n > 1$):
$$K(P) = 1 + 3g_{\mathbb{CP}^n}(X, JY)^2$$
where $\{X, Y\}$ is an orthonormal basis of the plane P, and J is the complex structure given at every point, i.e., $J_p : T_pM \to T_pM$ is such that $J_p^2 = -I$. For $n = 1$, the curvature is constant.

   Note that all the structures we are considering here are over $\mathbb{R}$. The analog of sectional curvature over $\mathbb{C}$ is the holomorphic sectional curvature, and in which case we take $Y = JX$ on the complex line P, and in that case we have constant holomorphic sectional curvature 4. In particular, we verify that statement of Remark 1.5.27: Einstein metric does not necessarily have constant sectional curvature.

Lemma 1.6.6. Let (M, g) be a Riemannian manifold and $X \in \mathfrak{X}(M)$, then the following are equivalent:

1. $\varphi_X^t : M \to M$ is a local isometry;
2. $L_X g = 0$;
3. $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all $Y, Z \in \mathfrak{X}(M)$.

Definition 1.6.7. A vector field X satisfying any condition in Lemma 1.6.6 is called a *Killing vector field* or an *infinitesimal isometry* of (M, g). We denote $\mathfrak{X}(M, g) \subseteq \mathfrak{X}(M)$ to be the linear subspace of Killing vector fields.

End of Lecture 12

Proof.

1. $\iff$ 2.: Note that
$$\begin{aligned}(\varphi_X^t)^* g = g &\iff \frac{d}{dt}(\varphi_X^t)^* g = 0\\ &\iff (\varphi_X^t)^*(L_X g) = 0\\ &\iff L_X g = 0.\end{aligned}$$

1. $\iff$ 3.: We have
$$\begin{aligned}(L_X g)(Y, Z) = 0 &\iff X(g(Y, Z)) = g(L_X Y, Z) + g(Y, L_X Z)\\ &\iff X(g(Y, Z)) = g([X, Y], Z) + g(Y, [X, Z])\\ &\qquad\qquad\qquad = g(\nabla_X Y - \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X)\\ &\iff X(g(Y, Z)) - g(\nabla_X Y - Z) - g(Y, \nabla_X Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)\\ &\iff g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0\end{aligned}$$

where the last equivalence holds since $(\nabla_X g)(Y, Z) = 0$.

□

[3] A Lie group always has a right-invariant structure. We can also construct the left-invariant structure instead, but not a bi-invariant one, c.f., Homework 1.

Remark 1.6.8. $\mathfrak{X}(M,g)$ is a Lie subalgebra of $\mathfrak{X}(M)$:

$$L_{[X,Y]} = L_X L_Y - L_Y L_X.$$

Corollary 1.6.9. Let G be a connected Lie group, then a G-action on (M,g) is by isometries if and only if the *infinitesimal action*

$$\rho : \mathfrak{g} \to \mathfrak{X}(M)$$

takes values in $\mathfrak{X}(M,g)$.

Remark 1.6.10. Given a point on the corresponding Lie algebra, the infinitesimal action is really defined as in the proof below: we take a one-parameter family of Lie group, therefore taking the derivative at 0 we get the infinitesimal action which gives the tangent space at the point: in this case we retrieve the vector field in $\mathfrak{X}(M)$.

Proof. G is generated by elements of the form $\exp(X)$ with $X \in \mathfrak{g}$. Since $\rho(X)$ is the infinitesimal action

$$\rho(X)|_X = \frac{d}{dt} \exp(-tx) \cdot x|_{t=0} = \frac{d}{dt} \varphi^t_{\rho(X)}(X)\Big|_{t=0},$$

then we can apply Lemma 1.6.6. □

Definition 1.6.11. We define the *group of isometries* to be

$$I(M,g) \equiv \{\varphi : (M,g) \to (M,g) \mid \varphi^* g = g\}.$$

This has a natural topology generated by open sets

$$V(K,U) = \{\varphi \in I(M,g) : \varphi(K) \subseteq U\}$$

for compact K and open U, which is the *compact-open topology*.

Remark 1.6.12. It is not hard to show that

$$\begin{aligned} I(M,g) \times I(M,g) &\to I(M,g) \\ (\varphi,\psi) &\mapsto \varphi \circ \psi \end{aligned}$$

and

$$\begin{aligned} I(M,g) &\to I(M,g) \\ \varphi &\mapsto \varphi^{-1} \end{aligned}$$

are continuous under the given topology. Therefore, the group of isometries is a topological group. Moreover, one can show that this is a finite-dimensional Lie group.

Theorem 1.6.13 (Myers-Steenrod)**.** For Riemannian manifold (M,g), the group $I(M,g)$ is a finite-dimensional Lie group, and the $I(M,g)$-action on M is a proper action. Moreover, if (M,g) is complete, then the corresponding Lie algebra of $I(M,g)$ is the Killing vector field $\mathfrak{X}(M,g)$.

Remark 1.6.14.

- Given a fixed point x, since the action is proper, then the isotropy group $I(M,g)_x$ at x is a compact Lie group.
- If (M,g) is not complete, then in general the corresponding Lie algebra is strictly contained in $\mathfrak{X}(M,g)$.
- if (M,g) is compact, then it is complete and $I(M,g)$ is a compact Lie group.

Theorem 1.6.15. Let G be a Lie group and G acts on M properly and *effectively*, i.e., given G-action on X, the kernel of the given map $G \to \Sigma(X)$ is trivial, then there exists a Riemannian metric g such that the action is by isometries. Therefore, G can be identified with a subgroup of $I(M,g)$ for some g.

1.7 Cartan's Structure Equations

Definition 1.7.1. A *local frame* in a manifold M over an open set $U \subseteq M$ is a family of vector fields $\{X_1, \dots, X_n\} \subseteq \mathfrak{X}(U)$ such that for any $x \in U$,

$$\{ X_1|_x, \dots, X_n|_x \}$$

is a basis for T_xM.

Dually, a *local coframe* in a manifold M over an open set $U \subseteq M$ is a family of vector fields $\{\omega^1, \dots, \omega^n\} \subseteq \mathfrak{X}(U)$ such that for any $x \in U$,

$$\{ \omega^1|_x, \dots, \omega^n|_x \}$$

is a basis for T_x^*M.

Remark 1.7.2.

- Any frame determines a coframe by duality, and vice versa, by

$$\omega^i(X_j) = \delta^i_j$$

  given by the Kronecker δ-function.

- Local frames and coframes always exist: given a chart (U, φ_i), we get the vector fields associated to the chart

$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$$

  and the dual components

$$\{dx^1, \dots, dx^n\}.$$

  But globally they may not exist: for instance, for $M = \mathbb{S}^2$, the global vector fields do not exist, c.f., hairy ball theorem.

- For any local frame, the Lie brackets satisfy

$$[X_i, X_j] = c^k_{ij} X_k$$

  for some $c^k_{ij} \in C^\infty(U)$. Dually, the de Rham differentials satisfy

$$d\omega^k = -\frac{1}{2} c^k_{ij} \omega^i \wedge \omega^j$$

  for local coframes.

Given a connection ∇, fixing a local frame (with dual coframe) gives

$$\nabla_{X_i} X_j = \Gamma^k_{ij} X_k$$

for some functions $\Gamma^k_{ij} \in C^\infty(U)$. In local charts, this gives the definition of the Christoffel symbols. It is worth noting that the frames may not commute in general, therefore we do not always have to take the usual local frame/coframe as given in Remark 1.7.2. More precisely, if $[X_i, X_j] \neq 0$, then if $T^\nabla = 0$, we have $\Gamma^k_{ij} \neq \Gamma^k_{ji}$.

Now consider the 1-forms

$$\omega^k_j = \Gamma^k_{ij} \omega^i \in \Omega^1(U)$$

on U, so in terms of matrices, we have

$$[\omega^k_j] \in \Omega^1(U, \mathfrak{gl}_m(\mathbb{R}))$$

as a *connection* 1*-form*. Therefore, we always have

$$\nabla_Z X_j = \omega^k_j(Z) X_k.$$

This encodes the information of the vector field locally in terms of frames. Now we can characterize the torsion and curvature in a similar way.

$$T^\nabla(X_i, X_j) = T^k_{ij} X_k$$

for $T_{ij}^k \in C^\infty(U)$. We can then define

$$\theta^k = \frac{1}{2} T_{ij}^k \omega^i \wedge \omega^j$$

so θ is a family of vector-valued 2-forms on U, inducing the *torsion 2-form* $[\theta^k] \in \Omega^2(U, \mathbb{R})$. Moreover,

$$T^\nabla(X, Y) = \theta^k(X, Y) X_k.$$

End of Lecture 13

Recall that we have the following setting.

- Given (M, ∇), we fix a local frame $\{X_1, \ldots, X_m\}$ with dual coframe $\{\omega^1, \ldots, \omega^m\}$ over $U \subseteq M$.
- $\nabla_{X_i} X_j = \Gamma_{ij}^k X_k$ for $\Gamma_{ij}^k \in C^\infty(U)$. Set $\omega_j^k = \Gamma_{ij}^k \omega^i$, then $[\omega_j^k] \in \Omega^1(U; \mathfrak{gl}_m)$ is known as the connection 1-form. Therefore, $\nabla_Y X_j = \omega_j^k(Y) X_k$ by definition.
- Set $T(X_i, X_j) = T_{ij}^k X_k$, then the torsion 2-forms $[\theta^k] \in \Omega^2(U, \mathbb{R}^n)$ are defined by $\theta^k = \frac{1}{2} T_{ij}^k \omega^i \wedge \omega^j$. Therefore $T(Y, Z) = \theta^k(Y, Z) X_k$.

We can now write $R^\nabla(X_i, X_j) X_k = R_{ijk}^\ell X_\ell$, then we define the *curvature 2-form* to be $[\Omega_k^\ell] \in \Omega^2(U, \mathfrak{gl}_m)$, defined as

$$\Omega_k^\ell = \frac{1}{2} R_{ijk}^\ell \omega^i \wedge \omega^j.$$

Proposition 1.7.3. *Cartan's structural equations* are then defined by

$$d\omega^i = -\omega_j^i \wedge \omega^j + \theta^i \tag{1.7.4}$$

and

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k + \Omega_j^i \tag{1.7.5}$$

In particular, this gives

$$\begin{cases} d\omega &= -[\omega_j^i] \wedge \omega + \theta \\ d[\omega_j^i] &= [\omega_j^i] \wedge [\omega_j^I] + \Omega \end{cases}.$$

Proof. We have

$$\begin{aligned} d\omega^i(X_r, X_s) &= X_r(\omega^i(X_s)) - X_s(\omega^i(X_r)) - \omega^i([X_r, X_s]) \\ &= \omega^i(T(X_r, X_s) - \nabla_{X_r} X_s + \nabla_{X_s} X_r) \\ &= \omega^i(\theta^k(X_r, X_s) X_k - \omega_r^k(X_s) X_k + \omega_s^k(X_r) X_k) \\ &= \theta^i(X_r, X_s) - \omega_r^k(X_s) + \omega_s^k(X_r) \\ &= \theta^i(X_r, X_s) - \omega_j^i(X_s) \omega^j(X_k) + \omega_j^i(X_r) \omega^j(X_s) \\ &= (\theta^i - \omega_j^i \wedge \omega^j)(X_s, X_r). \end{aligned}$$

Moreover,

$$\begin{aligned} \nabla_{X_r} \nabla_{X_s} X_j &= \nabla_{X_r}(\omega_j^i(X_s) X_i) \\ &= X_r(\omega_j^i(X_s)) X_i + \omega_j^k(X_s) \omega_k^i(X_r) X_i, \end{aligned}$$

but

$$\nabla_{X_s} \nabla_{X_r} X_j = X_s(\omega_j^i(X_r)) X_i + \omega_j^k(X_r) \omega_k^i(X_s) X_i,$$

and

$$\nabla_{[X_s,X_r]}X_j = -\omega_j^i([X_s,X_r])X_i.$$

Therefore,

$$\begin{aligned}\nabla_{X_r}\nabla_{X_s}X_j - \nabla_{X_s}\nabla_{X_r}X_j - \nabla_{[X_s,X_r]}X_j &= (X_r(\omega_j^i(X_s))X_i - X_s(\omega_j^i(X_r))X_i - \omega_j^i([X_s,X_r])X_i) \\ &\quad + (\omega_j^k(X_s)\omega_k^i(X_r)X_i - \omega_j^k(X_r)\omega_k^i(X_s)X_i) \\ &= d\omega_j^i(X_r,X_s)X_i + \omega_j^k \wedge \omega_k^i(X_s,X_r)X_i \\ &= R(X_r,X_s)X_j \\ &= \Omega_j^i(X_r,X_s)X_i.\end{aligned}$$

□

Remark 1.7.6. For Levi-Civita connection ∇ of (M,g), Equation (1.7.4) corresponds to

$$d\omega^i = -\omega_j^i \wedge \omega^j, \tag{1.7.7}$$

and Equation (1.7.5) corresponds to

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k + \Omega_j^i. \tag{1.7.8}$$

Moreover, the metric corresponds to

$$dg_{ij} = g_{ik}\omega_j^k + g_{kj}\omega_i^k \tag{1.7.9}$$

where $g_{ij} = g(X_i,X_j)$. But on a metric, we get to take an orthonormal frame instead of an ordinary one, and in such cases g_{ij} is constant, therefore Equation (1.7.9) becomes

$$\omega_j^i + \omega_i^j = 0.$$

In particular, this means $[\omega_j^i] \in \Omega^1(U,\mathfrak{so}_m)$, where $\mathfrak{so}_m = \{A \in \mathfrak{gl}_m : A + A^t = 0\}$, that is, the skew-symmetric matrices in the orthogonal group. Moreover, in that case Equation (1.7.8) becomes

$$\Omega_j^i + \Omega_i^j = 0,$$

therefore $[\Omega_j^i] \in \Omega^2(U,\mathfrak{so}_m)$.

Proof. We will now prove that Equation (1.7.9) holds. This is given by

$$\begin{aligned}dg_{ij}(X_r) &= X_r(g_{ij}) \\ &= X_r(g(X_i,X_j)) \\ &= g(\nabla_{X_r}X_i,X_j) + g(X_i,\nabla_{X_r}X_j) \\ &= g(\omega_i^k(X_r)X_k,X_j) + g(X_i,\omega_j^k(X_r)X_k) \\ &= g_{kj}\omega_i^k(X_r) + g_{ik}\omega_j^k(X_r).\end{aligned}$$

□

What happens if we are dealing with a surface with a metric?

Corollary 1.7.10. If (M,g) is 2-dimensional, then for any orthonormal coframe, we get

$$\begin{cases} d\omega^1 &= -\Omega_2^1 \wedge \omega^2 \\ d\omega^2 &= \omega_2^1 \wedge \omega^1 \\ d\omega_2^1 &= \Omega_2^1 \end{cases}$$

where

$$[\omega_j^i] = \begin{pmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{pmatrix}$$

and
$$[\Omega^i_j] = \begin{pmatrix} 0 & \Omega^1_2 \\ -\Omega^1_2 & 0 \end{pmatrix}.$$
Because this is a coframe, then the 2-form can be written as a function in terms of $\omega^1 \wedge \omega^2$ as the unique 2-form, i.e., $d\omega^1_2 = \Omega^1_2 = K\omega^1 \wedge \omega^2$, where K is the Gaussian curvature, also known as the sectional curvature. In particular, if $\{X_i\}$ is orthonormal, then
$$K = R(X_1, X_2, X_2, X_1),$$
so
$$\Omega^1_2 = K\omega^1 \wedge \omega^2.$$
In this case,
$$R(X, Y, Z, W) = K(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)).$$

Remark 1.7.11. In the case where $g_{\mathbb{S}^2_R} = R^2(\sin^2 \varphi(d\theta)^2 + (d\varphi)^2)$, then
$$\begin{cases} \omega^1 & = R\sin\varphi d\theta \\ \omega^2 & = Rd\varphi \end{cases}.$$
In the orthonormal coframe, $\omega^1_2 = \cos\varphi d\varphi$, therefore $d\omega^1_2 = \frac{1}{R^2}\omega^1 \wedge \omega^2$, and therefore $K = \frac{1}{R^2}$.

End of Lecture 14

Recall:

- let (M, g) be a Riemannian manifold, then consider the frame (and corresponding coframe) $\{X_1, \dots, X_n\}$ and $\{\theta^1, \dots, \theta^n\}$ for $\theta \in \Omega^1(U, \mathbb{R}^n)$ that is orthogonal over $U \subseteq M$, then we have
- the connective 1-form $\omega = [\omega^i_j] \in \Omega^1(U, \mathfrak{so}_m)$, and
- the curvature 1-form $\Omega = [\Omega^i_j] \in \Omega^2(U, \mathfrak{so}_n)$.
- We then saw that the structural equations hold:
$$\begin{cases} d\theta^i & = -\omega^i_j \wedge \theta^j \\ d\omega^i_j & = \omega^i_k \wedge \omega^k_j + \Omega^i_j \end{cases}$$
or correspondingly,
$$\begin{cases} d\theta & = -\omega \wedge \theta \\ d\omega & = -\omega \wedge \omega + \Omega \end{cases}$$

We have not discussed the corresponding *Bianchi's identity*.

Proposition 1.7.12. There are two Bianchi's identities,

- the first Bianchi's identity: $\Omega \wedge \theta = 0$;
- the second Bianchi's identity: $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$.

Proof. By differentiating the first structural equation, we get
$$\begin{aligned} 0 &= d^2\theta \\ &= -d\omega \wedge \theta + \omega \wedge d\theta \\ &= \omega \wedge \omega \wedge \theta - \Omega \wedge \theta - \omega \wedge \omega \wedge \theta \\ &= -\Omega \wedge \theta \end{aligned}$$

by applying the two structural equations. Similarly, differentiating the second structural equation gives

$$\begin{aligned}0 &= d^2\omega \\ &= -d\omega \wedge \omega + \omega \wedge d\omega + d\Omega \\ &= \omega \wedge \omega \wedge \omega - \Omega \wedge \omega - \omega \wedge \omega \wedge \omega + \omega \wedge \Omega + d\Omega \\ &= d\Omega + \omega \wedge \Omega - \Omega \wedge \omega.\end{aligned}$$

□

Exercise 1.7.13. Check that the first Binachi's identity is equivalent to Theorem 1.5.3:

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0.$$

Remark 1.7.14. One can actually check that these are identities in the global sense.

Corollary 1.7.15. If (M, g) is isotropic, then

$$\Omega_i^j = K\theta^i \wedge \theta^j$$

where K is the sectional curvature.

Proof. This is a rephrasing of curvature for isotropic manifolds. Note that

$$R^\nabla(X, Y)Z = \Omega_k^\ell(X, Y)\theta^k(Z)X_\ell.$$

For orthonormal frames, we may compute

$$R^\nabla(X_i, X_j)X_k = \Omega_k^\ell(X_i, X_j)X_\ell,$$

so

$$R(X_i, X_j, X_k, X_\ell) = \Omega_k^\ell(X_i, X_j)$$

by contraction, which is really just a Kronecker delta function depending on choices of i and j. For isotropic Riemannian manifold (M, g), we know the curvature is given by

$$R(X, Y, Z, W) = -K(g(X, Z)g(Y, W) - g(X, W)g(Y, Z),$$

so for orthonormal frames, we get

$$R(X_i, X_j, X_k, X_\ell) = -K(\delta_i^k\delta_j^\ell - \delta_j^\ell\delta_j^k) = \Omega_k^\ell(X_i, X_j),$$

which is equivalent to saying

$$\Omega_k^\ell = K\omega^\ell \wedge \omega^k.$$

□

Exercise 1.7.16. Check that if dimension is at least 3, then K must be constant.

Example 1.7.17. Consider

$$i : \mathbb{S}_R^2 = \{(x, y, z) : x^2 + y^2 + z^2 = R^2\} \hookrightarrow \mathbb{R}^3$$

For spherical coordinates, we get

$$g_{\mathbb{S}^2} = i^*g_0 = R^2(\sin^2\varphi(d\theta)^2 + (d\varphi)^2)$$

in the usual spherical coordinate system (R, φ, θ). We then get

- $X_1 = \frac{1}{R\sin\varphi}\frac{\partial}{\partial\theta}$, $X_2 = \frac{1}{R}\frac{\partial}{\partial\varphi}$;
- $\theta^1 = R\sin\varphi d\theta$, and $\theta^2 = Rd\varphi$

as frames and coframes. Since we are in dimension 2, then the manifold is isotropic, and

$$[\omega_j^i] = \begin{bmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{bmatrix}$$

and

$$\Omega_2^1 = \begin{bmatrix} 0 & \Omega_2^1 \\ \Omega_2^1 & 0 \end{bmatrix}$$

and by the structural equations, we have

$$\begin{aligned} d\theta^1 &= -\omega_2^1 \wedge \theta^2 \\ d\theta^2 &= \omega_2^1 \wedge \theta^1 \end{aligned}$$

By directly differentiating the coframes, we get

$$\begin{aligned} d\theta^1 &= -R\cos\varphi d\theta \wedge d\varphi \\ d\theta^2 &= 0, \end{aligned}$$

which forces $\omega_2^1 = R\cos\varphi d\theta$, $\theta^2 = d\varphi$, and $\omega_2^1 \wedge \theta^1 = 0$. Moreover, we have $d\omega_2^1 = \Omega_2^1$, therefore by comparing with the differentiation, we get

$$d\omega_2^1 = R\sin\varphi d\theta \wedge d\varphi = \Omega_2^1$$

Because we are in dimension 2, then the manifold is isotropic, so by Corollary 1.7.15,

$$\Omega_2^1 = K\theta^1 \wedge \theta^2,$$

hence $K = \frac{1}{R^2}$. This gives

$$d\omega_2^1 = \frac{1}{R^2} R\sin\varphi d\theta \wedge R d\varphi,$$

hence

$$\theta^1 = R\sin\varphi d\theta$$

and

$$\theta^2 = Rd\varphi.$$

Given a Riemannian manifold (M, g) with constant curvature. Let $x \in M$ be a point, then there is an exponential map as follows: for $0_x \in V \subseteq T_xM$, there is some open subset $x \in U \subseteq M$ such that

$$\exp_x : V \xrightarrow{\simeq} U.$$

Pick a basis $\{e_1, \dots, e_n\}$ for T_xM, we can do parallel transport for straight lines in V, and since we can reach any point by the exponential map, we then get a frame $\{X_1, \dots, X_n\}$ over U. We can use this frame to write down the structural equations, but because the curvature is constant, we can write down a much more simplified version of structural equations in this frame. Eventually, the exponential map $\exp_x$ is a local isometry between constant-curvature metrics in $\mathbb{R}^n$ (respectively, $\mathbb{S}_K^n$, H_K^n). After even a bit more work, we have the following global conclusions.

Theorem 1.7.18 (Killing-Hopf)**.** Let (M, g) be a complete Riemannian manifold of constant curvature, then

i. if M is simply connected, then there exists an isometry between M and

- $(\mathbb{R}^n, g_0)$, if $K = 0$;
- $(\mathbb{S}_K^n, g_{\mathbb{S}^n})$, if $K > 0$;
- $(\mathbb{H}_K^n, g_{\mathbb{H}_K^n})$, if $K < 0$.

This gives a precise classification.

In the case where M is not simply connected, we recover the classification by quotients: recall that the action of fundamental group on the manifold is induced by the deck transformations on the universal covering, via concatenation of paths, then

ii. if $\pi_1(M) = \Gamma$, then M is isometric to a quotient of the form $\tilde{M}/\Gamma$, with Γ acting freely and properly on $\tilde{M}$ by isometries, and where $\tilde{M}$ is one of the constant-curvature model spaces mentioned above.

We now discuss the change of frames and coframes. Consider two frames $(U, X_1, \dots, X_m)$ and $(\bar{U}, \bar{X}_1, \dots, \bar{X}_m)$, with two dual coframes $(U, \theta^1, \dots, \theta^m)$ and $(\bar{U}, \bar{\theta}^1, \dots, \bar{\theta}^m)$, such that $U \cap \bar{U} \neq \varnothing$. Therefore,

$$\begin{cases} \bar{X}_i &= X_k A_i^k \\ \bar{\theta}^i &= A_k^i \theta^k \end{cases},$$

where $A = [A_i^k] : U \cap \bar{U} \to \mathrm{O}(m)$. Now further assuming the frames/coframes are orthogonal, then we have $AA^T = A^T A = I$, so we write

$$\begin{cases} \bar{X} &= XA \\ \bar{\theta} &= A^T \theta \end{cases}$$

compactly.

Proposition 1.7.19.

- $\bar{\omega} = A^T \omega A + A^T dA$, where $\bar{\omega}_j^i = A_k^i \omega_\ell^k A_j^\ell + A_k^i dA_j^k$;
- $\bar{\Omega} = A^T \Omega A$, where $\Omega_j^i = A_k^i \Omega_\ell^k A_j^\ell$

Proof. Since $A^T A = I$, then $\theta = A\bar{\theta}$, and by differentiation,

$$(dA)^T A + A^T dA = 0.$$

We have

$$\begin{aligned} d\bar{\theta} &= (dA)^T \wedge \theta + A^T d\theta \\ &= (dA)^T \wedge A\bar{\theta} - A^T(\omega \wedge \theta) \\ &= (dA)^T \wedge \bar{\theta} - A^T(\omega \wedge A\bar{\theta}) \\ &= -(A^T dA + \bar{A}\omega A) \wedge \bar{\theta} \end{aligned}$$

by Proposition 1.7.3. Therefore $\bar{\omega} = A^T dA + \bar{A}\omega A$. The second equality can be done similarly, but in a more involved manner. We have

$$\begin{aligned} d\bar{\omega} &= d(A^T dA + A^T \omega A) \\ &= (dA)^T \wedge dA + (dA)^T \omega A + A^T d\omega A - A^T \omega dA \\ &= (dA)^T \omega A + A^T d\omega A - A^T \omega dA \\ &= (dA)^T A \wedge A^T \omega A - A^T \omega \wedge \omega A + A^T \Omega A - A^T \omega A \wedge A^T dA \\ &= -A^T dA \wedge A^T \omega A - A^T \omega A \wedge A^T \omega A + A^T \Omega A - A^T \omega A \wedge A^T dA. \end{aligned}$$

Since

$$\begin{aligned} A^T dA \wedge A^T dA &= -A dA^T \wedge A^T dA \\ &= -dA^T \wedge dA \\ &= 0, \end{aligned}$$

we have

$$\begin{aligned} d\bar{\omega} &= -A^T dA \wedge A^T \omega A - A^T \omega A \wedge A^T \omega A + A^T \Omega A - A^T \omega A \wedge A^T dA \\ &= A^T dA \wedge A^T dA - A^T dA \wedge A^T \omega A - A^T \omega A \wedge A^T \omega A + A^T \Omega A - A^T \omega A \wedge A^T dA \end{aligned}$$

$$= -(A^T dA + A^T \omega A) \wedge (A^T dA + A^T \omega A) + A^T \Omega A$$
$$= -\bar{\omega} \wedge \bar{\omega} + A^T \Omega A,$$

so we must have

$$\bar{\Omega} = A^T \Omega A.$$

□

End of Lecture 15

Recall that, given orthonormal frames $\{X_i\}$ and $\{\bar{X}_i\}$, which gives rise to coframes $\{\theta^i\}$ and $\{\bar{\theta}^i\}$, then they are related by $\bar{\theta} = A\theta$ for $A = (a^i_j) \in \mathrm{O}(m)$. In turn, we have connection 1-form $\bar{\omega} = A^T \omega A + A^T dA$ and curvature 2-form $\bar{\Omega} = A^T \Omega A$. In the case of dimension 2, if $\{\bar{\theta}\}$ and $\{\theta\}$ have the same orientation, then we can write $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ for some $\varphi : U \cap \bar{U} \to \mathbb{S}^1$. Therefore, we can write $\omega = \begin{pmatrix} 0 & \omega^1_2 \\ -\omega^1_2 & 0 \end{pmatrix}$ and $\Omega = \begin{pmatrix} 0 & \Omega^1_2 \\ -\Omega^1_2 & 0 \end{pmatrix}$. In the view that $A \in C^\infty(U, \mathrm{O}(m))$ and therefore $dA \in \Omega^1(U, \Omega(m))$, we have $\bar{\omega}^1_2 = \omega^1_2 - d\varphi$ and $\bar{\Omega}^1_2 = \Omega^1_2$ and

$$A^T dA = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} -\sin\varphi d\varphi & -\cos\varphi d\varphi \\ \cos\varphi d\varphi & -\sin\varphi d\varphi \end{pmatrix} = \begin{pmatrix} 0 & -d\varphi \\ d\varphi & 0 \end{pmatrix}.$$

1.8 Gauss-Bonnet Theorem

Theorem 1.8.1 (Gauss-Bonnet)**.** Let (M, g) be an compact (i.e., without boundary) oriented Riemannian 2-manifold, then

$$\int_M K_g V_g = 2\pi\chi(M)$$

where K_g is the Gaussian curvature of g, V_g is the Riemannian volume form of (M, g), and $\chi(M)$ is the Euler characteristic of M.

Remark 1.8.2. This is a result that connects geometry with topology.

1. For any manifold M, $\chi(M) = \sum_{i=0}^{m} (-1)^i \beta_i$ where $\beta_i = \dim(H^i(M))$ is the Betti number. In particular, for an oriented surface, we recover the Riemann-Roch theorem $\chi(M) = 2 - 2g$, where g is the genus of M. For instance, $\chi(\mathbb{S}^2) = 2$, $\chi(\mathbb{T}^2) = 0$, and $\chi(M) = -2$ for a manifold with 2 punctures.
2. This result generalizes as follows. For any even-dimensional manifold M, Chern proved that

   $$\int_M P(R) V_g = C_m \chi(M)$$

   where $P(R)$ is a polynomial in terms of the curvature R of g, and C_m is a constant that only depends on $\dim(M) = m$.
3. There is a version of Theorem 1.8.1 for compact oriented 2-manifolds with boundaries:

   $$\int_M K_g V_g + \int_{\partial M} k_g = 2\pi\chi(M)$$

   where the geodesic curvature k_g on ∂M coincides with the covariant derivative $D_{\gamma_i} \dot{\gamma}_i(t)$ for $\partial M = \bigcup\{\gamma_i\}$.

Corollary 1.8.3. $\mathbb{S}^2$ and $\mathbb{T}^2$ do not admit a metric with negative Gaussian curvature.

Proof. If such g exists, then

$$\int_M K_g dV_g < 0$$

which is impossible since $\chi(\mathbb{S}^2), \chi(\mathbb{T}^2) \geqslant 0$. □

We take a detour into Riemannian volume forms.

Lemma 1.8.4. Let (M, g) be an oriented Riemannian manifold, then there exists a unique volume form V_g such that for any positive, orthonormal frame $\{X_1, \dots, X_n\}$: $V_g(X_1, \dots, X_n) = 1$.

Proof. Suppose $\{X_i\}$ and $\{\bar{X}_i\}$ are both positively-oriented orthonormal frames, where $\bar{X} = XA$ with $A : U \cap \bar{U} \to \mathrm{SO}(n)$, then we can pick some volume form $\mu \in \Omega^n(M)$ defining the given orientation, and calculation of this volume form on the given frame shows that

$$\mu(\bar{X}_1, \dots, \bar{X}_m) = \det(A)\mu(X_1, \dots, X_m) = \mu(X_1, \dots, X_m) = c,$$

so set

$$V_g = \frac{1}{c}\mu \in \Omega^m(M).$$

□

Remark 1.8.5.

- If the manifold is not oriented, then there is no longer a volume form, but we may recover the notion of density.
- If $\{\theta^i\}$ is a positively-oriented orthonormal coframe, then $V_g = \theta^1 \wedge \cdots \wedge \theta^m$.
- If (U, x^i) is a positively-orientated chart, i.e., mapping the orientation of the chart to the standard orientation of $\mathbb{R}^n$, then $g = g_{ij}dx^i dx^j$, hence $V_g = \deg(g_{ij})^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n$.

Proof of Theorem 1.8.1. This makes use of Theorem 1.8.6 which we will prove later on in the course.

Theorem 1.8.6 (Poincaré-Hopf)**.** Let M be a compact, connected, oriented manifold. Suppose $X \in \mathfrak{X}(M)$ has finite number of zeros, say $\{p_1, \dots, p_N\}$, then

$$\chi(M) = \sum_{i=1}^{N} \mathrm{ind}_{p_i}(X).$$

In particular,

$$\chi(TM) = \chi(M)\mu$$

where μ is the *orientation class*, which will be defined later.

The index is a notion of rotation of vector field around each zero. To compute the index at a zero $p \in M$, we choose a chart (U, x^i) of p, pick a ball $B_\varepsilon(p)$. We can then look at the Gauss map

$$G : \partial B_\varepsilon(p) \to \mathbb{S}^{m-1}$$
$$x \mapsto \frac{X(x)}{||X(x)||}$$

and define $\mathrm{ind}_p(X) = \deg(G)$, where the degree is the unique integer such that

$$\int_{\partial B_\varepsilon} G^*\alpha = \deg(G) \int_{\mathbb{S}^{m-1}} \alpha$$

for closed form $\alpha \in \Omega^{m-1}(\mathbb{S}^{m-1})$.

To prove Theorem 1.8.1, we choose $X \in \mathfrak{X}(M)$ with zeros $\{p_1, \dots, p_N\}$. On $M \backslash \{p_1, \dots, p_N\}$, there is a positively-oriented orthonormal frame $\{X_1 = \frac{X}{||X||}, X_2\}$ with dual (positively-oriented orthonormal) coframe $\{\theta^1, \theta^2\}$. We choose some small enough balls $B_{\varepsilon_i}(p_i)$'s for all i, so that it contains only zero p_i, and is contained in some chart $(U_i i, \varphi_i)$. Therefore,

$$\begin{aligned}
\int\limits_{M\setminus\bigcup_i B_{\varepsilon_i}(p_i)} K_g V_g &= \int\limits_{M\setminus\bigcup_i B_{\varepsilon_i}(p_i)} K_g \theta^1 \wedge \theta^2 \\
&= \int\limits_{M\setminus\bigcup_i B_{\varepsilon_i}(p_i)} d\omega_2^1 \\
&= \int\limits_{\partial(M\setminus\bigcup_i B_{\varepsilon_i}(p_i))} \omega_2^1 \text{ by Stokes' theorem} \\
&= \sum_{i=1}^N \int\limits_{\partial B_{\varepsilon_i}(p_i)} \omega_1^2.
\end{aligned}$$

It then suffices to show that taking $\varepsilon_i \to 0$ for arbitrary i gives

$$\lim_{\varepsilon_i\to 0} \int\limits_{\partial B_{\varepsilon_i}(p_i)} \omega_1^2 = 2\pi \operatorname{ind}_{p_i}(X).$$

Now choose frame $\{\bar{X}_1, \bar{X}_2\}$ on each U_i that is positively-oriented and orthonormal, then we have $\bar{X} = XA$ for

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin(\theta) & \cos\theta \end{pmatrix}$$

with $\theta : U_i\setminus\{p_i\} \to \mathbb{S}^1$ as the angle between $\bar{X}_i$ and X_i, therefore

$$\theta|_{\partial B_{\varepsilon_i}(p_i)} = G$$

with $G(x) = \frac{x}{||x||}$. At the start of the lecture, we saw $\bar{\omega}_1^2 = \omega_1^2 - d\theta$, then

$$\bar{\omega}_1^2|_{\partial B_{\varepsilon_i}(p_i)} = \omega_1^2|_{\partial B_{\varepsilon_i}(p_i)} - G^* d\theta$$

where $d\theta$ is the standard angle function on $\mathbb{S}^1$. This shows us that

$$\begin{aligned}
\int\limits_{\partial B_{\varepsilon_i}(p_i)} \omega_1^2 &= \int\limits_{\partial B_{\varepsilon_i}(p_i)} G^* d\theta - \int\limits_{\partial B_{\varepsilon_i}(p_i)} \bar{\omega}_1^2 \\
&\xrightarrow[\varepsilon_i\to 0]{} \int\limits_{\partial B_{\varepsilon_i}(p_i)} G^* d\theta \\
&= \deg(G) \int\limits_{\mathbb{S}^1} d\theta \\
&= 2\pi \operatorname{ind}_{p_i}(X),
\end{aligned}$$

where the second term vanishes whenever $\varepsilon_i \to 0$ just like integrating a smooth function. □

End of Lecture 16

1.9 Hodge Decomposition

Let $(V, \langle -,-\rangle)$ be an Euclidean vector space with an inner product, which defines an inner product

$$\langle -,-\rangle : \Lambda^k V \times \Lambda^k V \to \mathbb{R}$$

that is uniquely determined by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle = \det(\langle v_i, w_j\rangle).$$

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis, then we get an orthonormal basis on $\Lambda^k V$ using the set

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : i_1 < \cdots < i_k\}.$$

Lemma 1.9.1. If $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are orthonormal bases that define the same orientation, then $e_1 \wedge \cdots \wedge e_n = f_1 \wedge \cdots \wedge f_n$.

Proof. We have seen a similar proof last time: write $f_i = \sum_j a_i^j e_j$, then $A = (a_i^j) \in \mathrm{SO}(n)$, i.e., it has determinant 1, since they define the same orientation, therefore

$$f_1 \wedge \cdots \wedge f_n = \det(A) e_1 \wedge \cdots \wedge e_n = e_1 \wedge \cdots \wedge e_n.$$

□

Fix some orientation V, given by $\mu = e_1 \wedge \cdots \wedge e_n$ as a notion of unit n-vector, where $\{e_i\}$'s give a positively-oriented orthonormal basis.

Proposition 1.9.2. There is a unique linear map $* : \Lambda^k V \to \Lambda^{n-k} V$ such that

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle \mu \tag{1.9.3}$$

for any $\alpha, \beta \in \Lambda^k V$.

Proof. If Equation (1.9.3) holds, then if $\{e_i\}$ is a positively-oriented orthonormal basis, then we find that

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \pm e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}, \tag{1.9.4}$$

where $\{e_1, \ldots, e_{i_k}, e_{j_1 \ldots, e_{j_{n-k}}}\}$ is basis, and the sign $\pm$ is determined by whether this basis is positively- or negatively-oriented. Therefore, $*$ is unique if it exists. But Equation (1.9.4) defines $*$ on a basis. □

Remark 1.9.5. The operator in Proposition 1.9.2 satisfies the following properties.

1. $*1 = e_1 \wedge \cdots \wedge e_n$.
2. $*$ satisfies Equation (1.9.4).
3. $*(Av_1 \wedge \cdots \wedge Av_k) = \det(A) *(v_1 \wedge \cdots \wedge v_k)$.
4. $*(\alpha \wedge *\beta) = \langle \alpha, \beta\rangle = \langle \beta, \alpha\rangle = *(\beta \wedge *\alpha)$.
5. $** = (-1)^{k(n-k)}$ defines an operator $\Lambda^k V \to \Lambda^k V$.

Assuming $\{e_1, \ldots, e_{i_k}, e_{j_1}, \ldots, e_{j_{n-k}}\}$ is positively-oriented, then we know Equation (1.9.4) holds, therefore

$$**(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \pm e_{i_1} \wedge \cdots \wedge e_{i_k}$$

where the sign $\pm$ depends on the number of sign changes required to reach $\{e_{j_1}, \ldots, e_{j_{n-k}}, e_{i_1}, \ldots, e_{i_j}\}$, i.e., $(-1)^{k(n-k)}$.

Remark 1.9.6. If $\{v_1, \ldots, v_n\}$ is any positively-oriented basis (that is not assumed to be orthonormal), then

$$*1 = \frac{1}{\det(\langle v_i, v_j\rangle)} v_1 \wedge \cdots \wedge v_n.$$

Suppose (M,g) is a Riemannian manifold with a fixed choice of orientation. For any point $x \in M$, there is a notion of inner product g_x on T_*M, so there is an identification $T_xM \simeq T_x^*M$ of vector spaces given by $v \mapsto g_x(v,-)$, which therefore transforms the inner product into T_x^*M, now denoted g_x^*. In local charts, if we write

$$g = g_{ij}dx^i dx^j,$$

then $g_x = (g_{ij}(x))$ and $g_x^* = (g_{ij})^{-1} = (g^{ij})$.

Performing the operator $*$ on each cotangent space, we get an operator

$$* : \Omega^k(M) \to \Omega^k(M).$$

Definition 1.9.7. The operator $* : \Omega^k(M) \to \Omega^k(M)$ defined in called the *Hodge star operator*.

If $\{\theta^i\}$ is a positively-oriented orthonormal coframe, then

$$*(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) = \pm\theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}},$$

where the choice of sign follows from the previous choices. In particular,

$$*1 = \theta^1 \wedge \cdots \wedge \theta^n = V_g$$

is the Riemannian volume form. More particularly, if M is a compact manifold, then the *volume* of M is defined by

$$\mathrm{Vol}(M) = \int_M *1.$$

Definition 1.9.8. We define L^2*-inner product* on the differential k-forms $\Omega_c^k(M)$ with compact support as

$$(\alpha,\beta) = \int_M \langle\alpha,\beta\rangle V_g = \int_M (\alpha \wedge *\beta).$$

We now assume M is compact, i.e., $\Omega_c^k(M) = \Omega^k(M)$.

Proposition 1.9.9. Given a oriented Riemannian manifold (M,g), the de Rham differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ has a formal adjoint $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$, i.e., $(d\alpha,\beta) = (\alpha, d^*\beta)$ for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$, called the *codifferential*, defined by

$$d^*\beta = (-1)^{nk} {*d*}\beta.$$

Proof. We have

$$\begin{aligned} d(\alpha \wedge *\beta) &= (d\alpha) \wedge *\beta + (-1)^k \alpha \wedge d*\beta \\ &= (d\alpha) \wedge *\beta + (-1)^k \alpha \wedge (-1)^{k(n-k)} **d*\beta \\ &= \langle d\alpha, \beta\rangle + (-1)^{kn} \langle \alpha, d^*\beta\rangle . \end{aligned}$$

By Stokes' theorem,

$$\begin{aligned} 0 &= \int_M d(\alpha \wedge *\beta) \\ &= (d\alpha,\beta) + (-1)^{kn}(\alpha, d^*\beta). \end{aligned}$$

□

Definition 1.9.10. The *Laplace-Beltrami operator* is

$$\Delta = dd^* + d^*d : \Omega^k(M) \to \Omega^k(M).$$

Proposition 1.9.11. Δ satisfies the following properties.

i. Δ is formally self-adjoint, i.e., $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$ for all $\alpha, \beta \in \Omega^*(M)$.

ii. $\Delta\alpha = 0$ if and only if $d\alpha = d^*\alpha = 0$.

iii. $\Delta * = * \Delta$.

Proof.

i. We have

$$\begin{aligned}(\Delta\alpha, \beta) &= (dd^*\alpha, \beta) + (d^*d\alpha, \beta)\\ &= (d^*\alpha, d^*\beta) + (d\alpha, d\beta)\\ &= (\alpha, \Delta\beta).\end{aligned}$$

ii. If $d^*\alpha = d\alpha = 0$, then $\Delta\alpha = 0$. Conversely, if $\Delta\alpha = 0$, then $||d^*\alpha||^2 + ||d\alpha||^2 = (\Delta\alpha, \alpha) = 0$, therefore $d^*\alpha = d\alpha = 0$.

iii. Direct computation.

□

Definition 1.9.12. The harmonic k-forms are defined by $\mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) : \Delta\alpha = 0\}$.

Remark 1.9.13. From the definition, the harmonic functions (still under the assumption that M is compact) are the ones that are constant on each connected component of the manifold. Therefore, $\mathcal{H}^0(M)$ is the vector space of dimension the number of connected components.

Let us now express the Laplace-Beltrami operator in local coordinates.

Example 1.9.14. Let $M = \mathbb{R}^n$ and $g_0 = \sum(dx^i)^2$ be the flat metric, under the usual orientation, then we have $df = \frac{\partial f}{\partial x^i} dx^i$ using a basis $\{dx^1, \ldots, dx^n\}$. Therefore,

$$*df = \sum_{i=1}^{n} (-1)^i \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

and

$$*dx^i = (-1)^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

therefore

$$\begin{aligned}\Delta f &= dd^*f + d^*df\\ &= *d\left(\sum_{i=1}^{n} (-1)^i \frac{\partial f}{\partial x^i} dx^1 \cdots \widehat{dx^i} \cdots dx^n\right)\\ &= -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial (x^i)^2}\end{aligned}$$

which is the negative of the usual Laplacian, since $dd^*f = 0$. Similarly,

$$\Delta\omega = \sum_{j=1}^{n} \frac{\partial^2 \omega_{i_1 \cdots i_k}}{\partial (x^j)^2} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Note that this does not use the compactness assumption, because this does not involve the L^2-inner product defined above.

End of Lecture 17

Recall that

- we defined the Hodge star operator
$$*: \Omega^k(M) \to \Omega^{n-k}(M).$$
On orthogonal positively-oriented coframe, this gives
$$*(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) = \pm\theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}}$$
where the sign depends on whether the set $\{\theta^{i_1}, \ldots, \theta^{i_k}, \theta^{j_1}, \ldots, \theta^{j_{n-k}}\}$ is oriented;
- the L^2-inner product is defined by
$$(\alpha, \beta) = \int_M \alpha \wedge *\beta;$$
- and we defined the codifferential to be
$$d^*: \Omega^k(M) \to \Omega^{k-1}(M)$$
to be $d^* = -(-1)^{m(k+1)} {*d*}$, which is the formal adjoint of de Rham differential d, i.e., $(d^*\alpha, \beta) = (\alpha, d\beta)$;
- we defined the Laplace-Beltrami operator to be
$$\Delta = dd^* + d^*d : \Omega^k(M) \to \Omega^k(M)$$
which is self-adjoint;
- we define the harmonic k-forms to be the set
$$\mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) : \Delta\alpha = 0\}.$$

Exercise 1.9.15. In a local chart (U, x^i),
$$\Delta f = -\frac{1}{(\det(g))^{\frac{1}{2}}} \frac{\partial}{\partial x^i} \left((\det(g))^{\frac{1}{2}} g^{ij} \frac{\partial f}{\partial x^j} \right)$$
for $g = g_{ij} dx^i dx^j$ and $(g^{ij}) = (g_{ij})^{-1}$.

Theorem 1.9.16 (Hodge Decomposition)**.** There is an orthogonal decomposition
$$\begin{aligned} \Omega^k(M) &= \Delta(\Omega^k(M)) \oplus \mathcal{H}^k(M) \\ &= d(d^*\Omega^k(M)) \oplus d^*(d\Omega^k(M)) \oplus \mathcal{H}^k(M). \end{aligned}$$
In particular, $\Delta\omega = \alpha$ has solutions if and only if $\alpha \in \mathcal{H}^k(M)^\perp$.

We first list a few consequences of Theorem 1.9.16.

Definition 1.9.17. Let $H : \Omega^k(M) \to \mathcal{H}^k(M)$ be the orthogonal projection. The *Green operator* is a linear operator defined by
$$\begin{aligned} G : \Omega^k(M) &\to \mathcal{H}^k(M)^\perp \\ \alpha &\mapsto \omega, \end{aligned}$$
where ω is the unique solution of the equation $\Delta\omega = \alpha - H(\alpha)$.

Lemma 1.9.18. G commutes with any linear operator $T : \Omega^*(M) \to \Omega^*(M)$ that commutes with Δ. In particular, G commutes with differential d, codifferential d^*, and Δ itself.

Proof. Assume that $T\Delta = \Delta T$, then $T(\mathcal{H}^k(M)) \subseteq \mathcal{H}^k(M)$, and since $\mathcal{H}(M)^\perp = \operatorname{im}(\Delta)$, therefore $T(\mathcal{H}^k(M)^\perp) \subseteq \mathcal{H}^k(M)^\perp$. By the description of the Green operator, we can write
$$G = \left(\Delta|_{\mathcal{H}(M)^\perp}\right)^{-1} \circ \operatorname{pr}_{\mathcal{H}^k(M)^\perp}.$$
This gives $G \circ T = T \circ G$. ∎

Corollary 1.9.19. The de Rham cohomology $H^*(M)$ of a manifold M is finite-dimensional, and every class in $H^k(M)$ has a unique harmonic representative.

Proof. Given $\alpha \in \Omega^k(M)$, then

$$\begin{aligned}\alpha &= \Delta G(\alpha) + H(\alpha) \text{ by Theorem 1.9.16}\\ &= dd^*G(\alpha) + d^*dG(\alpha) + H(\alpha)\\ &= dd^*G(\alpha) + d^*G(d\alpha) + H(\alpha) \text{ by Lemma 1.9.18}.\end{aligned}$$

In particular, if $d\alpha = 0$, then $\alpha = d(G(d^*\alpha)) + H(\alpha)$, so $[\alpha] = [H(\alpha)]$. One should now check that the harmonic forms are well-defined in representatives: given $[\alpha_1] = [\alpha_2]$, we should have $H(\alpha_1) = H(\alpha_2)$. Assume that $[\alpha_1] = [\alpha_2]$ and $\Delta\alpha_1 = \Delta\alpha_2 = 0$, then it suffices to show that $\alpha_1 = \alpha_2$. We see that $\alpha_1 - \alpha_2 = d\beta$ is exact, so

$$(\alpha_1 - \alpha_2, d\beta) = (d^*(\alpha_1 - \alpha_2), \beta),$$

but having $\Delta\alpha_1 = \Delta\alpha_2 = 0$, it is equivalent to saying that $d(\alpha_1 - \alpha_2) = d^*(\alpha_1 - \alpha_2) = 0$, therefore

$$(\alpha_1 - \alpha_2, d\beta) = (d^*(\alpha_1 - \alpha_2), \beta) = 0.$$

Now $||\alpha_1 - \alpha_2||^2 = (\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) = (\alpha_1 - \alpha_2, d\beta) = 0$, therefore $\alpha_1 = \alpha_2$. □

Lemma 1.9.20. In Theorem 1.9.16, the first decomposition implies the second decomposition.

Proof. Say $\Delta\alpha = 0$, or equivalently $d\alpha = d^*\alpha = 0$, then

$$\begin{aligned}(\alpha, d\beta) &= (d^*\alpha, \beta) = 0\\ (\alpha, d^*\beta) &= (d\alpha, \beta) = 0\end{aligned}$$

for any $\beta \in \Omega^k(M)$. Therefore, the harmonic forms is orthogonal to images of d and d^*. Finally,

$$(d\beta_1, d^*\beta_2) = (d^2\beta, d\beta) = 0.$$

This shows that all three factors in the second decomposition are pairwise orthogonal. □

It then remains to show the first decomposition. Suppose $(V, (\cdot,\cdot))$ is a Euclidean vector space, then for any $v \in V$, we can look at the functionals

$$\begin{aligned}\ell_v : V &\to \mathbb{R}\\ w &\mapsto (v, w)\end{aligned}$$

such that $|\ell_v(w)| = |(v, w)| \leqslant ||v|| \cdot ||w|| = c||w||$ for some constant c, therefore ℓ_v is a bounded linear function. If $\dim(V) < \infty$, then

- any functional $\ell : V \to \mathbb{R}$ is bounded, and in fact
- any functional $\ell : V \to \mathbb{R}$ is of the form $\ell(w) = (v, w)$ for some $v \in V$.

However, if $\dim(V) = \infty$, both properties may fail. The space of differential forms is one such space, therefore causing us problems. Regardless, we have

Theorem 1.9.21 (Riesz Representation Theorem)**.** If $(V, (\cdot,\cdot))$ is a Hilbert space, and $\ell : V \to \mathbb{R}$ is a bounded linear functional, then $\ell(w) = (v, w)$ for some unique $v \in W$.

We may want to apply this theorem, but the issue being, $\Omega^*(M)$ is not a Hilbert space, since it is not complete. To take the completion, another issue occurs: the notion of completion is then not unique. We want to find the right notion of completion $(W, (\cdot,\cdot))$ with $V \subseteq W$ and $\bar{V} = W$, which is given by

$$W = \{\alpha : \alpha, d\alpha, d^*\alpha \in L^2\},$$

whatever this means. The correct way of doing this is using the notion of a Sobolev space, but we digress. After completion, we look at the solutions $w \in W$ such that $\Delta\omega = \alpha$. Assuming that a solution exists, then

$$(\Delta\omega, \varphi) = (\alpha, \varphi)$$

for any $\varphi \in \Omega^k(M)$). To define this, we note that Δ is still self-adjoint after the completion, therefore this is equivalent to

$$(\omega, \Delta\varphi) = (\alpha, \varphi)$$

for any $\varphi \in \Omega^k(M)$). This is really the definition of α above, i.e., in the weak sense. The point being, the solutions ω of $\Delta\omega = \alpha$ are exactly the linear functionals

$$\ell_w : \Omega^k(M) \to \mathbb{R}$$

such that $\ell_\omega(\Delta\varphi) = (\alpha, \varphi)$. These are known as weak solutions, i.e., a solution in W by Theorem 1.9.21.

Definition 1.9.22. A *weak solution* of $\Delta\omega = \alpha$ is a bounded linear functional $\ell_w : \Omega^k(M) \to \mathbb{R}$ such that $\ell_\omega(\Delta\varphi) = (\alpha, \varphi)$.

Remark 1.9.23. ℓ_w should then be thought of as a function on W by Theorem 1.9.21, i.e., taking a completion on k-forms.

Any soluition now gives rise to a weak solution. We still need to connect weak solutions back to the regular solutions.

Theorem 1.9.24 (Regularity)**.** Given $\alpha \in \Omega^k(M)$ and weak solution $\ell_w : W \to \mathbb{R}$, then there exists $\omega \in \Omega^k(M)$ such that

$$\ell_\omega(\varphi) = (\omega, \varphi)$$

for all $\varphi \in \Omega^k(M)$.

Theorem 1.9.25. If $\{\alpha_n\} \subseteq \Omega^k(M)$ is a sequence of smooth functions that is bounded, and whose Laplacian is also bounded, i.e., $||\alpha_n|| \leqslant C$ and $||\Delta\alpha_n|| \leqslant C$ for some C for all $n \in \mathbb{N}$, then there exists a Cauchy subsequence $\{\alpha_{n_k}\}$.

End of Lecture 18

Proof of Theorem 1.9.16. We first show that $\mathcal{H}^k(M)$ is finite-dimensional. Assume not, then let $\{\alpha_n\} \subseteq \mathcal{H}^k(M)$ be such that $||\alpha_n|| = 1$ and $(\alpha_n, \alpha_n) = 0$ for all $n \neq m$, then this sequence has no Cauchy subsequences, which contradicts Theorem 1.9.25.

Given Lemma 1.9.20, it suffices to prove the first decomposition of Theorem 1.9.16. Fix orthonormal basis $\{\omega_1, \ldots, \omega_N\}$ for $\mathcal{H}^k(M)$. For any $\alpha \in \Omega^k$, we can write

$$\alpha = \beta + \sum_{i=1}^{N} (\alpha, \omega_i)\omega_i,$$

therefore

$$\Omega^k(M) = \mathcal{H}^k(M)^\perp \oplus \mathcal{H}^k(M).$$

It remains to show $\Delta(\Omega^k(M)) = \mathcal{H}^k(M)^\perp$. One direction is easy: to show $\Delta(\Omega^k(M)) \subseteq \mathcal{H}^k(M)^\perp$, note that for any $\varphi \in \mathcal{H}^k(M)$, we get

$$(\Delta\omega, \varphi) = (\omega, \Delta\varphi) = 0.$$

To show the other inclusion $\mathcal{H}^k(M)^\perp \subseteq \Delta(\Omega^k(M))$, we need the following lemma, stating that the inverse of Laplacian is continuous, assuming such inverse exists.

Lemma 1.9.26. There exists some $c > 0$ such that $||\varphi|| \leqslant c||\Delta\varphi||$ for all $\varphi \in \mathcal{H}^k(M)^\perp$.

Let $\alpha \in \mathcal{H}^k(M)^\perp$, we define

$$\begin{aligned} \ell : \Delta(\Omega^k(M)) &\to \mathbb{R} \\ \Delta\varphi &\mapsto (\varphi, \alpha) \end{aligned}$$

We first show that this is well-defined. Suppose $\Delta\varphi_1 = \Delta\varphi_2$, then $\Delta(\varphi_1 - \varphi_2) = 0$, hence $\varphi_1 - \varphi_2 \in \mathcal{H}^k(M)$, thus $(\varphi_1 - \varphi_2, \alpha) = 0$. Now we check that ℓ is bounded. By Lemma 1.9.26, we have

$$\begin{aligned}|\ell(\Delta\varphi)| &= |\ell(\Delta(\varphi - H(\varphi))| \\ &= (\varphi - H(\varphi), \alpha) \\ &\leqslant ||\alpha|| \cdot ||\varphi - H(\varphi)|| \\ &\leqslant c||\alpha|| \cdot ||\Delta(\varphi - H(\varphi))|| \\ &= c||\alpha|| \cdot ||\Delta\varphi||.\end{aligned}$$

By Hahn–Banach theorem, we know that bounded operator in the closed subspace can be extended to a bounded operator on the entire space, therefore there exists an extension $\ell : W \to \mathbb{R}$ which is bounded. Hence, ℓ is a weak solution of the equation. Finally, by Theorem 1.9.24, $\ell(\varphi) = (\omega, \varphi)$ with $\omega \in \Omega^k$, therefore $\Delta\omega = \alpha$. This proves the inclusion. Finally, we give a proof of Lemma 1.9.26.

Proof of Lemma 1.9.26. Suppose not, then there exists a sequence $\{\alpha_n\} \subseteq \mathcal{H}^k(M)^\perp$ such that the norm is constant, i.e., we may assume $||\alpha_n|| = 1$, and $||\Delta\alpha_n|| \to 0$ as $n \to \infty$. By Theorem 1.9.25, it has a Cauchy subsequence $\alpha_{n_k} \subseteq \mathcal{H}^k(M)^\perp$. That is, for any $\varphi \in \Omega^k(M)$, $(\alpha_{n_k}, \varphi) \in \mathbb{R}$ is Cauchy, hence $\lim_{k\to\infty}(\alpha_{n_k}, \varphi)$ exists. Now we define a linear operator

$$\begin{aligned}\ell : \Omega^k(M) &\to \mathbb{R} \\ \varphi &\mapsto \lim_{k\to\infty}(\alpha_{n_k}, \varphi)\end{aligned}$$

We claim that ℓ is bounded. Indeed,

$$\begin{aligned}|\ell(\varphi)| &= |\lim_{k\to\infty}(\alpha_{n_k}, \varphi)| \\ &= \lim_{k\to\infty}|(\alpha_{n_k}, \varphi)| \\ &\leqslant \lim_{k\to\infty}||\alpha_{n_k}|| \cdot ||\varphi|| \\ &= ||\varphi||.\end{aligned}$$

Moreover, we check that ℓ is a weak solution. Indeed,

$$\begin{aligned}\ell(\Delta\varphi) &= \lim_{k\to\infty}(\alpha_{n_k}, \Delta\varphi) \\ &= \lim_{k\to\infty}(\Delta\alpha_{n_k}, \varphi) \\ &= 0.\end{aligned}$$

By Theorem 1.9.24, we can write $\ell(\varphi) = (\omega, \varphi)$ for some smooth form $\omega \in \Omega^k$, such that $\Delta\omega = 0$. Therefore, $\alpha_{n_k} \to \omega$, so $||\omega|| = 1$ and $\omega \in \mathcal{H}(M)^\perp$. However, since $\Delta\omega = 0$, we note $\omega \in \mathcal{H}^k(M)$, which is a contradiction. ■

□

Remark 1.9.27. There is also a complex version of Theorem 1.9.16, which involves $\bar{\partial}$. For instance, c.f., [GH14].

2 Bundle Theory

2.1 Vector Bundles

Definition 2.1.1. For a map $\pi : E \to M$, a *trivializing chart* of dimension/rank r is a chart (U, ϕ) where

- $U \subseteq M$ is open, and
- $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ is a diffeomorphism,

such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\phi\quad} & U \times \mathbb{R}^r \\ & {}_{\pi}\searrow \quad \swarrow_{\mathrm{pr}_U} & \\ & U & \end{array}$$

commutes.

Notation. Fix $p \in M$,

- we denote $E_p = \pi^{-1}(p)$ to be the fiber over p;
- we denote ϕ^p to be the diffeomorphism given by the composition

$$E_p \xrightarrow{\phi} \{p\} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$$

Therefore, under this notation, E_p is a vector space. Unpacking all of this, we note that ϕ^p and the projection determines ϕ itself, via

$$\phi(v) = (\pi(v), \phi^{\pi(v)}(v)).$$

Definition 2.1.2. An *atlas* of trivializing charts for $\pi : E \to M$ is a collection of trivializing charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ such that

i. $\{U_\alpha\}_{\alpha \in I}$ is an open cover of M, and

ii. given any $\alpha \neq \beta$, for any $p \in U_\alpha \cap U_\beta$, we have a linear isomorphism

$$\phi_\alpha^p \circ (\phi_\beta^p)^{-1} : \mathbb{R}^r \to \mathbb{R}^r$$

demonstrating compatibility.

Remark 2.1.3. From the definition, it is clear that π is a surjective submersion.

Definition 2.1.4. A *vector bundle* $\xi = (E, \pi, M)$ is a map $\pi : E \to M$ together with a maximal atlas $\mathcal{C}$ of trivializing charts.

Remark 2.1.5.

- By a maximal atlas, we mean that if (U, ϕ) is any trivializing chart such that for any $p \in U_\alpha \cap U$, $\phi_\alpha^p \circ (\phi^p)^{-1}$ and $\phi^p \circ (\phi_\alpha^p)^{-1}$ are linear isomorphisms, then $(U, \phi) \in \mathcal{C}$.
- Any atlas is contained in a unique maximal atlas, therefore determining a unique vector bundle. Therefore, to define a vector bundle, it suffices to give an atlas.
- In the case of complex vector bundles, we should change all instances of $\mathbb{R}^r$ to $\mathbb{C}^r$, and ($\mathbb{R}$-)linear isomorphisms are now complex linear isomorphisms. However, the manifold is still a real manifold. This is different from holomorphic vector bundles, which are complex vector bundles over complex manifolds.

End of Lecture 19

From now on, we will call E the total space, M the base space, and π the projection. Recall that by definition, π is a surjective submersion.

Definition 2.1.6. Given the vector bundles $\xi_i = (E_i, p_i, M_i)$ for $i = 1, 2$, a *morphism* $(\Psi, \psi) : \xi_1, \xi_2$ is a pair of maps such that

i. the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\Psi} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow[\psi]{} & M_2 \end{array}$$

commutes, and

ii. for any $p \in M_1$, $\Psi : E_p \to E_{\psi(p)}$ is a linear map.

Moreover, we say (Ψ, ψ) is

- an *equivalence* if Ψ is a diffeomorphism, and
- an *isomorphism* if $M_1 = M_2$, $\psi = \mathrm{id}$, and Ψ is a diffeomorphism.

Furthermore, we denote **Vec** to be the category of vector bundles, and $\mathbf{Vec}(M)$ to be the category of vector bundles over M, with morphisms as covering $\psi = \mathrm{id}_M$.

We denote

$$\Gamma_U(E) = \{s : U \to E \mid \pi \circ s = \mathrm{id}_U\}$$

to be the *sections* of $\pi : E \to M$ over U, and $\Gamma(E) = \Gamma_M(E)$ to be the *global sections* of $\pi : E \to M$. A *frame* for $\pi : E \to M$ over U is then a collection $\{s_1, \dots, s_r\} \subseteq \Gamma_U(E)$ such that for any $p \in U$, $\{s_1(p), \dots, s_r(p)\}$ is a basis for E_p.

Example 2.1.7.

a. Tangent bundle TM, dual bundle T^*M, tensor products of them of the form $\bigotimes^r TM \bigotimes^s T^*M$, and exterior products of the form $\bigwedge^k TM$ and $\bigwedge^k T^*M$, and so on, are all examples of vector bundles.

b. The vector bundle

$$\mathcal{E}_M^r = (M \times \mathbb{R}^k, \mathrm{pr}_M, M)$$

is called the *trivial vector bundle* of rank r.

Proposition 2.1.8. A vector bundle $\xi = (E, p, M)$ has a global frame if and only if it is isomorphic to a trivial vector bundle $\mathrm{pr}_M : M \times \mathbb{R}^r \to M$.

Proof. The ($\Leftarrow$)-direction is obviously. Conversely, if we can get a global frame $\{s_1, \dots, s_r\}$ for ξ, then we can define an isomorphism

$$\begin{aligned} \Psi : \xi &\xrightarrow{\simeq} \mathcal{E}_M^r \\ v_x = \sum_{i=1}^r \lambda^i s_i(x) &\mapsto (x, (\lambda^1, \dots, \lambda^r)). \end{aligned}$$

□

Definition 2.1.9. A manifold is called *parallelizable* if TM is trivial.

Example 2.1.10.

a. The only parallelizable spheres $\mathbb{S}^n$ are when $n = 0, 1, 3, 7$. Note that these four cases correspond to elements of unit norm in the normed division algebras of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, respectively, on which we can create parallelism.

b. Any Lie group is parallelizable. For example, $\mathbb{S}^3$ can be viewed as the Lie group $\mathrm{SU}(2)$.

c. $\mathbb{S}^2$ is not parallelizable by the hairy ball theorem.

Example 2.1.11. Vector bundles of rank one are called *line bundles*. Up to isomorphism, there are exactly two line bundles over $\mathbb{S}^1$, namely $\mathcal{E}^1_{\mathbb{S}^1}$ and $\gamma^1_1 = \{([x], v) : v = \lambda x, \lambda \in \mathbb{R}\}$, along with the map

$$\begin{aligned} \gamma^1_1 &\to \mathbb{S}^1 = \mathbb{RP}^1 \\ ([x], \gamma) &\mapsto [x]. \end{aligned}$$

More generally, one can define a line bundle γ^1_d over $\mathbb{RP}^d$ via $\gamma^1_d = \{([x], v) \in \mathbb{RP}^d \times \mathbb{R}^{d+1} : \gamma = \lambda x, \lambda \in \mathbb{R}\}$ with map $\pi : \gamma^1_d \to \mathbb{RP}^d$. These bundles γ^1_d's are called the *tautological line bundles*.

Exercise 2.1.12. Check that this is a smooth manifold and π satisfies triviality.

Remark 2.1.13. We will see later that a line bundle over a 1-connected manifold is trivial.

We will now use cocycles to describe the vector bundles.

Definition 2.1.14. Let $\xi = (E, \pi, M)$ be a vector bundle, and choose an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$. For any $p \in U_\alpha \cap U_\beta$, we recall that $\varphi^p_\alpha \circ (\varphi^p_\beta)^{-1} : \mathbb{R}^r \to \mathbb{R}^r$ is always a linear isomorphism. Therefore, we can define

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\to \mathrm{GL}_r(\mathbb{R}) \\ p &\mapsto \varphi^p_\alpha \circ (\varphi^p_\beta)^{-1} \end{aligned}$$

Note that this collection of $g_{\alpha\beta}$'s satisfies the following properties:

i. for any $p \in U_\alpha$, $g_{\alpha\alpha}(p) = I$;

ii. for any $p \in U_\alpha \cap U_\beta$, we have $g_{\beta\alpha}(p) g_{\alpha\beta}(p) = I$;

iii. for any $p \in U_\alpha \cap U_\beta \cap U_\gamma$, we have $g_{\alpha\beta}(p) g_{\beta\gamma}(p) g_{\gamma\alpha}(p) = I$.

In particular, note that (iii) implies the first two properties. We call (iii) the *cocycle condition*, and $\{g_{\alpha\beta}\}$'s the *cocycle* associated with the atlas $\{(U_\alpha, \varphi_\alpha)\}$.

Lemma 2.1.15. Let $\{g_{\alpha\beta}\}$ and $\{\bar{g}_{\alpha\beta}\}$ be cocycles associated with atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(U_\alpha, \bar{\varphi}_\alpha)\}$with the same open cover, then

$$\bar{g}_{\alpha\beta}(p) = \lambda_\alpha(p) g_{\alpha\beta}(p) \lambda_\beta(p)^{-1}$$

for smooth maps $\lambda_\alpha : U_\alpha : \mathrm{GL}_r(\mathbb{R})$.

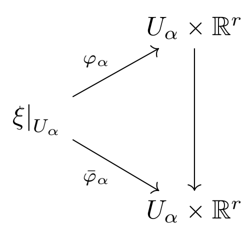
Proof. Set

$$\begin{aligned} \lambda_\alpha : U_\alpha &\to \mathrm{GL}_r(\mathbb{R}) \\ p &\mapsto \bar{\varphi}^p \circ (\varphi^p)^{-1}, \end{aligned}$$

then this builds an assignment

$$\begin{aligned} U_\alpha \times \mathbb{R}^r &\to U_\alpha \times \mathbb{R}^r \\ (p, v) &\mapsto (p, \lambda_\alpha(p)(v)) \end{aligned}$$

such that the diagram



commutes. Therefore, for any $p \in U_\alpha \cap U_\beta$, we have

$$\begin{aligned}\bar{g}_{\alpha\beta}(p) &= \bar{\varphi}_\alpha^p \circ (\bar{\varphi}_\beta^p)^{-1} \\ &= \lambda_\alpha(p) \circ \varphi_\alpha^p \circ (\varphi_\beta^p)^{-1} \circ \lambda_\beta(p)^{-1} \\ &= \lambda_\alpha(p) g_{\alpha\beta}(p) \lambda_\beta^{-1}(p).\end{aligned}$$

□

Note that if we have cocycles $\{g_{\alpha\beta}\}$ and $\{\bar{g}_{\bar{\alpha}\bar{\beta}}\}$ associated with atlases with different covers, then one can restrict cocycles to $U_\alpha \cap \bar{U}_{\bar{\alpha}}$, and then apply Lemma 2.1.15, hence allowing a generalization.

Definition 2.1.16. Let M be a manifold.

i. A *cocycle subordinated to a cover* $\{U_\alpha\}$ is a family of maps satisfying the cocycle condition.

ii. Two cocycles are called *equivalent* if there is a refinement of their covers such that their restrictions satisfy Lemma 2.1.15 for some family $\{\lambda_\alpha\}$.

Theorem 2.1.17. For a manifold M, there is a one-to-one correspondence between isomorphism classes of vector bundles of rank r over M, and the equivalence classes of cocycles $\{g_{\alpha\beta}\}$.

Remark 2.1.18. The latter is usually known as $H^1(M, \mathrm{GL}_r)$, the non-abelian cohomology with coefficients in GL_r.

Proof. Given cocycle $\{g_{\alpha\beta}\}$ subordinated to a cover $\{U_\alpha\}$, we construct a vector bundle

$$\begin{aligned}\pi : E := \left(\bigcup_\alpha U_\alpha \times \mathbb{R}^r\right) / \sim &\to M \\ (x, v) &\mapsto x\end{aligned}$$

with equivalence relation $(x, v) \sim (y, w)$ defined by

$$\begin{cases} x = y \in U_\alpha \cap U_\beta \\ w = g_{\alpha\beta}(x) v \end{cases}.$$

We then equip E with quotient topology. We give a local trivialization $(U_\alpha, \varphi_\alpha)$ of E via

$$\begin{aligned}\varphi_\alpha : \pi^{-1}(U_\alpha) &\to U_\alpha \times \mathbb{R}^r \\ [(x, v)] &\mapsto (x, v),\end{aligned}$$

then for any local chart (V, ψ) of M, we have

$$\begin{aligned}\pi^{-1}(U_\alpha \cap V) &\to \mathbb{R}^n \times \mathbb{R}^r \\ [(x, v)] &\mapsto (\psi(x), \varphi_\alpha([x, v]))\end{aligned}$$

as a local chart for E. Therefore,

- E is a smooth manifold, and $\pi : E \to M$ is a surjective submersion, and
- $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is a vector bundle atlas with cocycle $\{g_{\alpha\beta}\}$ defined above.

To check that this is well-defined, given another equivalent cocycle $\{\bar{g}_{\alpha\beta}\}$, we can assume it has the same open cover by Lemma 2.1.15, then we can define a vector bundle isomorphism

$$\begin{aligned}\Phi : E &\to \bar{E} \\ [(p, v)] &\mapsto [(p, \lambda_\alpha(p)(v)]\end{aligned}$$

which shows that this is well-defined. □

End of Lecture 20

2.2 Constructions with Vector Bundles

Let $\xi = (E, \pi, M)$ be a vector bundle and $N \subseteq M$ be a submanifold, then the restriction $\xi|_N = (\pi^{-1}(N), \pi, N)$ to the submanifold is also a vector bundle. Moreover, $i : \xi|_N \hookrightarrow \xi$ is a vector bundle morphism covering $N \hookrightarrow M$.

Definition 2.2.1. A vector bundle $\eta = (F, \tau, N)$ is called a *vector subbundle* of $\xi = (E, \pi, M)$ if $F \subseteq E$ is a submanifold and $\eta \hookrightarrow \xi$ is a vector bundle morphism.

Given a morphism $(\Psi, \mathrm{id}) : \eta \to \xi$ covering id with constant rank, then

- the kernel $\ker(\Psi) = \{v \in E : \Psi(v) = 0\} \subseteq \eta$ is a vector subbundle;
- the image $\operatorname{im}(\Psi) = \{\Psi(v) : v \in E\} \subseteq \xi$ is a vector subbundle;
- the cokernel $\operatorname{coker}(\Psi) = \xi/\operatorname{im}(\Psi)$ is a vector bundle over M, whose fiber is defined pointwise by this quotient $E_p/\operatorname{im}(\Psi^p)$.

Remark 2.2.2. In general, given a vector bundle $\xi = (E, \pi, M)$ and a vector subbundle $\eta = (F, \tau, M)$,[4] then the quotient vector bundle ξ/η has fibers E_p/F_p defined pointwise. In turn, there is a natural map

$$\begin{aligned} q : \xi &\to \xi/\eta \\ v &\mapsto [v] \end{aligned}$$

Proposition 2.2.3. The quotient vector bundle fits into a short exact sequence of vector bundles

$$0 \longrightarrow \eta \longrightarrow \xi \xrightarrow{q} \xi/\eta \longrightarrow 0$$

where injectivity and surjectivity are defined fiberwise.

Proof. Exercise. □

Example 2.2.4.

1. Let $N \subseteq M$ be a submanifold, then there is an inclusion $TN \subseteq T_N M$. For the vector bundles to have the same base, we consider $TN \subseteq T_N M$ to be the vector subbundle, and the quotient is the *normal bundle* to N in M, denoted $\nu(N) = T_N M/TN$.
2. Suppose $\mathcal{F}$ is a foliation of M, then $T\mathcal{F} \subseteq TM$ is a subbundle of TM. The *normal bundle of the foliation* $\mathcal{F}$ is the quotient $\nu(\mathcal{F}) = TM/T\mathcal{F}$.

We can also build up vector bundles from Whitney sum.

Definition 2.2.5. Given vector bundles $\xi_1 = (E_1, \pi_1, M)$ of rank r_1 and $\xi_2 = (E_2, \pi_2, M)$ of rank r_2, the *Whitney sum* of ξ_1 and ξ_2 is the vector bundle $\xi_1 \oplus \xi_2 = (E_1 \times_M E_2, \pi, M)$, of rank $r_1 + r_2$

Remark 2.2.6. Note that the *product* $E_1 \times E_2$ is also a vector bundle, but it is given by the structure map $(\pi_1, \pi_2) : E_1 \times E_2 \to M \times M$. In this language, the Whitney sum is the restriction of the product bundle to the diagonal $\Delta \subseteq M \times M$.

Fixing local trivializations $\{g^1_{\alpha\beta}\}$ and $\{g^2_{\alpha\beta}\}$ from $(U_\alpha, \varphi^1_\alpha)$ and $(U_\alpha, \varphi^2_\alpha)$, respectively, then we can build a local trivailization $(U_\alpha, \varphi_\alpha)$ for $\xi_1 \oplus \xi_2$, where

$$\varphi_\alpha(v_1, v_2) = (x, (\varphi^1_\alpha)^p(v_1), (\varphi^2_\alpha)^p(v_2))$$

with cocycle

$$g_{\alpha\beta} = g^1_{\alpha\beta} \oplus g^2_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(\mathbb{R}^{r_1} \oplus \mathbb{R}^{r_2})$$

hence of the form

$$\begin{pmatrix} g^1_{\alpha\beta} & 0 \\ 0 & g^2_{\alpha\beta} \end{pmatrix}$$

[4]Assuming this is given by an inclusion, so that the vector bundles have the same base.

Remark 2.2.7. The idea being, any constructions we can do for vector spaces can be done on vector bundles.

Definition 2.2.8. The *tensor product* $\xi_1 \otimes \xi_2$ of vector bundles ξ_1 of rank r_1 and ξ_2 of rank r_2 has fibers $(E_1)_p \otimes (E_2)_p$ at each point p, and the local trivialization gives

$$g^1_{\alpha\beta} \otimes g^2_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^{r_1} \otimes \mathbb{R}^{r_2}).$$

Definition 2.2.9. The kth *wedge power* $\bigwedge^k \xi$ of a vector bundle ξ of rank r has fibers $\bigwedge^k E_p$ at each point p, and the local trivialization gives

$$\bigwedge^k g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}\left(\bigwedge^k \mathbb{R}^r\right).$$

Definition 2.2.10. The *dual vector bundle* ξ^* of ξ has fiber $(E_p)^*$ defined pointwise, with $g^*_{\alpha\beta} = (g^T_{\alpha\beta})^{-1}$.

Definition 2.2.11. Given vector bundles ξ and η, the *hom set* $\mathrm{Hom}(\xi, \eta)$ is a vector bundle with fiber $\mathrm{Hom}(E_p, F_p)$ defined pointwise.

Exercise 2.2.12. There is a canonical isomorphism

$$\mathrm{Hom}(\xi, \eta) \simeq \xi^* \otimes \eta.$$

We will introduce another operation, namely the pullback of a vector bundle, later on in the course.

Definition 2.2.13. A vector bundle $\xi = (E, \pi, M)$ of rank r is *orientable* if the top wedge power $\bigwedge^r \xi$ is a trivial line bundle, i.e., has a non-vanishing section.

We say two non-vanishing sections $s_1, s_2 \in \Gamma\left(\bigwedge^r \xi\right)$ are *equivalent* if there exists $0 < f \in C^\infty(M)$ such that $s_2 = f s_1$.

An *orientation* of a vector bundle ξ of rank r is an equivalence class $[s]$ for a non-vanishing section $s \in \Gamma\left(\bigwedge^r \xi\right)$.

Remark 2.2.14.

1. If $\xi = TM$, then this recovers the usual notion of orientation.
2. A vector bundle ξ is orientable if and only if it admits a local trivialization $\{(U_\alpha, \varphi_\alpha)\}$ for which the cocycle $\{g_{\alpha\beta}\}$ taking values in $\mathrm{GL}_+(\mathbb{R}^r)$.
3. $\xi = (E, \pi, M)$ can be orientable without E and M being orientable. For instance, consider the trivial bundle over a non-orientable manifold.

Proposition 2.2.15. If two among ξ, E, and M are orientable, then so is the third.

Proof Sketch.

- Step 1: one can show that a short exact sequence

$$0 \longrightarrow \xi \longrightarrow \eta \longrightarrow \theta \longrightarrow 0$$

of vector bundles also exhibits the 2-out-of-3 property on orientability, i.e., if any two bundles out of the three are orientable, so is the third.

- Step 2: consider the zero section

$$\begin{aligned} s_0 : M &\hookrightarrow E \\ x &\mapsto 0_x \end{aligned}$$

we exhibit M as a submanifold of E, then the differential of the projection

$$T_M E \to TM$$

is surjective as a vector bundle map, therefore it extends to a short exact sequence with kernel isomorphic to ξ:

$$0 \longrightarrow \xi \longrightarrow T_M E \xrightarrow{d\pi} TM \longrightarrow 0$$

Here $T_M E = \{v_p : E_p : p \in M\}$.

- Step 3: note that $T_M E$ is orientable if and only if TE is orientable.

□

Let us now consider the Riemannian metrics on vector bundles. Note that we just want an inner product $\langle -, - \rangle_p : E_p \times E_p \to \mathbb{R}$ for each point $p \in M$, which varies smoothly with p. More formally,

Definition 2.2.16. A *Riemannian metric on a vector bundle* $\xi = (E, \pi, M)$ is an inner product $\langle -, - \rangle : \Gamma(\xi^* \otimes \xi^*) \to \mathbb{R}$.

Exercise 2.2.17. Show that a vector bundle ξ of rank r has a Riemannian metric if and only if one can choose a local trivialization $\{(U_\alpha, \varphi_\alpha)\}$ for which the cocycle $\{g_{\alpha\beta}\}$ takes values in $\mathrm{O}(r) \subseteq \mathrm{GL}_r(\mathbb{R})$. This is quite surprising in general, but the deep reason is that $\mathrm{GL}(\mathbb{R}^r) \cong \mathrm{O}(r) \times P(r)$ where $P(r)$ are the $r \times r$ positive-definite symmetric matrices, which is contractible.

Proposition 2.2.18. Every vector bundle admits a Riemannian metric.

Proof. Choose a local trivialization $\{(U_\alpha, \varphi_\alpha)\}$ and a partition of unity $\rho = \{\rho_\alpha\}$ subordinated to $\{U_\alpha\}$, then

$$\langle v, w \rangle_p = \sum_\alpha \rho_\alpha(p) \langle \varphi_\alpha^p(v), \varphi_\alpha^p(w) \rangle_{\mathbb{R}^r}$$

defines a Riemannian metric. □

Remark 2.2.19. Given a vector subbundle $\eta \subseteq \xi$ where $\eta = (F, \tau, N)$ and $\xi = (E, \pi, M)$, we have an abstract quotient $\xi|_N / \eta$ since the bundles have the same base. Once we fix a Riemannian metric on ξ, we can write $\xi|_N = \eta \oplus \eta^\perp$, where $\eta^\perp$ is the vector bundle over N with fibers $F_p^\perp \subseteq E_p$ defined pointwise. Therefore, we identify $\xi|_N / \eta \simeq \eta^\perp$, therefore identifying it as a subbundle of ξ.

2.3 Thom Class and Euler Class

Recall that

$$H^*(M \times \mathbb{R}^r) \simeq H^*(M),$$

where we view $\mathbb{R}^r$ as a trivial vector bundle, thereby identifying the product as a total space. We have the following generalization.

Proposition 2.3.1. For any vector bundle $\xi = (E, \pi, M)$, we have

$$H^*(E) \simeq H^*(M).$$

Proof. Consider the zero section $s_0 : M \to E$, then

$$\begin{cases} \pi \circ s_0 = \mathrm{id} \\ s_0 \circ \pi \sim_h \mathrm{id} \end{cases}$$

given by a homotopy

$$\begin{aligned} h : E \times [0,1] &\to E \\ (v, t) &\mapsto tv \end{aligned}$$

that contracts the fibers. Therefore, on the level of cohomology, π and s_0 are inverses of one another.

$$H^*(E) \underset{\pi^*}{\overset{s_0^*}{\rightleftarrows}} H^*(M)$$

□

We will now see what happens if we consider the cohomology in terms of compact support.

End of Lecture 21

Recall that for cohomology of compact support, we have

$$H_c^*(M \times \mathbb{R}^r) \simeq H_c^{*-r}(M).$$

Does this generalizes to arbitrary vector bundles? The answer is no.

Example 2.3.2. Consider the tautological bundle $\gamma_1^1 \to \mathbb{S}^1$, with total space E isomorphic to the Möbius band. Recall that if a connected manifold M has dimension n, then

$$H_c^n(M) = \begin{cases} \mathbb{R}, & M \text{ orientable} \\ 0, & M \text{ non-orientable} \end{cases}$$

along with the general fact that the Möbius band is non-orientable, therefore $H_c^2(E) = 0$, but $H_c^1(\mathbb{S}^1) = \mathbb{R}$.

Definition 2.3.3. A manifold M is *of finite type* if it has a finite open cover $\{U_1, \ldots, U_N\}$ such that any subcollection $U_{i_1} \cap \cdots \cap U_{i_k} \simeq \mathbb{R}^n$. Such subcollection is called a *good cover*.

Remark 2.3.4.

- Any manifold admits a good cover, but only the ones with finite good cover are called of finite type.
- A manifold of finite type has finite-dimensional cohomology.
- If M is an oriented manifold of finite type of dimension n, then there is a non-degenerate bilinear pairing

$$\begin{aligned} H^*(M) \times H_c^{n-*} &\to \mathbb{R} \\ ([\omega], [\eta]) &\mapsto \int_M \omega \wedge \eta \end{aligned}$$

  It then follows that $H^*(M) \simeq (H_c^{n-*})^* \simeq H_c^{n-*}$ as finite-dimensional vector spaces, retrieving *Poincaré duality*.

Theorem 2.3.5 (Thom Isomorphism)**.** Let $\xi = (E, \pi, M)$ be a vector bundle of rank r, where both E and M are oriented of finite type, then

$$H_c^*(E) \simeq H_c^{*-r}(M).$$

In particular, this is valid if ξ is an oriented vector bundle over an oriented manifold.

Proof. Suppose M has dimension n, then by Poincaré duality,

$$H_c^*(E) \simeq H^{(n+r)-*}(E) \simeq H^{(n+r)-*}(M) \simeq H_c^{*-r}(M).$$

□

We now give an explicit construction of this isomorphism in Theorem 2.3.5, given by *fiber integration*. Instead of a pullback map, we have a pushforward

$$\pi_* : \Omega_c^*(E) \to \Omega_c^{*-r}(M)$$

defined as follows. Fixing a positively-oriented local chart (U_α, x^i) as well as a local trivialization $(U_\alpha, \varphi_\alpha)$ of the vector bundle, again we can assume this is also positively-oriented, then this gives local coordinates (x^i, t^j) on the total space on $\pi^{-1}(U_\alpha) \subseteq E$.

Given $\omega \in \Omega_c^*(E)$, the restriction $\omega_\alpha = \omega|_{\pi^{-1}(U_\alpha)}$ is a sum of two types of forms,

- functions $f(x,t)\pi^*(\theta_\alpha)dt^{i_1} \wedge \cdots dt^{i_k}$, with degree given by the sum of degree of θ as well as $i_1, \ldots, i_k$, with the condition that $k < r$;
- functions $f(x,t)\pi^*(\theta_\alpha)dt^1 \wedge \cdots \wedge dt^r$,

where f is compactly-supported on fibers, and θ_α is a form on the base U_α. We now define the isomorphism by sending the forms of first type to 0, and the forms of second type to $\left(\int_{\pi^{-1}(x)} f dt^1 \cdots dt^r\right)\theta_\alpha$, via π_*. This does not depend on the choices we made above.

Fiber integration exhibits two important properties.

- π_* is a chain map: $\pi_* d = d\pi_*$, therefore inducing a map on the level of cohomology, as in Theorem 2.3.5.
- π_* satisfies the *projection formula*: given a form θ on the base, then $\pi_*(\pi^*(\theta) \wedge \omega) = \theta \wedge \pi_*(\omega)$.

We know that if M is compact, connected, oriented, and of dimension n, then

$$H_c^n(M) = H^n(M) \simeq \mathbb{R}.$$

This isomorphism is canonical, given by the *orientation class* $\mu \in H^n(M)$, determined by the property that $\mu = [\omega]$ for any form $\omega \in \Omega^n(M)$ such that $\int_M \omega = 1$. Therefore, this canonical isomorphism maps μ to the number $1 \in \mathbb{R}$.

Remark 2.3.6. In terms of Poincaré duality, we see that

$$H^n(M) \simeq H^0(M),$$

which is the number of (constant functions on) connected components, therefore the orientation class corresponds to the constant function 1 in this case, as opposed to the number $1 \in \mathbb{R}$.

Definition 2.3.7. The *Thom class* of an oriented vector bundle $\xi = (E, \pi, M)$ of rank r over a compact, oriented, connected manifold M is the unique class $U \in H_c^r(E)$ that satisfies $\pi_*(U) = 1$.

Remark 2.3.8.

a. Using the Thom class, the inverse of Theorem 2.3.5 can be expressed explicitly as the assignment $[\theta] \mapsto [\pi^*(\theta)] \smile U$ as a cup product. Applying fiber integration to this assignment using the projection formula, we recover $[\theta]$.

b. Given a morphism $\Phi : \xi_1 \to \xi_2$ of vector bundles that is an orientation-preserving isomorphism on the fibers, of Thom classes U_1 and U_2, respectively, then $\Phi^*(U_2) = U_1$. Therefore, the Thom class is an invariant of the vector bundle.

We now see that gluing the orientation class (given by pullback of the Thom class) of each fiber of the morphism together, we recover 1, which is the idea of the following theorem.

Theorem 2.3.9. The Thom class is the unique cohomology class $U \in H_c^r(E)$ with the property that $i_p^* U \in H_c^r(E_p)$ is the orientation class of the fiber for all $p \in M$, where $i_p : E_p \hookrightarrow E$ is the inclusion of fiber over p. In particular, if $U = [u] \in H_c^*(E)$, then $\int_{E_p} i_p^*(u) = 1$ for all $p \in M$.

Proof. Since $\pi_*(U) = 1$, then $i_p^*(U) = 1$, therefore $\int_M i_p^*(u) = 1$. Now suppose there is another class $\bar{U} = [\bar{u}] \in H_c^*(E)$ with the same property, then by the projection formula, for every form $\theta \in \Omega^0(M)$ on the base, $\pi_*(\pi^*(\theta) \wedge \bar{u}) = \theta \wedge \pi_*(\bar{u}) = \theta$. In particular, this class $\bar{U}$ defines an inverse $[\theta] \mapsto \pi^*[\theta] \smile \bar{U}$ to fiber integration π_*, which is the defining property of the Thom class U. ☐

We see that the Thom class is defined over the total space. If we pull this back down to the base space via the zero section, we obtain the Euler class.

Definition 2.3.10. Suppose $\xi = (E, \pi, M)$ is an oriented vector bundle of rank r over a compact, oriented, connected manifold M. The *Euler class* $\chi(\xi)$ of ξ is

$$s_0^*(U) \in H^r(M)$$

where $s_0 : M \hookrightarrow E$ is the zero section.

Remark 2.3.11. The choice of section does not really matter, since any section is homotopic to the zero section. That is, if s is any section, then we can define a homotopy $H(t,x) = ts(x)$ from the zero section s_0 to the chosen section s. Therefore, $s^* = s_0^* : H^*(E) \to H^*(M)$.

End of Lecture 22

Recall that

- let $\xi = (E, \pi, M)$ be an oriented vector bundle of rank r over a compact connected oriented manifold, then we have fiber integration
$$\pi_* : H_c^*(E) \simeq H^{*-r}(M).$$
The Thom class is then the class $U \in H_c^r(E)$ such that $\pi_*(U) = 1$. The Euler class is the pullback $\chi(\xi) = \xi_0^* U$ of the zero section $\pi : M \to E$ of the Thom class. Again, recall that the choice of section does not matter.

The following theorem shows that the Euler class is an obstruction to the non-vanishing sections.

Theorem 2.3.12. If ξ has a non-vanishing section, then the Euler class $\chi(\xi) = 0$.

Proof. Since the base space is compact, if a non-vanishing section $s : M \to E$ exists, then choosing a representative $U = [u]$ of the Thom class for some compactly-supported form $u \in \Omega_c^r(E)$, so the image is compact, therefore by multiplying some large enough $\lambda > 0$, then
$$\mathrm{im}(\lambda s) \cap \mathrm{supp}(u) = \varnothing.$$
For such λ, we have $(\lambda s)^* u = 0$, therefore $\chi(\xi) = 0$. □

Remark 2.3.13.

1. The converse does not hold in general: there exists ξ such that $\chi(\xi) = 0$ but does not admit non-vanishing sections.
2. However, if the rank of ξ is $\dim(M)$, then one can show that $\chi(\xi) = 0$ implies the existence of a non-vanishing section.

Proposition 2.3.14.

i. If $\Phi : \xi_1 \to \xi_2$ is an orientation-preserving morphism that is also a fiberwise isomorphism, then the pullback $\varphi^* \chi(\xi_2) = \chi(\xi_1)$ for
$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow[\varphi]{} & M_2 \end{array}$$

ii. If $\bar{\xi}$ is ξ with opposite orientation, then $\chi(\bar{\xi}) = -\chi(\xi)$.

iii. If r is odd, then $\chi(\xi) = 0$.

iv. Note that the Whitney sum $\xi_1 \oplus \xi_2$ admits a natural orientation from the orientations of ξ_1 and ξ_2, then we have $\chi(\xi_1 \oplus \xi_2) = \chi(\xi_1) \smile \chi(\xi_2)$.

Proof.

i. Let $U_i = (\pi_i)_*^{-1}(1)$ be the Thom class of ξ_i for $i = 1, 2$, then for
$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow[\varphi]{} & M_2 \end{array}$$

taking pullback gives

$$(\pi_1)_*\Phi^*\omega = \varphi^*(\pi_2)_*\omega.$$

Applying this to the definition, we get

$$\begin{aligned}\Phi^*U_2 &= \Phi^*(\pi_2)_*^{-1}(1)\\ &= (\pi_1)_*^{-1}\varphi^*(1)\\ &= (\pi_1)_*^{-1}(1)\\ &= U_1.\end{aligned}$$

ii. Omitted.

iii. Follows from ii.

iv. Omitted.

□

Theorem 2.3.15. If M is a compact connected oriented manifold of dimension n, and $X \in \mathfrak{X}(M)$ has zeros $\{p_1, \dots, p_N\}$, then

$$\chi(TM) = \sum_{i=1}^{N} \mathrm{ind}_{p_i}(X)\mu$$

where $\mu \in H^n(M)$ is the orientation class.

Before proving that, we will take a detour to the degree and index of a vector field. Suppose we have a map $\varphi : M_1 \to M_2$ between (compact oriented connected) manifolds of the same dimension. There is a then a map on the level of top cohomology

$$\mathbb{R} \simeq H^n(M_2) \xrightarrow{\varphi^*} H^n(M_1) \simeq \mathbb{R}$$

which is thereby determined by a real number $\deg(\varphi) \in \mathbb{R}$, called the *degree* of φ, such that $\varphi^*[\omega] = \deg(\varphi)[\omega]$. This is equivalent to the fact that, given any $\omega \in \Omega^n(M_2)$, we have

$$\int_{M_1} \varphi^*\omega = (\deg(\varphi)) \int_{M_2} \omega.$$

Surprisingly, $\deg(\varphi)$ is an integer.

Remark 2.3.16. Throughout the discussion today, compactness is not necessary: we can work with compactly-supported cohomology instead. However, in such cases, the map φ here has to be proper.

Remark 2.3.17. If φ_1 and φ_2 are homotopic, then $\deg(\varphi_1) = \deg(\varphi_2)$.

Example 2.3.18. Let us look at the antipodal map

$$\begin{aligned}\varphi : \mathbb{S}^n &\to \mathbb{S}^n\\ p &\mapsto -p\end{aligned}$$

for some orientation of the sphere, then $\deg(\varphi) = (-1)^{n-1}$. To see why, we can take the volume form on the sphere induced by

$$\omega = \sum_{i=1}^{n+1} (-1)^{i+1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge x^{n+1} \in \Omega^n(\mathbb{R}^{n+1}).$$

Pulling back ω along φ, then all x^j's are now $-x^j$'s, therefore $\varphi^*\omega = (-1)^{n-1}\omega$, hence

$$\int_{\mathbb{S}^n} \varphi^*\omega = (-1)^{n-1} \int_{\mathbb{S}^n} \omega,$$

thus $\deg(\varphi) = (-1)^{n-1}$.

Remark 2.3.19. As an application, over an even-dimensional sphere, every vector field vanishes somewhere. Assume not, then take $X \in \mathfrak{X}(\mathbb{S}^{2d})$ that is nowhere vanishing, i.e., $X_p \neq 0$ for all $p \in \mathbb{S}^{2d}$. Therefore, at each point p, there is a unique half diameter $\gamma_p : [0,1] \to \mathbb{S}^{2d}$ that connects p with $-p$, i.e., $\gamma_p(0) = p$, $\gamma_p(1) = -p$, and $\dot{\gamma}_p(0) = X_p$. Therefore, there exists a homotopy

$$\begin{aligned} H : [0,1] \times \mathbb{S}^{2d} &\to \mathbb{S}^{2d} \\ (t,p) &\mapsto \gamma_p(t) \end{aligned}$$

between the identity and antipodal map φ. This is a contradiction, since the identity has degree 1, but the antipodal map has degree -1.

Remark 2.3.20. More explicitly, considering an odd-dimensional sphere $\mathbb{S}^{2d+1} \subseteq \mathbb{R}^{2d+2}$, we have a vector field

$$X = \sum_{i=1}^{d+1} \left(x^{2i} \frac{\partial}{\partial x^{2i+1}} - x^{2i+1} \frac{\partial}{\partial x^{2i}} \right)$$

that is nowhere vanishing, for any $p \in \mathbb{S}^{2d+1}$.

Theorem 2.3.21. If $q \in M_2$ is a regular value of $\varphi : M_1 \to M_2$ under the assumptions we made above, then

$$\deg(\varphi) = \sum_{i=1}^{N} \operatorname{sgn}_{p_i}(\varphi)$$

where $\{p_1, \ldots, p_N\}$ is the set $\varphi^{-1}(q)$ of preimages of q, which is finite by compactness assumptions, and

$$\operatorname{sgn}_p(\varphi) = \begin{cases} 1, & \text{if } d_p\varphi \text{ preserves orientation} \\ -1, & \text{if } d_p\varphi \text{ reverses orientation} \end{cases}$$

Remark 2.3.22. One can check that if φ is not surjective, then $\deg(\varphi) = 0$. Therefore, if q is not in the image of φ, then the set of preimages is empty, therefore the degree is 0 by convention. Indeed, one can take an open set around the point that is disjoint with the image, then once we pullback we can compute the degree.

Proof. Let $q \in \operatorname{im}(\varphi)$ be a regular value, and let $\varphi^{-1}(q) = \{p_1, \ldots, p_N\}$, so the differential is an isomorphism, therefore the map is a local diffeomorphism at each p_i. That is, we can choose (connected) open neighborhood $U_i \ni p_i$ and open (connected) subsets $V_i \subseteq M_2$ such that $\varphi : U_i \simeq V$ is a diffeomorphism. Assume further that V is the domain of some chart $(y^1, \ldots, y^n)$ of M_2, then

$$\omega = f dy^1 \wedge \cdots \wedge dy^n$$

for $f \geqslant 0$ such that $\operatorname{supp}(f) \subseteq V$, then

$$\begin{aligned} \int_{M_1} \varphi^*\omega &= \sum_{i=1}^{N} \int_{U_i} \varphi^*\omega \\ &= \sum_{i=1}^{N} (\pm 1) \int_{V} \omega \end{aligned}$$

where ± 1's are determined by whether $\varphi|_{U_i}$ preserves or reverses orientation. But over the connected subsets, this agrees with $\operatorname{sgn}_{p_i}(\varphi)$ for each i. □

We now move on to the index of a vector field. Let us first assume that $X \in \mathfrak{X}(M)$ has a unique zero, i.e., $X_p = 0$ if and only if $p = 0$. Given $\varepsilon > 0$, we can now construct Gauss maps

$$\begin{aligned} G_\varepsilon^X : \mathbb{S}_\varepsilon^{n-1} &\to \mathbb{S}^{n-1} \\ x &\mapsto \frac{X(x)}{|X(x)|} \end{aligned}$$

that normalizes the sphere of radius ε. Choosing the orientation of the spheres to be the one induced by $\mathbb{R}^n$, then the degree of the map is well-defined, and we define the *index* to be $\operatorname{ind}_0(X) := \deg(G_\varepsilon^X)$. This index satisfies the following properties.

Lemma 2.3.23.

i. It is independent of ε.

ii. If $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism such that $\varphi(0) = 0$, then $\deg(G_\varepsilon^{\varphi_* X}) = \deg(G_\varepsilon^X)$, therefore $\text{ind}_0(\varphi_* X) = \text{ind}_0(X)$.

This is true because the degree is invariant under homotopy.

Proof.

i. Given $\varepsilon_1, \varepsilon_2 > 0$, we can define a homotopy

$$H(t, x) = G^X_{t\varepsilon_2 + (1-t)\varepsilon_1}(x)$$

between the Gauss maps G_{ε_1} and G_{ε_2}.

ii. We define

$$\varphi_t(x) = \begin{cases} \frac{1}{t}\varphi(tx), & t > 0 \\ d_0\varphi(x), & t = 0 \end{cases}.$$

This is a homotopy between the linear map $d_0\varphi(x)$ and φ. For $X_t = (\varphi_t)_* X$, we note that $\deg(G_\varepsilon^{X_t})$ is constant. This reduces to the case where the diffeomorphism is a linear map, which is homotopic to some orthogonal transformation. Therefore, we can further assume that $d_0\varphi = A \in \mathrm{O}(n)$, and we compute

$$G_\varepsilon^{A_* X}(x) = \frac{AX(A^{-1}x)}{|AX(A^{-1}x)|} = AX(A^{-1}x) = A \circ G_\varepsilon^X \circ A^{-1}(x).$$

But A is one-to-one, so by Theorem 2.3.21, the degree of $G_\varepsilon^{A_* X}$ is 1.

□

Remark 2.3.24. The construction of the index does not depend on the orientation. The only thing we need is that the manifold is isolated.

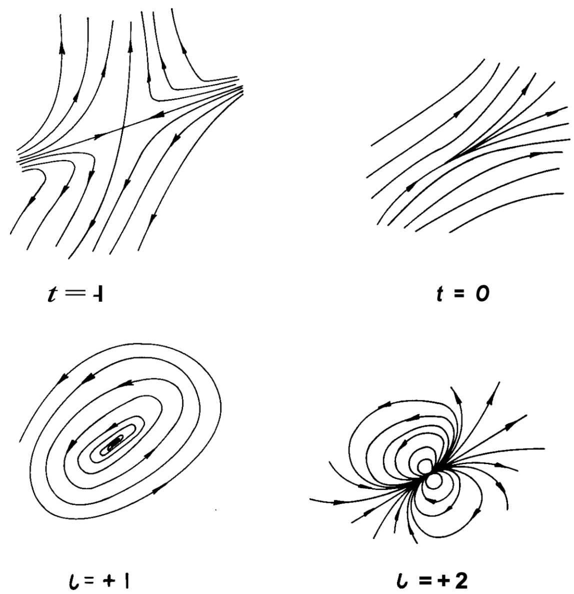
We will use this to extend the definition of index to general vector fields (of finitely many zeros), therefore proving Theorem 2.3.15.

End of Lecture 23

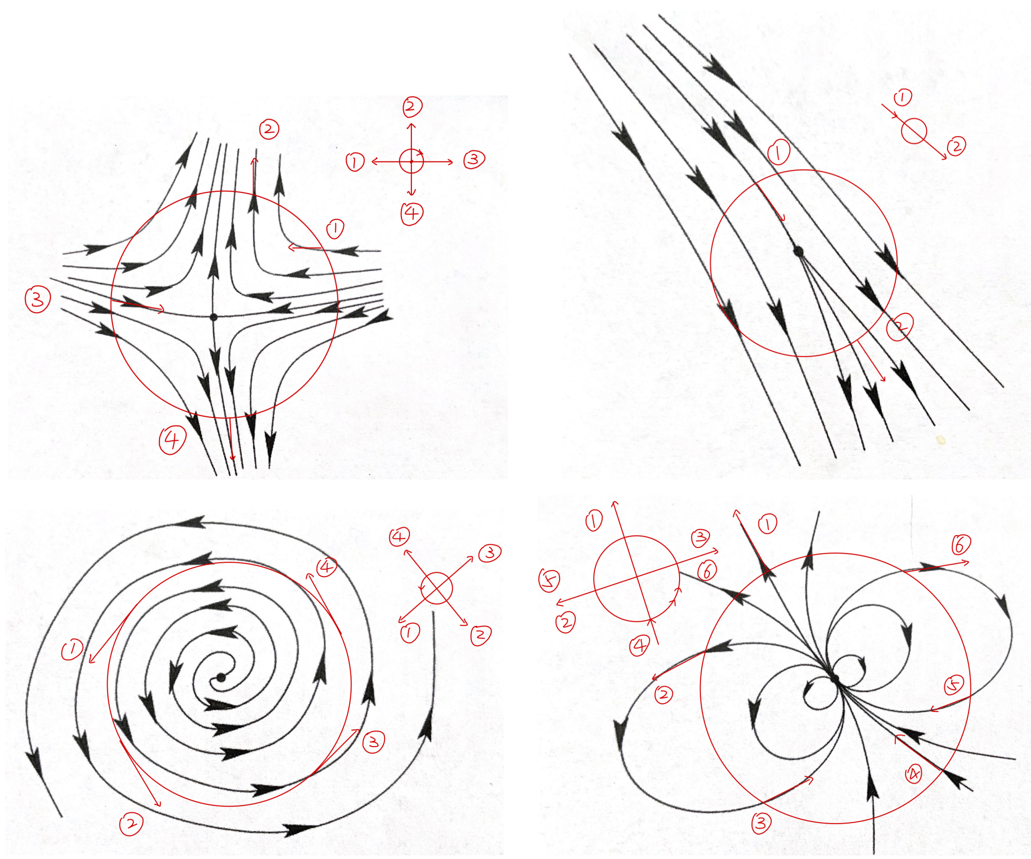
Lemma 2.3.23 allows the following definition.

Definition 2.3.25. If $X \in \mathfrak{X}(M)$ has an isolated zero at $x_0 \in M$, then we define the *index* to be the pushforward $\text{ind}_{x_0}(X) = \text{ind}_0(\varphi_*(X))$ via the chart (U, φ) centered at x_0.

The following are a few examples, taken from Figure 12 of [MW97], where we compute the index at the given point p_0 of singularity.



Draw a ball of radius ε large enough around the point (that intersects the flow), we check the direction of rotations. Calculation from the Gauss map shows that the orientation of the sphere goes counterclockwise, therefore turning around the singularity once counterclockwise gives an index of 1.



Now we should ask how to compute the index efficiently. For $X : M \to TM$, note that $X_{x_0} = 0$ if and only if

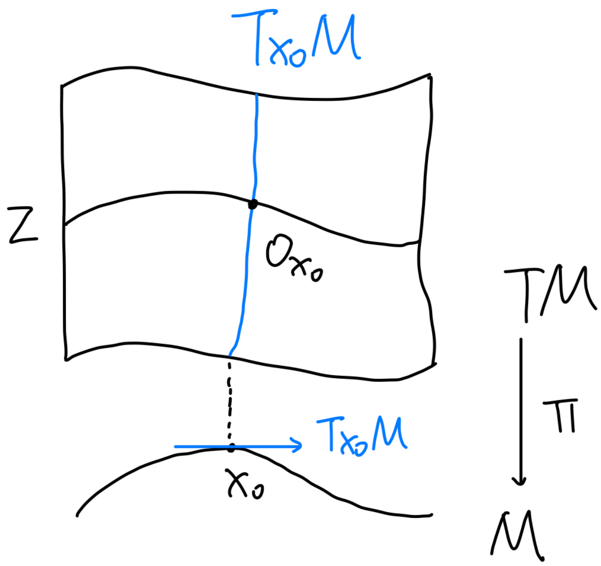
$x_0 \mapsto 0_{x_0}$ maps to the zero section at x_0, then we obtain a map

$$d_{x_0}X : T_{x_0}M \to T_{0_{x_0}}(TM).$$

For zero section $Z = \{0_x \in TM : x \in M\}$, we have a decomposition

$$T_{0_{x_0}}(TM) = T_{0_{x_0}}(Z) \oplus T_{0_{x_0}}(T_{x_0}M) \simeq T_{x_0}M \oplus T_{x_0}M,$$

giving horizontal and vertical directions of the vector field. Therefore, the differential d_{x_0} defined above has a vertical component and a horizontal component.



- The mapping on the first component, i.e., vertically, is given by $d\pi$, and the mapping on the second component is the identification for projection $\pi : TM \to M$.
- The mapping on the second component, i.e., horizontally, is the identification $T_{0_{x_0}}(V) \simeq V$ of vector spaces.

Since the section corresponds to the projection $\pi : TM \to M$ and $X : M \to TM$ satisfies $X_{x_0} = 0$, then the mapping on the first component is just identity, thus we have a description

$$(d_{x_0}X)(v) = (v, L_{x_0}(v)),$$

where $L_{x_0} : T_{x_0}M \to T_{x_0}M$ is a linear approximation to X at x_0 given horizontally. We can also view L_{x_0} as a vector field on $T_{x_0}M$. Locally on a chart (U, x^i) centered at x_0, we have

$$X = X^i \frac{\partial}{\partial x^i}, \quad X^i(0) = 0.$$

Writing the linear map L_{x_0} as a matrix

$$\left(\frac{\partial X^i}{\partial x^j}(0)\right),$$

it corresponds to a vector field

$$\frac{\partial X^i}{\partial x^j}(0)x^j \left.\frac{\partial}{\partial x^i}\right|_{x_0}.$$

Definition 2.3.26. A zero of $X \in \mathfrak{X}(M)$ is *non-degenerate* if the linear approximation $d_{x_0}X$ is invertible.

Proposition 2.3.27. If x_0 is a non-degenerate zero of $X \in \mathfrak{X}(M)$, then it is isolated with index

$$\operatorname{ind}_{x_0} X = \begin{cases} 1, & \text{if } \det(d_{x_0}X) > 0 \\ -1, & \text{if } \det(d_{x_0}X) < 0 \end{cases}.$$

Proof. Exercise. □

We now come back to the stated theorem Theorem 2.3.15.

Proof of Theorem 2.3.15. Take $u \in \Omega_c^n(TM)$ be a representative of the Thom class $U = [u]$. We want to show that

$$\int_M X^* u = \sum_{i=1}^N \operatorname{ind}_{p_i}(X).$$

Choose local charts (V_i, φ_i) centered at zero p_i, and denote $D_i = \varphi_i^{-1}(B_{\varepsilon_i}(0))$ for some $\varepsilon_i > 0$, for $i = 1, \dots, N$. Note that

- $M \setminus \bigcup_{i=1}^p D_i$ is compact since the complement is closed, and
- u by definition has compact support.

Recall from the proof of Theorem 2.3.12 that, when the vector bundle has a non-vanishing section, then we can find some large enough $\lambda > 0$ so that the induced image is disjoint from the support of u. Note that we have zeros in the section, so we cannot directly multiply large enough λ like before, since zeros will still be zeros, but we can say that $\lambda X \cap \operatorname{supp}(u) = \varnothing$ in $M \setminus \bigcup_{i=1}^p D_i$ if $\lambda \gg 0$. Now λX and X are homotopic vector fields with the same zeros, therefore $\operatorname{ind}_{p_i}(\lambda X) = \operatorname{ind}_{p_i}(X)$, so

$$\int_{M \setminus \bigcup_{i=1}^p D_i} X^* u = \int_{M \setminus \bigcup_{i=1}^p D_i} (\lambda X)^* u = 0,$$

therefore

$$\int_M X^* u = \sum_{i=1}^N \int_{D_i} X^* u.$$

It now suffices to show that

$$\int_{D_i} X^* u = \operatorname{ind}_{p_i}(X).$$

Computing in the local charts

$$X = X^i \frac{\partial}{\partial x^i}$$

with $TV_i = V_i \times \mathbb{R}^n$, then the Gauss map

$$\begin{aligned} G_i : \partial D_i &\to \mathbb{S}^{n-1} \\ x &\mapsto \frac{X(x)}{|X(x)|} \end{aligned}$$

We let us write the compactly-supported form u as $u|_{TV_i} \in \Omega^n(TV_i)$. Note that θ is not compactly-supported, but $d\theta$ is. Now since $\operatorname{supp}(d\theta) \subseteq B_{\varepsilon_i}(0)$, so by the definition of the Thom class we can write $u = \operatorname{pr}_{\mathbb{R}^n}^* d\theta$, and by Stokes' theorem we may assume $\int_{\mathbb{S}^{n-1}} \theta = 1$. The Gauss map $G_i : \partial D_i \to \mathbb{R}^n$ is homotopic to $X : \partial D_i \to \mathbb{R}^n$, therefore

$$\int_{D_i} X^* u = \int_{D_i} X^* d\theta$$

$$
\begin{aligned}
&= \int_{D_i} d(X^*\theta) \\
&= \int_{\partial D_i} X^*\theta \\
&= \deg(X) \int_{\mathbb{S}^{n-1}} \theta \\
&= \deg(G_i) \int_{\mathbb{S}^{n-1}} \theta \\
&= \operatorname{ind}_{p_i}(X).
\end{aligned}
$$

□

The following corollary is straightforward.

Corollary 2.3.28. If M is a compact connected oriented manifold of dimension n, and $X, Y \in \mathfrak{X}(M)$ are vector fields with zeros $\{p_1, \ldots, p_N\}$ and $\{q_1, \ldots, q_{N'}\}$, respectively, then

$$\sum_{i=1}^{N} \operatorname{ind}_{p_i}(X) = \sum_{j=1}^{N'} \operatorname{ind}_{q_j}(Y).$$

Definition 2.3.29. The *Euler characteristic* of M is

$$\chi(M) = \sum_{i=1}^{\dim(M)} (-1)^i \dim(H^i(M)).$$

Euler characteristics can be computed by triangulation of the manifold and then *Euler's formula*, i.e.,

$$\chi(M) = \sum_{i=1}^{\dim(M)} (-1)^i r_i. \tag{2.3.30}$$

where r_i is the number of faces of dimension i. We can now give a long-postponed proof of the Poincaré-Hopf theorem.

End of Lecture 24

Proof of Theorem 1.8.6. The second statement follows from what we proved last time. By Corollary 2.3.28, it suffices to construct a vector field X for which Theorem 1.8.6 holds. This is constructed using a triangulation $\Delta = \{G_1, \ldots, G_r\}$ of M. We construct a vector field $X \in \mathfrak{X}(M)$ with the following properties:

i. X has a unique non-degenerate zero in each open face of Δ, and

ii. $\operatorname{ind}_{p_i}(X) = (-1)^k$ where k is the dimension of the face containing the zero.

Then

$$\sum_{i=1}^{N} \operatorname{ind}_{p_i}(X) = \sum_k (-1)^k (\#\text{faces of dimension } k) = \chi(M)$$

by Equation (2.3.30). To construct X, we proceed by induction.

- We set each index to be a zero.
- Place zero at the barycenter of the face of dimension 1, and make them attractors in the face of dimension 1.

- For each face of dimension k, we place a zero at the barycenter, then make them attractors in the face. Since each zero p_i is non-degenerate, if p_i is in the face with dimension K, then

$$d_{p_i}(X) = \begin{cases} k, & \text{if eigenvalue is } -1 \\ n-k, & \text{if eigenvalue is } 1 \end{cases}$$

therefore

$$\operatorname{ind}_{p_i}(X) = \det(d_{p_i}(X)) = (-1)^k.$$

□

Remark 2.3.31. The only reason why the manifold has to be oriented is for the existence of the orientation class. Even though Corollary 2.3.28 is stated for oriented manifolds, since the definition of index is local, i.e., it does not depend on a global orientation, we can state that without the oriented assumption. In particular, the main statement of Theorem 1.8.6 does not require the manifold to be oriented.

Example 2.3.32. Let us study vector bundles of rank 2. Let $\xi = (E, \pi, M)$ be a vector bundle equipped with a Riemannian structure. Suppose $\{(U_\alpha, x^i_\alpha)\}$ is an atlas given by positively-oriented charts, and suppose $\{e^1_\alpha, e^2_\alpha\}$ is a positively-oriented orthonormal frame over each U_α. This gives a chart $(x^i_\alpha, r_\alpha, \theta_\alpha)$ as chart for $\pi^{-1}(U_\alpha)\backslash 0_M$ where 0_M is the zero section, and $(r_\alpha, \theta_\alpha)$ is the polar coordinates on $\mathbb{R}^2$. On any double intersection $U_\alpha \cap U_\beta$, we have

$$\begin{cases} r_\alpha & = r_\beta \\ \theta_\alpha - \theta_\beta & = \pi^* \varphi_{\alpha\beta} \end{cases}$$

where $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta : \mathbb{S}^1$. On triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, then

$$\varphi_{\alpha\beta} + \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}.$$

Choosing a partition of unity $\{\rho_\alpha\}$ subordinate to cover $\{U_\alpha\}$, we have

$$e_\alpha = \sum_\gamma \rho_\gamma d\varphi_{\alpha\gamma} \in \Omega^1(U_\alpha)$$

and therefore

$$\begin{aligned} e_\alpha - e_\beta &= \sum_\gamma \rho_\gamma (d\varphi_{\alpha\gamma} - d\varphi_{\beta\gamma}) \\ &= \sum_\gamma \rho_\gamma d(\varphi_{\alpha\gamma} + \varphi_{\gamma\beta}) \\ &= \sum_\gamma \rho_\gamma d(\varphi_{\alpha\beta}) \\ &= d\varphi_{\alpha\beta}. \end{aligned}$$

Note that this is true on $U_\alpha \cap U_\beta$. Therefore, there exists $e \in \Omega^2(M)$ such that

$$e|_{U_\alpha} = de_\alpha.$$

Claim 2.3.33. $-\frac{1}{2\pi}[e] \in H^2(M)$ is the Euler class of ξ.

Exercise 2.3.34. Using Claim 2.3.33, find the Euler class for the normal bundle of

$$\begin{aligned} \mathbb{S}^2 = \mathbb{CP}^1 &\subseteq \mathbb{CP}^2 \\ [x:y] &\mapsto [x:y:0] \end{aligned}$$

Proof. On $(E\backslash 0_M)|_{U_\alpha \cap U_\beta}$, we have

$$d\theta_\alpha - d\theta_\beta = \pi^* d\varphi_{\alpha\beta} = \pi^* e_\alpha - \pi^* e_\beta.$$

We conclude that

$$d\theta_\alpha - \pi^* e_\alpha = d\theta_\beta - \pi^* e_\beta$$

on $\pi^{-1}(U_\alpha \cap U_\beta)$. This means there exists a globally defined 1-form $\phi \in \Omega^1(E\backslash 0_M)$ such that $\phi|_{U_\alpha} = d\theta_\alpha - \pi^* e_\alpha$. Choosing $\rho_0 : \mathbb{R} \to \mathbb{R}$ so that we construct a bump function ρ_0' such that

$$\int_0^\infty \rho_0'(t)dt = \frac{1}{2\pi},$$

then

$$\rho_0(r) = \begin{cases} -\frac{1}{2\pi}, & r < \varepsilon \\ \int_0^r \rho_0'(t)dt, & r \geqslant \varepsilon \end{cases}.$$

This constructs an assignment

$$\begin{aligned} \rho : E &\to \mathbb{R} \\ v &\mapsto \rho_0(||v||) \end{aligned}$$

Set $u = d(\rho\phi)$ to be a global 2-form on $\Omega^2(E)$, then

$$\begin{aligned} u &= d(\rho\phi) \\ &= d\rho \wedge \phi + \rho d\phi \\ &= d\rho \wedge \phi - \rho\pi^* e. \end{aligned}$$

Note that u satisfies

i. $u|_{E_p} = (d\rho \wedge \phi)|_{E_p} = d\rho \wedge d\theta_\alpha$ has compact support. This implies that $u \in \Omega_c^2(E)$;

ii. by the identification $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, we assume the circle has radius 1, therefore

$$\begin{aligned} \int_{E_p} u &= \iint_{\mathbb{R}^2} d\rho \wedge d\theta_\alpha \\ &= \int_0^\infty d\rho \int_0^1 d\theta_\alpha \\ &= \frac{1}{2\pi} \end{aligned}$$

so $2\pi \cdot u$ is the Thom class;

iii. the zero section $s_0 : M \to E$ satisfies $(s_0)^* u = -e$.

□

2.4 Pullbacks of Vector Bundles

Definition 2.4.1. Let $\xi = (E, \pi, N)$ be a vector bundle of rank r and suppose $\psi : M \to N$ is a map of manifolds. The *pullback* of ξ along ψ is the vector bundle $\psi^*\xi$ of rank r over M with

i. total space $M \times_N E = \{(x, v) : \psi(x) = \pi(v)\} \equiv \psi^* E$;

ii. projection

$$\begin{aligned}\hat{\pi} : \psi^* E &\to M \\ (x, v) &\mapsto x\end{aligned}$$

such that we have a commutative diagram

$$\begin{CD} \psi^* E @>{\Psi}>> E \\ @V{\hat{\pi}}VV @VV{\pi}V \\ M @>>{\psi}> N \end{CD}$$

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Remark 2.4.2. We will check that this is well-defined.

1. Note that $M \times_N E$ is a manifold: given by the assignment

$$\begin{aligned}M \times E &\to N \times N \\ (x, v) &\mapsto (\psi(x), \pi(v))\end{aligned}$$

which is traversal to the diagonal $\Delta \subseteq N \times N$, this follows from the fact that π is a submersion.

2. Local triviality. Suppose $\{g_{\alpha\beta}\}$ gives a cocycle for ξ associated with some cover $\{U_\alpha\}_{\alpha \in I}$, then $\{g_{\alpha\beta} \circ \psi\}$ is a cocycle for the pullback $\psi^* \xi$ associated for the cover $\{\psi^{-1}(U_\alpha)\}$ of the preimages.

Example 2.4.3. Let $\xi = (E, \pi, N)$ with submanifold $i : S \hookrightarrow N$, then the restriction $\xi|_S = i^* \xi$ to submanifold is the same as the pullback along the inclusion.

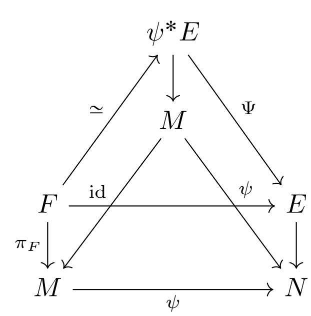
Remark 2.4.4. We observe the assignment $(x, v) \mapsto v$ given by the commutative square above is an isomorphism on the fibers.

Proposition 2.4.5. Consider a vector bundle morphism

$$\begin{CD} F @>{\Psi}>> E \\ @VVV @VVV \\ M @>>{\psi}> N \end{CD}$$

such that it is an isomorphism on the fibers, then there exists a natural isomorphism $F \simeq \psi^* E$.

Proof. We have a commutative diagram



and then the assignment is given by $w \mapsto (\pi_F(w), \Psi(w))$. □

The pullback satisfies the universal property of a pullback.

Proposition 2.4.6 (Universal Property)**.** Given the commutative square

$$\begin{CD} \psi^*E @>{\Psi}>> E \\ @V{\hat{\pi}}VV @VV{\pi}V \\ M @>>{\psi}> N \end{CD}$$

and any manifold F with maps $\Psi : F \to E$ and $\pi_F : F \to M$, then there exists a unique map $\hat{\Psi} : F \to \psi^*E$ such that the diagram below commutes.

$$\begin{array}{ccccc} F & & \xrightarrow{\quad\Psi\quad} & & E \\ & \overset{\exists!}{\searrow} & & & \\ & & \psi^*E & \xrightarrow{\Psi} & E \\ & {\scriptstyle \pi_F}\searrow & \downarrow{\scriptstyle \hat{\pi}} & & \downarrow{\scriptstyle \pi} \\ & & M & \xrightarrow[\psi]{} & N \end{array}$$

Proposition 2.4.7 (Homotopy Invariance)**.** Given a vector bundle $\xi = (E.\pi, N)$ and homotopic maps $\varphi, \psi : M \to N$, then $\varphi^*\xi \simeq \psi^*\xi$.

Proof. Let $H : M \times [0,1] \to N$ be the homotopy between $H(x,0) = \varphi(x)$ and $H(x,1) = \psi(x)$. Therefore, we have

$$\varphi^*\xi = H^*\xi|_{M\times\{0\}}, \quad \psi^*\xi = H^*\xi|_{M\times\{1\}}.$$

It suffices to show that for any vector bundle η over $M \times [0,1]$, $\eta|_{M\times\{0\}} \simeq \eta|_{M\times\{1\}}$.

To show this, we construct a C^0-morphism of vector bundles $(\Delta, \delta) : \eta \to \eta$ where

$$\begin{aligned} \delta : M \times [0,1] &\to M \times [0,1] \\ (x,t) &\mapsto (x,1) \end{aligned}$$

which is an isomorphism for each $t \in [0,1]$. That is, for any $t \in [0,1]$,

$$\Delta_t : \eta|_{M\times\{t\}} \to \eta|_{M\times\{1\}}$$

is an isomorphism. Using approximation theory, this can then be upgraded to a C^∞-morphism.

There now exists an open cover $\{U_\alpha\}$ of M such that the restriction

$$\eta|_{U_\alpha\times[0,1]}$$

of the vector bundle is trivial. Recall that we already know this is true for $\eta|_{U_\alpha\times[a,b]}$ of small intervals in $[0,1]$, i.e., locally, so to prove the statement, note that if we know $\eta|_{U_\alpha\times[a,b]}$ and $\eta|_{U_\alpha\times[b,c]}$ are both trivial, then $\eta|_{U_\alpha\times[a,c]}$ is also trivial.

Now fix countable open cover $\{U_k\}$ that is locally finite, i.e., for each point $x \in M$, there exists an open neighborhood that only intersects finitely many U_k's, and such that there is a trivialization

$$\begin{array}{ccc} E|_{U_k\times[0,1]} & \xrightarrow[\simeq]{\varphi_k} & (U_k \times [0,1]) \times \mathbb{R}^r \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & U_k \times [0,1] & \end{array}$$

given on each U_k, argued as above. Now choose a partition of unity $\{\tilde{\rho}_k\}$ subordinated to $\{U_k\}$, then set

$$\rho_k(x) = \frac{\tilde{\rho}_k(x)}{\max\limits_{m\in\mathbb{N}} \tilde{\rho}_m(x)}$$

Since the cover is locally finite, the maximum is well-defined. This is an envelope of 1, i.e., $1 \geqslant \rho_k(x) \geqslant 0$ and $\max_k \rho_k(x) = 1$. Now set

$$\begin{array}{ccc} \eta & \xrightarrow{\Delta_k} & M \\ \downarrow & & \downarrow \\ M \times [0,1] & \xrightarrow[\delta_k]{} & M \times [0,1] \end{array}$$

where the base is given by

$$\delta_k(x,t) = (x, \max\{\rho_k(x), t\})$$

and

$$\Delta_k(\varphi_k^{-1}(x,t,v)) = \begin{cases} \varphi_k^{-1}(x, \max\{\rho_k(x), t\}, v), & \text{if } \varphi_k^{-1}(x,t,v) \in \pi^{-1}(U_k \times [0,1]) \\ \text{id}, & \text{else} \end{cases}$$

Now define $\Delta = \cdots \circ \Delta_k \circ \cdots \circ \Delta_1$, then Δ is a morphism of vector bundles that

1. covers $\delta = \cdots \circ \delta_k \circ \cdots \circ \delta_1$, and
2. $\eta|_{M \times \{t\}} \to \eta|_{M \times \{1\}}$ is an isomorphism.

□

Corollary 2.4.8. Every vector bundle over a contractible manifold is trivial.

Proof. Suppose we have

$$\begin{aligned} \varphi : M &\to M \\ x &\mapsto x_0 \end{aligned}$$

then it is homotopic to the identity $\text{id} : M \to M$. The pullback $\varphi^*\xi$ is the trivial vector bundle with fiber given by copies of x_0. Therefore, $\varphi^*\xi \simeq M \times \mathbb{R}^r$. By Proposition 2.4.7, we have

$$\begin{aligned} M \times \mathbb{R}^r &\simeq \varphi^*\xi \\ &\simeq \text{id}^* \xi \\ &= \xi. \end{aligned}$$

□

Remark 2.4.9.

1. Most of the operations we have seen at this point are preserved under pullbacks. However, whenever we identify the pullback of a vector bundle with another vector bundle by an equal sign, it is really given by a canonical isomorphism.
2. We can also pullback the sections of a vector bundle. Given a map

$$\begin{array}{ccc} \psi^*E & \xrightarrow{\Psi} & E \\ {\scriptstyle\hat{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[\psi]{} & N \end{array}$$

Taking a section $s : N \to E$ of π induces a section $\psi^*s : M \to \psi^*E$, which is defined in the obvious way by

$$(\psi^*s)(x) = (x, s(\psi(x))).$$

Example 2.4.10. Suppose $\xi = (E, \pi, \mathbb{S}^1)$ is a vector bundle. Taking stereographical projection of $\mathbb{S}^1 = U \cup V$ for $U = \mathbb{S}^1 \backslash \{p_N\}$ and $V = \mathbb{S}^1 \backslash \{p_S\}$. Note that both open sets are contractible, therefore $\xi|_U \simeq U \times \mathbb{R}$ and $\xi|_V \simeq V \times \mathbb{R}$ are both trivial. Looking at the intersection, there are two connected components, therefore the cocycle given by

$$g_{U \cap V} : U \cap V \to \mathrm{GL}_1(\mathbb{R}) = \mathbb{R} \backslash \{0\}$$

has two possibilities: $g_{U \cap V}$ either has the same sign on both connected components, or it has opposite signs. In the first case, we retrieve the trivial vector bundle $\eta \simeq \mathcal{E}^1_{\mathbb{S}^1}$; for the second case, we get $\eta \simeq \gamma^1_{\mathbb{S}^1}$ which is the Möbius band. That is to say, the category $\mathrm{Vect}_1(\mathbb{S}^1)$ of line bundles of $\mathbb{S}^1$ has only two objects.

Example 2.4.11. Now let $\xi = (E, \pi, \mathbb{S}^1)$ be a rank-1 vector bundle over a connected manifold. Note that this is associated to a group homomorphism

$$\begin{aligned} \omega : \pi_1(M) &\to \mathbb{Z}/2\mathbb{Z} \\ [\gamma] &\mapsto \begin{cases} 0, & \text{if } \gamma^*\xi \text{ is trivial} \\ 1, & \text{if } \gamma^*\xi \text{ is non-trivial} \end{cases} \end{aligned}$$

Therefore, ω sends commutators to commutators, therefore this induces a map

$$\tilde{\omega} : H_1(M) \simeq \pi_1(M)/[\pi_1(M), \pi_1(M)] \to \mathbb{Z}/2\mathbb{Z}$$

In particular, this means $[\bar{\omega}] \in H^1(M, \mathbb{Z}/2\mathbb{Z})$, which is the first *Stiefel-Whitney class*. This in fact completely classifies the rank-1 vector bundles.

End of Lecture 26

2.5 Connections on Vector Bundles

We will extend the definition of connections on tangent bundles to general vector bundles.

Definition 2.5.1. A *connection* ∇ on $\xi = (E, \pi, M)$ is a $\mathbb{R}$-bilinear map

$$\begin{aligned} \mathfrak{X}(M) \times \Gamma(\xi) &\to \Gamma(\xi) \\ (x, s) &\mapsto \nabla_X s \end{aligned}$$

such that

a. $\nabla_{fX} s = f \nabla_X s$, and

b. $\nabla_X (fs) = f \nabla_X s + X(f) s$.

Remark 2.5.2.

i. Set of connections on ξ is an affine space: for any $a, b \in \mathbb{R}$ such that $a + b = 1$, $a\nabla^1 + b\nabla^2$ is a connection for any connections ∇^1, ∇^2. This is modeled on $\Omega^1(M; \mathrm{End}(E))$.

ii. Let $v \in T_x M$ and s be a local section, then $\nabla_v s = \nabla_{\tilde{X}} \tilde{s}$ with $\tilde{X} \in \mathfrak{X}(M)$ such that $\tilde{X}\Big|_x = v$, and $\tilde{s} \in \Gamma(\xi)$ such that $\tilde{s}|_U = s$ for open neighborhood U of x.

iii. For any open subset $U \subseteq M$, ∇ induces a connection on $\xi|_U$.

Writing in terms of local coordinates, suppose we have a chart (U, x^i), then $\xi|_U$ has a frame $\{e_1, \ldots, e_r\}$, so

$$\nabla_{\frac{\partial}{\partial x^i}} e_a = \Gamma^b_{ia} e_b$$

for Christoffel symbols symbols $\Gamma^b_{ia} \in C^\infty(U)$. If $s|_U = s^e_a$ and $X|_U = X^i \frac{\partial}{\partial x^i}$, then

$$(\nabla_X S)|_U = X^i \nabla_{\frac{\partial}{\partial x^i}} (s^a e_a) = \left(X^i \frac{\partial s^b}{\partial x^i} + X^i s^a \Gamma^b_{ia} \right) e_b.$$

A local connection 1-form can be written as

$$\nabla_X e_a = \omega_a^b(X) e_b$$

where $\omega_a^b = \Gamma_{ia}^b dx^i$, and we can think of $[\omega_a^b] \in \Omega^1(U, \mathfrak{gl}_r(\mathbb{R}))$. Note that we replace the dimension of the manifold by the rank of the vector bundle.

Exercise 2.5.3. Under change of frames

$$\bar{e}_a = A_a^b e_b$$

for $A = [A_a^b] \in C^\infty(U, \mathrm{GL}_r(\mathbb{R}))$. We conclude that

$$\bar{\omega} = A\omega A^{-1} + A^{-1} dA.$$

Example 2.5.4. Suppose we have a global frame ξ, i.e., the vector bundle is trivial, then we can define a connection by declaring $\nabla_X e_a = 0$ for any vector field X. This is called the trivial connection on the trivial vector bundle, but note that this connection is not unique, since it depends on the frame: changing the frame changes the trivial connection!

Theorem 2.5.5. Every vector bundle admits a connection.

This is the same proof as the one we did for tangent bundles.

Proof. Consider an open cover $\{U_i\}$ such that $\xi|_{U_i}$ is trivial, and we choose a connection ∇^i on $\xi|_{U_i}$ for each i. We choose a partition of unity $\{\rho_i\}$ subordinated to the cover, then $\nabla = \sum_i e_i \nabla^i$ defines a connection. ☐

Recall that we have defined a lot of operations on vector bundles, and the point being there are corresponding connections defined on those bundles as well. That is, suppose ∇^i is a connection on vector bundle ξ^i over M for $i = 1, 2$, then

- there is a connection ∇ on $\xi^1 \oplus \xi^2$ defined by

$$\nabla_X(s^1 \oplus s^2) = \nabla_X s^1 \oplus \nabla_X s^2;$$

- there is a connection ∇ on $\xi^1 \otimes \xi^2$ defined by

$$\nabla_X(s^1 \otimes s^2) = \nabla_X s^1 \otimes s^2 + s^1 \otimes \nabla_X s^2;$$

- suppose ∇ is a connection on ξ, then there is a dual connection ∇^* defined on ξ^*, given by

$$\langle \nabla_X^* \eta, s \rangle = X \cdot \langle \eta, s \rangle - \langle \eta, \nabla_X s \rangle .$$

  We will suppress $()^*$ to avoid confusion.

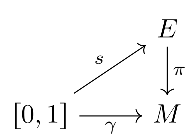
- suppose ∇ is a connection on ξ, then there is a connection defined on exterior power $\bigwedge^d \xi$ by

$$\nabla_X(s_1 \wedge \cdots \wedge s_d) = \sum_{i=1}^d s_1 \wedge \cdots \wedge \nabla_X s_i \wedge \cdots \wedge s_d;$$

- more importantly, suppose $\xi = (E, \pi, N)$ is a vector bundle with connection ∇, and let $\varphi : M \to N$ be a map of manifolds, then there is a pullback connection defined on $\varphi^*\xi$ via

$$\nabla_X(\varphi^* s)(x) = (x, \nabla_{d_x\varphi(X)} s).$$

Let us now discuss covariant derivatives along path $\gamma : [0, 1] \to M$. We have



replacing vector fields along a path by sections along the path. We note that $s : [0, 1] \to E$ such that $\pi(s(t)) = \pi(t)$ is now called a section along γ, but this is nothing more than a section of the pullback of vector bundle of E over M, i.e., $s \in \Gamma(\gamma^*\xi)$.

Definition 2.5.6. Given a section $s \in \Gamma(\gamma^*\xi)$, we define

$$D_\gamma s = \nabla_{\frac{d}{dt}}(\gamma^* s)$$

where ∇ is the pullback connection on $\gamma^*\xi$.

Exercise 2.5.7. This coincides with the definition in the time-dependent sense we saw before: if a time-dependent section[5] $\tilde{s}_t \in \Gamma(\xi)$ is such that it restricts along $\tilde{s}_t|_{\gamma(t)} = s(t)$ for $t \in [0, 1]$, then

$$(D_\gamma s)(t) = \nabla_{\dot{\gamma}(t)}\tilde{s}_t + \frac{d}{dt}\tilde{s}_t\bigg|_{\gamma(t)}.$$

Remark 2.5.8. $D_\gamma s$ satisfies the following properties:

- $D_\gamma(s_1 + s_2) = D_\gamma s_1 + D_\gamma s_2$ for sections s_1, s_2;
- $D_\gamma(fs) = fD_\gamma s + (f(\gamma(t)))'(s)$.

We can now write down a local expression for $D_\gamma s$. Let (U, x^i) be a local chart with local frame $\{e_1, \ldots, e_r\}$ on $\xi|_U$, and consider a path

$$\begin{aligned} \gamma : [0, 1] &\to U \\ t &\mapsto (\gamma^1(t), \ldots, \gamma^n(t)), \end{aligned}$$

with a section $s \in \Gamma(\gamma^*\xi)$ that can be written as $s(t) = s^a(t)\ e_a|_{\gamma(t)}$, then

$$(D_\gamma s)(t) = \left(\frac{ds^b}{dt}(t) + \Gamma^b_{ia}(\gamma(t))\dot{\gamma}^i(t)s^a(t)\right) e_b|_{\gamma(t)}.$$

Lemma 2.5.9. Given a curve $\gamma : [0, 1] \to M$ and $v_0 \in E_{\gamma(0)}$, then there exists a unique section $s \in \Gamma(\gamma^*\xi)$ such that

$$\begin{cases} D_\gamma s &= 0 \\ s(0) &= 0 \end{cases}. \tag{2.5.10}$$

Such s is called a *parallel section* along γ.

Proof. Note that the pullback along the vector bundle $\gamma^*\xi \to [0, 1]$ has a global frame $\{e_1(t), \ldots, e_r(t)\}$ as sections along γ, and the section s we want can be expressed as $s(t) = s^a(t)e_a(t)$ for some $s^a : [0, 1] \to \mathbb{R}$. To write this expression, let us write

$$(D_\gamma e_a)(t) = \omega^b_a(t)e_b(t)$$

where $\omega^b_a(t)$ are time-dependent functions and not forms. We can then write

$$D_\gamma s = \left(\frac{ds^b}{dt}(t) + \omega^b_a(t)s^a(t)\right) e_b(t).$$

For this to be zero, since $\omega^b_a(t)$ is completely determined by the underlying structure, we just need to solve

$$\begin{cases} \frac{ds^b}{dt}(t) &= -\omega^b_a(t)s^a(t) \\ s^a(0) &= v^a_0 \end{cases}$$

which always has a solution as a linear system. ∎

Definition 2.5.11. The *parallel transport* along a curve $\gamma : [0, 1] \to M$ is

$$\begin{aligned} \tau_\gamma : E_{\gamma(0)} &\to E_{\gamma(1)} \\ v_0 &\mapsto s(1) \end{aligned}$$

where $s(t)$ is the unique solution of Equation (2.5.10).

[5] We only require this to be defined on a neighborhood, but the flexibility of vector bundles allows us to define this globally.

Proposition 2.5.12.

i. τ_γ is a linear isomorphism.

ii. For any $s \in \Gamma(\xi)$, we have
$$\nabla_{\dot{\gamma}(0)} s = \lim_{t \to 0} \frac{\tau_{\gamma(t)}^{-1}(s(\gamma(t)) - s(\gamma(0))}{t}$$
where $\tau_{\gamma(t)}$ is the parallel transport along $\gamma : [0,t] \to M$.[6]

Proof.

i. We define $\tau_\gamma^{-1} = \tau_{\bar{\gamma}}$ where $\bar{\gamma}$ is the reverse path defined by $\bar{\gamma}(t) = \gamma(1-t)$. Therefore, τ_γ is a linear isomorphism.

ii. By Lemma 2.5.9, since we have a basis of the vector space, then by performing parallel transport, we get a frame $\{e_1(t), \dots, e_r(t)\}$ along γ consisting of parallel sections along γ such that $D_\gamma e_i = 0$. Now we write
$$s(\gamma(t)) = s^a(t)e_a(t),$$
we note $\tau_{\gamma_t}^{-1}(e_a(t)) = e_a(0)$, so the limit can be computed as
$$\begin{aligned}\lim_{t \to 0} \frac{\tau_{\gamma(t)}^{-1}(s(\gamma(t))) - s(\gamma(0))}{t} &= \lim_{t \to 0} \frac{s^a(t)e_a(0) - s^a(0)e_a(0)}{t} \\ &= \left.\frac{d}{dt}s^a(t)\right|_{t=0} e_a(0) \\ &= D_\gamma(s(\gamma(t)))(0) \\ &= \left(\nabla_{\dot{\gamma}(0)} s\right)\big|_{\gamma(t)}\end{aligned}$$
using local expressions and time-independency.

□

End of Lecture 27

Definition 2.5.13. Let us fix a vector bundle $\xi = (E, \pi, M)$ with connection ∇. The *curvature tensor* of the connection ∇ is
$$R^\nabla(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$
for $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$. In particular, the connection is *flat* if $R^\nabla \equiv 0$.

Note that R^∇ is $C^\infty(M)$-linear in each entry, and it is skew-symmetric in both X and Y, therefore
$$R^\nabla \in \Omega^2(M; \mathrm{End}(E)).$$

Using local expressions, suppose we have a frame $\{s_1, \dots, s_r\}$ for $\xi|_U$, recall we will get a form
$$\nabla_X s_\alpha = \omega_a^b(x) s_b$$
for local connection 1-form $\omega = [\omega_a^b] \in \Omega^1(U; \mathfrak{gl}_r)$, then
$$R(X,Y)s_a = \Omega_a^b(X,Y)s_b$$
for local curvature 2-form $\Omega = [\Omega_a^b] \in \Omega^2(U; \mathfrak{gl}_r)$.

[6] The point being $\tau_{\gamma_t}^{-1}$ now identifies the fiber at 0 with fiber at t of the original linear isomorphism, therefore it is well-defined to take the difference.

Remark 2.5.14. If U is a domain of a chart $(x^1, \dots, x^n)$, such that

$$\nabla_{\frac{\partial}{\partial x^i}} s_a = \Gamma_{ia}^b s_b$$

and so we get

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) s_a = R_{ija}^b s_b$$

where R_{ija}^b is some expression in terms of Γ_{ia}^b.

We can also interpret this geometrically. Let

$$\gamma : [0,1] \to M$$

be a path such that

$$\begin{aligned} \gamma_\varepsilon : [0,1] &\to M \\ t &\mapsto \gamma(t,\varepsilon) \end{aligned}$$

$$\begin{aligned} \gamma_t : [0,1] &\to M \\ \varepsilon &\mapsto \gamma(t,\varepsilon) \end{aligned}$$

and we choose a section

$$s : [0,1] \times [0,1] \to E$$

such that $s(t,\varepsilon) \in E_{\gamma(t,\varepsilon)}$. We recall that this is the same as choosing a section $s \in \Gamma(\gamma^* E)$. This allows us to give an easy proof of Proposition 1.5.4.

Proposition 2.5.15. We have

$$R^\nabla(\dot{\gamma}_\varepsilon, \dot{\gamma}_t)s = D_{\gamma_\varepsilon} D_{\gamma_t} s - D_{\gamma_t} D_{\gamma_\varepsilon} s.$$

Proof. Let $\bar{\nabla} = \gamma^* \nabla$ be the connection on $\gamma^* \xi \to I = [0,1] \times [0,1]$, then there are new sections

$$D_{\gamma_\varepsilon} s = \bar{\nabla}_{\frac{d}{dt}} s, \quad D_{\gamma_t} s = \bar{\nabla}_{\frac{d}{d\varepsilon}} s.$$

We now have

$$\begin{aligned} R^\nabla(\dot{\gamma}_\varepsilon, \dot{\gamma}_t)s &= R^\nabla\left(\frac{d}{dt}, \frac{d}{dt}\right) s \\ &= \bar{\nabla}_{\frac{d}{dt}} \bar{\nabla}_{\frac{d}{d\varepsilon}} s - \bar{\nabla}_{\frac{d}{d\varepsilon}} \bar{\nabla}_{\frac{d}{dt}} s \\ &= \bar{\nabla}_{\frac{d}{dt}} \bar{\nabla}_{\frac{d}{d\varepsilon}} s - \bar{\nabla}_{\frac{d}{d\varepsilon}} \bar{\nabla}_{\frac{d}{dt}} s - \nabla_{\left[\frac{d}{d\varepsilon}, \frac{d}{dt}\right]} s \\ &= D_{\gamma_\varepsilon} D_{\gamma_t} s - D_{\gamma_t} D_{\gamma_\varepsilon} s. \end{aligned}$$

□

Corollary 2.5.16. If ∇ is flat and $\gamma_0, \gamma_1 : [0,1] \to M$ are homotopic curves, then $\tau_{\gamma_0} = \tau_{\gamma_1}$.

Proof. This is the same as Corollary 1.5.6. □

Corollary 2.5.17. If (ξ, ∇) is a vector bundle with a flat metric, then for every point $x_0 \in M$, there exists an open neighborhood $x_0 \in U \subseteq M$ and a trivialization $\varphi : \xi|_U \to U \times \mathbb{R}^r$ of ξ that sends ∇ to the trivial connection.

Proof. We choose a chart (U, φ) centered at x_0 with $U \simeq B_1(0)$. This is contractible, and the vector bundle can be trivialized over this. We choose a basis $\{v_1, \dots, v_r\}$ for the fiber E_{x_0}, then this can be propagated into a frame $\{e_1, \dots, e_r\}$ over U, such that

$$e_a(x) = \tau_{\gamma_x}(v_a)$$

for $\gamma_x(t) = tx$. Using the proposition last time, we can check that

$$\nabla_{\frac{\partial}{\partial x^i}} e_a = 0.$$

This is equivalent to saying that the trivialization provided by the frames sends ∇ to the trivial connection. □

This says that locally a *flat bundle*, i.e., vector bundle with flat connection, is just trivial. What about globally?

Example 2.5.18. Let M be a connected manifold and fix a basepoint $x_0 \in M$, then the universal covering space of M can be explicitly realized as

$$\tilde{M} = \{[\eta] \mid \eta : I \to M, \eta(0) = x_0\},$$

which comes with a projection

$$\begin{aligned} p : \tilde{M} &\to M \\ [\eta] &\mapsto \eta(1) \end{aligned}$$

The fundamental group $\pi_1(M, x_0)$ acts on the right of $\tilde{M}$, via concatenation $[\eta] \cdot [\gamma] = [\eta \circ \gamma]$, freely and properly. Since the action is along the fiber of the map, Taking the quotient gives an isomorphism

$$\begin{array}{ccc} \tilde{M} & & \\ {\scriptstyle q}\downarrow & \searrow^{p} & \\ \tilde{M}/\pi_1(M, x_0) & \xrightarrow[\simeq]{} & M \end{array}$$

Suppose we are given a representation $\rho : \pi_1(M, x_0) \to \mathrm{GL}(\mathbb{R}^r)$, then we can build a flat bundle $\tilde{M} \times \mathbb{R}^r \to \tilde{M}$. This inherits a $\pi_1(M, x_0)$-action by

$$([\eta], v) \cdot [\gamma] = ([\eta] \cdot [\gamma], \rho([\gamma])^{-1}(v)).$$

This action is again free and proper by taking the diagonal action over a free and proper action. Passing to the quotient along the action again, we have a bundle $E \to M$. Since the representation acts linearly, the trivial vector bundles gives another vector bundle, and the flat connection now descends along the action.

Exercise 2.5.19. Check that there is a unique connection ∇^E such that

$$q^*(\nabla^E_X s) = \nabla_{\tilde{X}}(q^* s)$$

for trivial connection ∇, $s \in \Gamma(E)$, and $q_* \tilde{x} = x$.

Remark 2.5.20. Here q gives a pullback diagram

$$\begin{array}{ccc} \tilde{M} \times \mathbb{R}^r & \xrightarrow{q} & E \\ {\scriptstyle q^*s}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle s} \\ \tilde{M} & \xrightarrow[q]{} & M \end{array}$$

with induced pullback section.

This is a way of construction flat vector bundles. In fact, every flat bundle can be constructed in this way, using the holonomy representation.

Definition 2.5.21. Given a flat bundle (ξ, ∇), the *holonomy representation* based at $x_0 \in M$ is the parallel transport along a loop

$$\begin{aligned} \mathrm{hd} : \pi_1(M, x_0) &\to \mathrm{GL}(E_{x_0}) \\ [\gamma] &\mapsto \tau_\gamma \end{aligned}$$

Remark 2.5.22.

1. hd is a homomorphism because the concatenation behaves well with parallel transport, i.e.,

$$\tau_{\gamma_1 \circ \gamma_2} = \tau_{\gamma_1} \circ \tau_{\gamma_2}.$$

2. If x_1 is another basepoint, and choose a path $\eta : I \to M$ such that $\eta(0) = x_0$ and $\eta(1) = 1$, then a parallel transport along the path gives
$$\operatorname{hd}_{x_0}([\eta^{-1} \circ \gamma \circ \eta]) = (\tau_\eta)^{-1} \operatorname{hd}_{x_1}([\gamma])\tau_\eta.$$
Therefore the two holonomy representations are conjugates, and we do not care that much about the choice of a basepoint.

Theorem 2.5.23. For a fixed manifold M and a basepoint $x_0 \in M$, there is a one-to-one correspondence between

- isomorphism classes of flat bundles (ξ, ∇) of rank r over M, and
- $\operatorname{Hom}(\pi_1(M, x_0), \operatorname{GL}_r)/\operatorname{GL}_r$ where GL_r acts by conjugation.

Proof. If we start with a flat bundle, we get a homomorphism by taking the holonomy representation. If we get a representative of homomorphism, we get a representative of fundamental group, and we construct the flat bundle accordingly.

Exercise 2.5.24. Check that the two maps are inverses to each other up to isomorphism.

□

Remark 2.5.25. When the sets of isomorphism classes in Theorem 2.5.23 are identified as spaces, the first space is called the *moduli space of flat connections*, and the second space is called the *character variety*.

What happens if the bundle is not flat? We will associate them to characteristic classes using topological information independent of the connection. This leads to Chern-Weil homomorphism. We denote

- $\Omega^k(M, E)$ to be the *E-valued forms* $\Gamma(\Lambda^k T^*M \otimes E)$, so these are just maps $\eta : \mathfrak{X}(M)^k \to \Gamma(E)$ that are $C^\infty(M)$-multilinear alternating. This is the same as taking the wedges instead of products.
- Set $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M, E)$, then we can construct $\omega \wedge \eta \in \Omega^{k+\ell}(M, E)$ which is defined by $(\omega \wedge \eta)(x_1, \ldots, x_{k+\ell}) = \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)})$.

Definition 2.5.26. A linear map $d_0 : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$ is called a (degree-1) *graded derivation* if it satisfies
$$d_0(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge d_0\eta.$$

It turns out that giving a graded derivation is the same as giving a connection.

End of Lecture 28

Proposition 2.5.27. Let $\xi = (E, \pi, M)$ be a vector bundle. There is a one-to-one correspondence between

- connections ∇ on ξ, and
- graded derivations $d_0 : \Omega^* \to \Omega^{*+1}$.

Proof. We can write (not necessarily uniquely)
$$\eta = \sum_i \omega_i \wedge s_i$$
for $\omega_i \in \Omega^*(M)$ and $s_i \in \Gamma(E)$, so the graded derivation is completely determined by its behavior on degree 1, i.e., $d_0 : \Omega^0(M, E) \simeq \Gamma(E) \to \Omega^1(M, E)$. In degree 0, this means that the section $(d_0 s)(X)$ can be written as
$$(d_0 s)(X) = \nabla_X s,$$
but the property of d_0 indicates this is true if and only if d_0 is a derivation, i.e., $d_0(fs) = d_0 f \wedge s + f \wedge d_0 s$, but that is true if and only if $\nabla_X(fs) = f\nabla_X s + \langle df, X\rangle\, s = f\nabla_X s + X(f)s$. □

Remark 2.5.28. Given a connection ∇, then there is an explicit expression for the corresponding derivation d^∇:

$$(d^\nabla \eta)(X_0,\dots,X_k) = \sum_i (-1)^i \nabla_{X_i}(\eta(X_0,\dots,\hat{X}_i,\dots,X_k)) + \sum_{i<j}(-1)^{i+j}\eta([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k).$$

In particular, if $E = M \times \mathbb{R} \to \mathbb{R}$ is a trivial bundle with trivial connection $\nabla_X \frac{\partial}{\partial t} = 0$, we recover the usual de Rham differential. Note that the de Rham differential in general squares to zero, but not this derivation.

Proposition 2.5.29.

i. We have $(d^\nabla)^2\eta = R \wedge \eta$ for curvature $R \in \Omega^2(M, \mathrm{End}(E))$, $\eta \in \Omega^*(M, E)$.

   Remark 2.5.30. Here the wedge product $\wedge$ combines the usual wedge product with the action of the endomorphism, since our connection is only defined on E whereas R is defined over $\mathrm{End}(E)$. That is, if $R \in \Omega^k(M, \mathrm{End}(E))$ and $\eta \in \Omega^\ell(M, E)$, then

   $$(R \wedge \eta)(X_1,\dots,X_{k+\ell}) = \frac{1}{(k+\ell)!}\sum_{\sigma \in S_{k+\ell}} (-1)^\sigma R(X_{\sigma(1)},\dots,X_{\sigma(k)})(\eta(X_{\sigma(k+1)},\dots,X_{\sigma(k+\ell)}).$$

ii. Using the extra action from the endomorphism again, we have

   $$d^\nabla R = 0$$

   which is also called *Bianchi's identity*.

Proof. Since the graded derivation is generated on degree 0, we just need to prove this for a section, and extend this to forms of arbitrary degree.

i. Take $\eta = s \in \Omega^0(M,E) = \Gamma(E)$, then

   $$\begin{aligned}(d^\nabla)^2 s(X,Y) &= \nabla_X(d^\nabla s(Y)) - \nabla_Y(d^\nabla s(X)) + d^\nabla s([X,Y]) \\ &= \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s \\ &= R(X,Y)s.\end{aligned}$$

ii. Given by the way how the connection extended to endomorphism, i.e., $\mathrm{End}(E) \simeq E^* \otimes E$, we note

   $$\nabla_X(\sum_i \xi_i \otimes s_i) = \sum_i (\nabla_X \xi_i \otimes s_i + \xi_i \otimes \nabla_X s_i),$$

   then

   $$\begin{aligned}(d^\nabla R)(X,Y,Z)(s) &= \left(\nabla_X(R^\nabla(Y,Z)) + \mathrm{cycPerm}(X,Y,Z) - (R^\nabla([X,Y],Z) + \mathrm{cycPerm}(X,Y,Z))\right)(s) \\ &= (\nabla_X(R^\nabla(Y,Z)s) - R^\nabla(Y,Z)\nabla_X s + \mathrm{cycPerm}(X,Y,Z)) \\ &\quad - (R^\nabla([X,Y],Z)s + \mathrm{cycPerm}(X,Y,Z)) \\ &= 0.\end{aligned}$$

□

Remark 2.5.31.

- When the connection ∇ is flat, the curvature R^∇ is indeed 0, therefore $(d^\nabla)^2 = 0$. In this case, $(\Omega^*(M,E), d^\nabla)$ is a complex and gives rise to a cohomology $H^*(M,E)$ given by the system of local coefficients.
- If $R^\nabla \neq 0$, we have the ordinary cohomology classes attached to E.

2.6 Characteristic Classes

Theorem 2.6.1. Let V be a finite-dimensional vector space, then we can do two things:

- take symmetric multi-linear maps $P: V^k \to \mathbb{R}$, which are the k-symmetric tensor products S^kV^*, or
- take polynomial functions $\tilde{P}: V \to \mathbb{R}$, homogeneous of degree k, i.e., $\tilde{P}(\lambda v) = \lambda^k P(v)$.

In fact, they are in one-to-one correspondence.

Proof.

- Given $P \in S^kV^*$, we construct $\tilde{P}$ via $\tilde{P}(v) = P(v, \dots, v)$.
- Given $\tilde{P}$, fixing a basis $\{e_i\}$ for V and dual basis $\{\xi^i\}$ for V^*, so we can write

$$\tilde{P} = \sum_{i_1,\dots,i_j} a_{i_1,\dots,i_k} \xi^{i_1}(v) \cdots \xi^{i_k}(v)$$

with $a_{i_1,\dots,i_k}$ symmetric, then we construct P via

$$P(v_1, \dots, v_k) = \sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k} \xi^{i_1}(v_1) \cdots \xi^{i_k}(v_k).$$

□

Example 2.6.2. Let $V = \mathbb{R}^3$ with coordinates in (x, y, z), for homogeneous polynomial

$$\tilde{P}(x, y, z) = x^2 + xy + z^2 = x^2 + \frac{1}{2}(xy + yx) + z^2,$$

therefore

$$P(\bar{v}, \bar{w}) = v_1w_1 + \frac{1}{2}(v_1w_2 + v_2w_1) + v_3w_3.$$

This shows that product of polynomials corresponds to symmetric products, i.e., $\tilde{P}_1\tilde{P}_2$ corresponds to

$$(P_1 \circ P_2)(v_1, \dots, v_{k+\ell}) = \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} P_1(v_{\sigma(1)}, \dots, v_{\sigma(k)}) P_2(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

For a Lie group G and Lie algebra $\mathfrak{g}$ of G, then the adjoint G-action Ad on $\mathfrak{g}$ is given by

$$\mathrm{Ad}_g(v) = d_e i_g(v)$$

for

$$\begin{aligned} i_g : G &\to G \\ h &\mapsto ghg^{-1} \end{aligned}$$

This then determines the *invariant polynomials* $P : \mathfrak{g} \to \mathbb{R}$, i.e., satisfying $P(\mathrm{Ad}_g v) = P(v)$ for all $g \in G$ and $v \in V$. Finally, we define $I^k(V)$ to be the Ad-invariant homogeneous polynomials of degree k. This gives rise to a graded ring $I(G) = \bigoplus_k I^k(G)$ of all Ad-invariant polynomials.

Remark 2.6.3. By the correspondence before, $I^k(G)$ corresponds to the symmetric polynomials $P \in S^kV^*$ such that $P(\mathrm{Ad}_g v_1, \dots, \mathrm{Ad}_g v_k) = P(v_1, \dots, v_k)$.

We will focus our interest on general linear groups for now.

Example 2.6.4. Take $G = \mathrm{GL}_r(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}_r(\mathbb{R})$. The invariant polynomials are polynomials $P : \mathfrak{gl}_r(\mathbb{R}) \to \mathbb{R}$ such that $\mathrm{tr} \in I^1(\mathrm{GL}_r)$ and $\det \in I^r(\mathrm{GL}_r)$.

Remark 2.6.5. Recall that the characteristic polynomial $\det(A - \lambda I)$ of a matrix A is given by coefficients $\operatorname{tr}(A)$, some linear combinations of $\operatorname{tr}(A)$, and $\det(A)$. In fact, all invariant polynomials are generated by these coefficients.

Proposition 2.6.6. Let $\xi = (E, \pi, M)$ be a vector bundle of rank r with connection ∇. For a given $P \in I^k(\mathrm{GL}_r)$, there is a map

$$P : \Omega^*(M, \bigotimes^k \operatorname{End}(E)) \to \Omega^*(M)$$

from the differentials on the connection to the de Rham differential, such that

$$Pd^\nabla \eta = dP\eta$$

for any form η, i.e., we have a chain map.

Proof. We want to first define a multi-linear function $\tilde{P}$, and we will then use the correspondence and obtain P. Set

$$\bar{P}(R_1 \otimes \cdots \otimes R_k)(x) = \tilde{P}(R_1, \ldots, R_k)$$

for $R_i \in \operatorname{End}(E)$, thus $R_i|_x \in \operatorname{End}(E)_x \simeq \operatorname{End}(\mathbb{R}^r)$. Note that this isomorphism uses the existence of a basis, but the result is still independent of choice. Given $\eta \in \Omega^d(M, \bigotimes^k \operatorname{End}(E))$, we have

$$(P\eta)(X_1, \ldots, X_d) = \bar{P}(\eta(X_1, \ldots, X_d)).$$

Using multi-linearity and the definition of ∇ on $\bigotimes^k$, we have

$$\begin{aligned}
(dP(\eta))(X_0, \ldots, X_\ell) &= \sum_i (-1)^i X_i(P(\eta)(X_0, \ldots, \hat{X}_i, \ldots, X_\ell)) \\
&\quad + \sum_{i<j} (-1)^{i+j} P(\eta)([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_\ell) \\
&= \sum_i (-1)^i P(\nabla_{X_i} \eta(X_0, \ldots, \hat{X}_i, \ldots, X_\ell)) \\
&\quad + \sum_{i<j} (-1)^{i+j} P(\eta)([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_\ell) \\
&= P(d^\nabla \eta(X_0, \ldots, X_\ell)) \\
&= P(d^\nabla \eta)(X_0, \ldots, X_\ell).
\end{aligned}$$

This uses the fact that there is an action of the endomorphism on the tensor product, differentiating it termwise. □

For a connection ∇ with curvature $R = R^\nabla \in \Omega^2(M; \operatorname{End}(E))$, we define

$$R^k \in \Omega^{2k}(M, \bigotimes^k \operatorname{End}(E))$$

by

$$R^k(X_1, \ldots, X_{2k}) = \frac{1}{(2k)!} \sum_{\sigma \in S_{2k}} (-1)^\sigma R(X_{\sigma(1)}, X_{\sigma(2)}) \otimes R(X_{\sigma(3)}, X_{\sigma(4)}) \otimes \cdots \otimes R(X_{\sigma(2k-1)}, X_{\sigma(2k)}).$$

Therefore, if $P \in I^k(\mathrm{GL}_r)$, then the assignment defined in Proposition 2.6.6 gives $\mathrm{CW}^\nabla(\xi)(P) := P(R^k) \in \Omega^{2k}(M)$.

Proposition 2.6.7 (Bianchi's identity)**.** We have $dP(R^k) = 0$.

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Proof. We have

$$\begin{aligned} dP(R^k) &= P(d^\nabla R^k) \\ &= P(kd^\nabla R \wedge R^{k-1}) \\ &= 0 \end{aligned}$$

since $d^\nabla R = 0$ by Bianchi's identity. □

From Proposition 2.6.7, we get the following theorem.

Theorem 2.6.8 (Chern-Weil)**.** Let $\xi = (E, \pi, B)$ be a vector bundle of rank r. A connection ∇ defines maps

$$\begin{aligned} \mathrm{CW}^\nabla : I^k(G) &\to \Omega^{2k}(M) \\ P &\mapsto P(R^\nabla, \dots, R^\nabla) \end{aligned}$$

as above, then $\mathrm{CW}^\nabla(\xi)(P)$ is closed, and its cohomology class is independent of ∇. Moreover, we obtain a (Chern-Weil) ring homomorphism

$$\mathrm{CW} : I^*(G) \to H^*(M).$$

Proof. Given ∇^0 and ∇^1, we claim that

$$[\mathrm{CW}^{\nabla^0}(\xi)] = [\mathrm{CW}^{\nabla^1}(\xi)].$$

To see this, we take the projection $M \times [0,1] \to M$, then we combine both combinations into one single connection

$$\nabla = (1-t)p^*\nabla^0 + tp^*\nabla^1.$$

We have integration along fibers of these maps, therefore we define the *transgression form* of ∇^0 and ∇^1 to be

$$P(\nabla^0, \nabla^1) = \int_0^1 P((R^\nabla)^k) \in \Omega^{2k-1}(M)$$

for any $P \in I^k(\mathrm{GL}_r)$. We see that

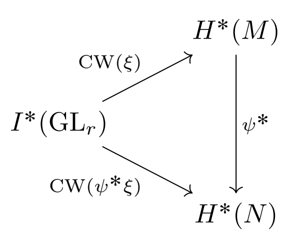
$$dP(\nabla^0, \nabla^1) = \mathrm{CW}^{\nabla^1}(\xi)(P) - \mathrm{CW}^{\nabla^0}(\xi)(P)$$

To see that this is a ring map, this mostly follows from the definition, but to see it preserves the structure, we note that the symmetric product on $I^*(\mathrm{GL}_r)$ is preserved as wedge product on $H^*(M)$. The symmetric product is symmetric but the wedge product is only graded symmetric, but since everything is in even degree, this automatically upgrades the wedge product to be symmetric as well. □

Remark 2.6.9. Let $\psi : N \to M$ be a smooth map of manifolds, and $\xi = (E, \pi, M)$ be a vector bundle on M. Suppose $P \in I^k(\mathrm{GL}_r)$, then

$$\psi^* P(R^\nabla, \dots, R^\nabla) = P(R^{\bar\nabla}, \dots, R^{\bar\nabla})$$

where $\bar\nabla$ is the pullback connection on $\psi^*\xi$. Once we check this, we see that the Chern-Weil homomorphism behaves well with respect to pullbacks. That is, the following is a commutative diagram



of ring homomorphisms.

We will use this to produce characteristic classes.

Definition 2.6.10. Any class in the image of $\mathrm{CW}(\xi)$ is called a *characteristic class* of ξ.

The collection of elements in $I^*(\mathrm{GL}_r)$ given by

$$X \mapsto \operatorname{tr}(X^k)$$

for $X \in \mathfrak{gl}_r$ and arbitrary k will generate $I^*(\mathrm{GL}_r)$. We want to pick the "smallest" subset of generators in some sense (which implies some choices of generators are better than others), which helps us to produce characteristic classes. It turns out, for $X \in \mathfrak{gl}_r$, we have

$$\det(\lambda I - X) = \lambda^r + \sigma_1(X)\lambda^{r-1} + \cdots + \sigma_r(X)$$

which means the elements $\sigma_i(X) \in I^r(\mathfrak{gl}_r)$ are invariant polynomials, and give a set of algebraic independent generators using Galois theory. Essentially, given a monic polynomial $P(x)$, we get a factorization

$$P(x) = \prod_{i=1}^{r}(x - x_i) = x^r - s_1 x^{r-1} + \cdots + (-1)^r s_r$$

where the s_i's are called the elementary symmetric functions. Therefore, for each s_i, we can express

$$\begin{cases} s_1 &= \sum\limits_{i=1}^{r} x_i \\ s_2 &= \sum\limits_{i<j} x_i x_j \\ \vdots \\ s_r &= \prod\limits_{i=1}^{r} x_i \end{cases}$$

and by applying this to the characteristic polynomial, we conclude that each σ_i is a function in terms of $\operatorname{tr}(X)$:

$$\sigma_1 = -\operatorname{tr}(X), \sigma_2 = \frac{1}{2}(\operatorname{tr}(X^2) - \operatorname{tr}(X)^2), \ldots, \sigma_r = \det(X).$$

We will define a few characteristic classes.

Definition 2.6.11. The *Pontryagin classes* of ξ are

$$P_k(\xi) = \left[\sigma_{2k}\left(\left(\frac{1}{2\pi}R\right)^{2k}\right)\right] \in H^{4k}(M)$$

for each k. The total Pontryagin class is defined by

$$P = 1 + P_1 + \cdots + P_{\lfloor \frac{r}{2} \rfloor}.$$

Remark 2.6.12.

1. By taking the power to $2k$, we really mean the usual notion: taking wedge products and then tensoring the endomorphism part.
2. There is a purely topological approach to define characteristic classes, giving every vector bundle a Chern-Weil homomorphism without referring to the connections. These classes are defined using universal bundles and classifying maps, but they are defined with integer coefficients. To connect these two constructions, we need to look at the map
$$H^*(M, \mathbb{Z}) \to H^*(M),$$
and the factor $\frac{1}{2\pi}$ ensures the $P_k(\xi)$'s belong to the image of this map. In this sense, $H^*(M, \mathbb{Z})$ is more powerful than $H^*(M)$, as they detect torsions.

3. For odd degrees, we note that
$$\left[\sigma_{2k+1}\left(\left(\frac{1}{2\pi}R\right)^{2k+1}\right)\right]=0,$$
therefore the classes in the odd degrees vanish. Fixing a metric $\langle -,-\rangle$, we can pick a connection ∇ that preserves the metric, i.e.,
$$X(\langle s_1,s_2\rangle)=\langle\nabla_X s_1,s_2\rangle+\langle s_1,\nabla_X s_2\rangle$$
for any s_1,s_2,X. This connection is not unique: we can glue connections that satisfy this equality, which become a connection that also satisfies the property. Giving a local orthonormal frame $\{s_1,\dots,s_r\}$, the curvature 2-form determined by
$$R^\nabla(X,Y)s_i=\Omega_i^j(X,Y)s_j,$$
must be skew-symmetric by the choices above, i.e., $\Omega_i^j(X,Y)=-\Omega_j^i(X,Y)$. Therefore, $\Omega\in\mathfrak{so}(r)\subseteq\mathfrak{gl}(r)$. But given $X\in\mathfrak{gl}_2$ such that $X=-X^T$, then
$$\sigma_{2k+1}(X)=\sigma_{2k+1}(X^T)=-\sigma_{2k+1}(X),$$
therefore $\sigma_{2k+1}(X)=0$. We conclude that
$$\mathrm{CW}^\nabla(\xi)(\sigma_{2k+1})=0.$$
In general, this does not mean it is zero for other connections, but it must be an exact form.

Proposition 2.6.13. Let P be the total Pontryagin class, then

i. $P(\xi_1\oplus\xi_2)=P(\xi_1)\smile P(\xi_2)$;

ii. $P(\xi)=1$ if ξ is flat;

iii. $P(\psi^*\xi)=\psi^*P(\xi)$ for any map ψ of manifolds.

Remark 2.6.14. In the special case of a tangent bundle $\xi=TM$ where M is compact and oriented (so there is a notion of integration), then we can define *Pontryagin numbers* as follows: for any $a_i\in\mathbb{N}_0$ such that
$$4\left(a_1+2a_2+\cdots+\lfloor\frac{r}{2}\rfloor a_{\lfloor\frac{r}{2}\rfloor}\right)=4\dim(M),$$
the integration
$$\int_M P_1^{a_1}\cdots P_{\lfloor\frac{r}{2}\rfloor}^{a_{\lfloor\frac{r}{2}\rfloor}}$$
is well-defined, and produces a Pontryagin number.

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Sometimes these properties are enough to determine the Pontryagin class.

Example 2.6.15.

1. Let $M=\mathbb{S}^n\subseteq\mathbb{R}^{n+1}$, then
$$T_{\mathbb{S}^n}\mathbb{R}^{n+1}=T\mathbb{S}^n\oplus\nu(\mathbb{S}^n),$$
but both the restricted tangent bundle $T_{\mathbb{S}^n}\mathbb{R}^{n+1}$ and the normal bundle $\nu(\mathbb{S}^n)$ are trivial, therefore
$$1=P(T\mathbb{S}^n)\smile 1,$$
which gives $P(T\mathbb{S}^n)=1$. Note that this does not mean $T\mathbb{S}^n$ is trivial: in fact, $T\mathbb{S}^n$ is trivial if and only if $n=1,3,7$.

2. Let $M = \mathbb{CP}^2$, using Fubini-Study metric, we can directly compute that

$$p_1(T\mathbb{CP}^2) = 3\mu,$$

where μ is the orientation class.

We will move on to Chern class, which is defined over complex vector bundles. A complex vector bundle $\xi = (E, \pi, M)$, behaving like a real vector bundle, has trivialization done over $\mathbb{C}$, that is, the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow[\simeq]{\varphi_\alpha} & U_\alpha \times \mathbb{C}^r \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}} & \\ & U_\alpha & \end{array}$$

commutes, i.e., fibers are $\mathbb{C}$-vector spaces, so for any $\lambda \in \mathbb{C}$ and $v \in E$, we have $\lambda v \in E$, and such that the transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{C}).$$

Since $\mathrm{GL}_r(\mathbb{C})$ is a real vector bundle of dimension $2r$, we recover a real structure on the complex vector bundle. Therefore, a complex vector bundle of rank r is also just a real vector bundle ξ of rank $2r$ with an endomorphism

$$\mathfrak{j} : \xi \to \xi$$

such that $\mathfrak{j}^2 = -\mathrm{id}$. On the fiber, to mimic the $\mathbb{C}$-vector space structure, for any $\lambda = a+ib$, we have $(a+ib)v = ab+b\mathfrak{j}(v)$.

	Base	Fiber
$\mathbb{R}$-vector bundle	Real Manifold	$\mathbb{R}$-vector bundle
$\mathbb{C}$-vector bundle	Real Manifold	$\mathbb{C}$-vector bundle
Holomorphic vector bundle	Complex Manifold	$\mathbb{C}$-vector bundle

Definition 2.6.16. A $\mathbb{C}$-*connection* on a complex vector bundle ξ is an ordinary connection ∇ such that

$$\nabla_X(\lambda s) = \lambda(\nabla_X s)$$

for all $\lambda \in \mathbb{C}$, or equivalently, $\nabla_X(\mathfrak{j}s) = \mathfrak{j}(\nabla_X s)$, which is just saying $\nabla_X \mathfrak{j} = 0$ for any $X \in \mathfrak{X}(M)$.

Remark 2.6.17. What do we mean by $\nabla_X \mathfrak{j}$? Given a connection ∇ defined over $E \to M$, it induces a connection ∇ over $\mathrm{End}(E) \to M$, defined by

$$(\nabla_X \mathfrak{j})(s) = \nabla_X(\mathfrak{j}(s)) - \mathfrak{j}(\nabla_X s).$$

Given a $\mathbb{C}$-connection ∇, we can choose a local $\mathbb{C}$-frame $\{s_1, \dots, s_r\}$, with

$$\nabla_X s_a = \omega_a^b(X),$$

so we have $\omega = [\omega_a^b] \in \Omega^1(U, \mathfrak{gl}_r(\mathbb{C}))$, and similarly

$$\Omega = [\Omega_a^b] \in \Omega^2(U, \mathfrak{gl}_r(\mathbb{C}).$$

Looking at $\mathrm{GL}_r(\mathbb{C})$-invariant polynomials $P : \mathfrak{gl}_r(\mathbb{C}) \to \mathbb{C}$, for any $X \in \mathfrak{gl}_r(\mathbb{C})$, we have

$$\det(\lambda I + X) = \lambda^r + \sigma_1(X)\lambda^{r-1} + \cdots + \sigma_r(X)$$

for some coefficients $\{\sigma_1, \dots, \sigma_r\}$ generating $\mathrm{GL}_r(\mathbb{C})$. We realize that this is the same story as before, so we can make the following definition.

Definition 2.6.18. The *Chern classes* of a $\mathbb{C}$-vector bundle $\xi = (E, \pi, M)$ of rank r is

$$C_k(\xi) = \left[\sigma_k\left(\left(\frac{R}{2\pi i}\right)^k\right)\right] \in H^{2k}(M, \mathbb{R}) = H^{2k}(M, \mathbb{R}).$$

The total Chern class is defined by

$$C(\xi) = 1 + C_1(\xi) + \cdots + C_r(\xi).$$

Remark 2.6.19. If $P \in I^*(\mathrm{GL}_r(\mathbb{C}))$, the Chern-Weil homomorphism is described as

$$\mathrm{CW}^\nabla(\xi) : I^*(\mathrm{GL}_r(\mathbb{C})) \to H^*(M, \mathbb{C}).$$

However, note that the Chern class lies in the $\mathbb{R}$-valued cohomology of M. Indeed, if chosen a Hermitian metric h on ξ, we can

- get a unitary local coframe $\{s_1, \dots, s_r\}$, i.e., $h(s_i, s_j) = \delta_{ij}$, and
- see that ∇ is a connection preserving h,

then the connection 1-form $\omega = [\omega_a^b] \in \mathfrak{u}(r) = \{X \in \mathfrak{gl}_r(\mathbb{C}) : X + \bar{X}^T = 0\}$ lies in the Lie algebra of the unitary group, and the curvature 2-form $\Omega \in \Omega^2(U, \mathfrak{u}(r))$. In particular, Ω is diagonalizable with imaginary eigenvalues, so $\frac{1}{2\pi i}\Omega$ has real eigenvalues. That is to say,

$$\sigma_k\left(\left(\frac{R}{2\pi i}\right)^k\right)$$

has real values.

Proposition 2.6.20. Let ξ, ξ_1, ξ_2 be $\mathbb{C}$-vector bundles, and let ψ be a map of manifolds, then

1. $C(\xi_1 \oplus \xi_2) = C(\xi_1) \smile C(\xi_2)$,
2. $C(\psi^*\xi) = \psi^* C(\xi)$,
3. $C(\xi) = 1$ if ξ has flat $\mathbb{C}$-connections,
4. $C(\gamma_1^1) = 1 - \mu$, where $\gamma_1^1 \to \mathbb{CP}^1$ is the canonical $\mathbb{C}$-line bundle, and μ is the canonical orientation of $\mathbb{CP}^1$.

Remark 2.6.21. We describe γ_1^1 as a complex vector bundle $\pi : E \to \mathbb{CP}^1$ as follows: we define $E = \{(\ell, x) : \ell \in \mathbb{CP}^1, x \in \ell\}$, so such pair $(\ell, x) \in \mathbb{CP}^1 \times \mathbb{C}^2$, then the assignment is defined by

$$\begin{aligned} \pi : E &\to \mathbb{CP}^1 \\ (\ell, x) &\mapsto \ell \end{aligned}$$

Remark 2.6.22. The properties in Proposition 2.6.20 characterizes the Chern class. That is, the Chern class is the unique map

$$\mathbf{Vec}^{\mathbb{C}}(M) \to H^*(M)$$

that satisfies these properties.

Note that $\pi : E \to \mathbb{CP}^1$ defines a section $s : \mathbb{CP}^1 \to E$, so we get a natural $\mathbb{C}$-connection on γ_1^1 by defining

$$(\nabla_X s)(\ell) = \mathrm{pr}_\ell(d_\ell s(X))$$

where $\mathrm{pr}_\ell : \mathbb{C}^2 \to \ell$ is the projection relative to the canonical Hermitian product.

For a local chart $U_0 = \{[z_0 : z_1] : z_0 \neq 0\}$, a $\mathbb{C}$-coframe over U_0 is just determined by a single section $\{s\}$. Therefore, the usual chart ψ on U_0 takes $[z_0 : z_1]$ to $\frac{z_1}{z_0} \in \mathbb{C}$, therefore s is determined by

$$s([1 : z]) = ([1 : z], (1, z)),$$

so it is a non-vanishing section. Now for

$$\nabla_X s = \omega(X)s,$$

computing $\nabla_{\frac{\partial}{\partial x}}$ and $\nabla_{\frac{\partial}{\partial y}}$ gives

$$\omega = \frac{1}{1 + x^2 + y^2}(xdx + ydy + i(xdy - ydx)).$$

Since we are working over a line bundle, then Cartan's formula gives

$$\Omega = d\omega = \frac{2i}{(1 + x^2 + y^2)^2} dx \wedge dy.$$

This is purely imaginary since the connection preserves Hermitian product. Computing the first Chern class, we have

$$\begin{aligned} C_1(\xi)|_{U_0} &= \frac{1}{2\pi i}\sigma_1(\Omega)\Big|_{U_0} \\ &= -\frac{1}{\pi(1+x^2+y^2)}dx\wedge dy \end{aligned}$$

since $\sigma_1(X) = -\operatorname{tr}(X)$.

Remark 2.6.23. To define the trace of $X \in \mathfrak{gl}_r$, note that $\mathfrak{gl}_r$ has a basis given by e_{ij}'s for $i,j \in \{1,\dots,r\}$, where e_{ij} is the matrix with (i,j)th entry as 1 and other entries as 0. Writing $X = \sum_{i,j} x_{ij}e_{ij}$ as a linear combination under this basis, we have $\operatorname{tr}(X) = \sum_{i=1}^{r} x_{ii}$.

To check that $C(\gamma_1^1) = 1-\mu$ in this case, we need

$$\int_M C_1(\xi) = -1.$$

Since U_0 is open and dense and the chart is positively-oriented, then it suffices to compute the integral in this chart, so

$$\begin{aligned} \int_M C_1(\xi) &= \int_{\mathbb{R}^2} \frac{1}{\pi(1+x^2+y^2)^2}dxdy \\ &= -\int_0^\infty\int_0^{2\pi} \frac{r}{\pi(1+r^2)^2}drd\theta \\ &= -\int_0^\infty \frac{2r}{(1+r^2)^2}dr \\ &= \frac{1}{1+r^2}\Big|_0^\infty \\ &= -1. \end{aligned}$$

There are usual two sources of $\mathbb{C}$-vector bundles, namely

1. from holomorphic manifolds M, e.g., $TM, T^*M, \bigotimes^k TM, \bigwedge^k TM$ are complex vector bundles over M, viewed as a real manifold. In this case, they all have a notion of Chern classes;

   Example 2.6.24.
   - The total Chern class of $\mathbb{CP}^1$ is $C(T\mathbb{CP}^1) = 1+2\mu$, and in general
   - $C(T\mathbb{CP}^n) = (1+a)^{n+1}$ for a certain element $a \in H^2(\mathbb{CP}^n)$.

2. from complexification of a real vector bundle. That is, given a $\mathbb{R}$-vector bundle, we take

   $$\xi\otimes\mathbb{C} = \xi\otimes\mathcal{E}_M^2$$

   with complex structure given by

   $$i(v\otimes z) = v\otimes iz.$$

   It is now natural to ask about the relation between the Chern class of $\xi\otimes\mathbb{C}$ and the Pontryagin class of ξ.

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Proposition 2.6.25. If ξ is a $\mathbb{R}$-vector bundle, then

$$P_k(\xi) = (-1)^k C_{2k}(\xi \otimes \mathbb{C}),$$

$$C_{2k+1}(\xi \otimes \mathbb{C}) = 0.$$

Proof. The first formula follows from a comparison of the definitions, where $(-1)^k$ comes from the power of i. For the second formula, given a complex vector bundle η, there exists a conjugate complex vector bundle $\bar{\eta}$ with the same underlying $\mathbb{R}$-vector bundle structure, but having complex structure $\bar{\mathfrak{j}} = -\mathfrak{j}$.

Exercise 2.6.26. $C_k(\eta) = (-1)^k C_k(\bar{\eta})$.

If $\eta = \xi \otimes \mathbb{C}$ for some $\mathbb{R}$-vector bundle ξ, then the assignment

$$\begin{aligned} \xi \otimes \mathbb{C} &\to \overline{\xi \otimes \mathbb{C}} \\ v \otimes \lambda &\mapsto v \otimes \bar{\lambda}. \end{aligned}$$

is an isomorphism of vector bundles. By Exercise 2.6.26,

$$C_k(\xi \otimes \mathbb{C}) = (-1)^k C_k(\overline{\xi \otimes \mathbb{C}}),$$

so all odd Chern classes vanish. □

There are also other characteristic classes we may define.

- In the context of K-theory, we can define the Chern character of a complex vector bundle. Applying Chern-Weil homomorphism to invariant function

  $$\begin{aligned} \chi : \mathfrak{gl}_r(\mathbb{C}) &\to \mathbb{R} \\ X &\mapsto \operatorname{tr}(\exp(X)), \end{aligned}$$

  we get *Chern character* $\operatorname{Ch}(\xi)$. This satisfies

  $$\operatorname{Ch}(\xi_1 \otimes \xi_2) = \operatorname{Ch}(\xi_1) \smile \operatorname{Ch}(\xi_2)$$

  $$\operatorname{Ch}(\xi_1 \oplus \xi_2) = \operatorname{Ch}(\xi_1) + \operatorname{Ch}(\xi_2)$$

  Therefore Chern character gives rise to a (semi-)ring homomorphism from complex vector bundles into cohomology.

- Let us revisit the Euler class. Let ξ be an oriented vector bundle of even rank $r = 2m$. Since ξ is oriented, we may choose a metric on ξ and then a positive, oriented local frame $\{s_1, \dots, s_{2m}\}$ over U, and then choose a connection ∇ that preserves the metric. With this,

  $$\nabla_X s_a = \omega_a^b(X) s_b$$

  for

  $$[\omega_a^b] \in \Omega^1(U, \mathfrak{so}(2m)).$$

  By applying invariance of the Lie algebra here, we may check the invariant polynomials $P : \mathfrak{so}(r) \to \mathbb{R}$, then $\{\sigma_1, \dots, \sigma_r\}$ is a generating set if r is odd, but this is not true if r is even. In that case, we need one extra function $\operatorname{Pf} \in I^m(\mathfrak{so}(2m))$, known as the *Pfaffian*. Once we know this, the Euler class can be expressed as

  $$e(\xi) = \left[\operatorname{Pf}\left(\left(\frac{R}{2\pi}\right)^m\right)\right] \in H^{2m}(M).$$

  Given $X \in \mathfrak{so}(2m)$, we can write $X = ADA^{-1}$, where D is a block matrix of (2×2)-components, where each component is of the form

  $$\begin{pmatrix} 0 & x_i \\ -x_i & 0 \end{pmatrix}$$

for $i = 1, \ldots, m$. The Pfaffian is then defined by

$$\mathrm{Pf}(X) = \det(A) \prod_{i=1}^{m} x_i.$$

Given any $B \in \mathrm{SO}(2m)$, we have

$$\mathrm{Pf}(BXB^{-1}) = \det(BA) \prod_{i=1}^{m} x_i = \det(A) \prod_{i=1}^{m} x_i = \mathrm{Pf}(A),$$

therefore Pfaffian is invariant under conjugation.

2.7 Fiber Bundles

Fix a manifold F.

Definition 2.7.1. A *locally-trivial fibration* with fiber type F is a surjective submersion $\pi : E \to M$ so that each $x \in M$ has a trivalizing neighborhood (U, φ) such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}} & \\ & U & \end{array}$$

Given a cover $C = \{(U_\alpha, \varphi_\alpha)\}$ by trivializing charts, we have

$$\begin{aligned} \varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F &\to (U_\alpha \cap U_\beta) \times F \\ (x, v) &\mapsto (x, g_{\alpha\beta}(x)(v)) \end{aligned}$$

where

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F).$$

Remark 2.7.2. We are replacing linear isomorphisms by diffeomorphisms, since we upgraded the structure from vector space to manifold.

We will now try to replace $\mathrm{Diff}(F)$ by an ∞-dimensional Lie group. Given an action $G \times F \to F$ of a Lie group G on F, we can think of it as a group homomorphism $G \to \mathrm{Diff}(F)$. In general, this is not an inclusion/injection, unless the action is effective. Fixing one such action, we have the following definition.

Definition 2.7.3. A *fiber bundle* with *structure group* G and fiber F is a locally-trivial fibration $\pi : E \to M$ with fiber F together with a trivializing cover $C = \{(U_\alpha, \varphi_\alpha\}$ such that

i. $\varphi_\alpha \circ \varphi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x)(v))$, with $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, and

ii. it forms a maximal atlas: given a trivializing chart (U, φ) such that $(\varphi \circ (\varphi_\alpha)^{-1})(x, v) = (x, g_\alpha(x)(v))$ with $g_\alpha : U \cap U_\alpha \to G$, then $(U, \alpha) \in C$.

Remark 2.7.4. There are two important special cases.

1. For vector bundles, we set $G = \mathrm{GL}(r)$ and $F = \mathbb{R}^r$.

2. For *G-principal bundles*, $G = F$ is some group that acts on itself by translation

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

so that the fibers are copies of the Lie group G.

Definition 2.7.5. Two cocycles $g_{\alpha\beta}, h_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ are *equivalent* if there exists some collection $\{\lambda_\alpha : U_\alpha \to G\}$ such that
$$h_{\alpha\beta}(x) = \lambda_\alpha(x) g_{\alpha\beta}(x) \lambda_\beta(x)^{-1}.$$

Remark 2.7.6. We note that this is a generalization from the case of vector bundles. However, one may need to pass to a refinement, therefore the cocycles give a cohomology group $H^1(M, G)$, namely the G-cocycles quotient by equivalence with refinement.

Proposition 2.7.7. Fixing a manifold M and an action $G \times F \to F$, there is a one-to-one correspondence between

- isomorphism classes of G-fiber bundles with fiber F, and
- $H^1(M, G)$.

Proof. The proof is the same as Theorem 2.1.17. For $\pi : E \to M$ where $E = \bigcup_\alpha (U_\alpha \times F)/\sim$, the equivalence is defined by $(x, v) \sim (x, g_{\alpha\beta}(x)v)$ for any $x \in U_\alpha \cap U_\beta$. □

Remark 2.7.8. Note that a morphism of G-fiber bundles is a pair of maps

$$\begin{array}{ccc} E & \xrightarrow{\Psi} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[\psi]{} & M' \end{array}$$

such that
$$\varphi_\beta \circ \Psi \circ \varphi_\alpha^{-1}(x, v) = (x, g(x)v)$$
with $g(x) \in G$, where $g : U_{\alpha\beta} \to G$ is a smooth map.

Definition 2.7.9. Let $H \subseteq G$ be a closed subgroup. Given a G-fiber bundle ξ, we say the structure group G of ξ can be *reduced* to H if there exists some cocycle $\{g_{\alpha\beta}\}$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to H$.

Example 2.7.10.

1. A G-fiber bundle is trivial, i.e., isomorphic to the trivial bundle $M \times F \to M$, if and only if the structure group G can be reduced to the trivial group.
2. For a vector bundle ξ, that is, where $G = \mathrm{GL}(r)$,
   i. it is orientable if and only if G can be reduced to $\mathrm{GL}_+(r)$;
   ii. it has a metric if and only if G can be reduced to $\mathrm{O}(r)$. Therefore, G can always be reduced to $\mathrm{O}(r)$;
   iii. G can be reduced to $\mathrm{SO}(r)$ if the vector bundle is orientable.

Remark 2.7.11. We note that in Proposition 2.7.7, $H^1(M, G)$ does not actually concern F. We see that the action is built in using the language of G-principal bundles.

End of Lecture 32

Definition 2.7.12. Let $G \times P \to P$ be a Lie group action for some manifold P, then it is

- *free* if $G_p = \{g \in G : g \cdot p = p\} = \{e\}$ for all $p \in P$, and
- *proper* if
$$\begin{aligned} G \times P &\to P \times P \\ (g, p) &\mapsto (g \cdot p, p) \end{aligned}$$
is a proper action.

Theorem 2.7.13 (Slice)**.** If $G \times P \to P$ is a proper action for a manifold P, then for each $p \in P$, there exists an embedded submanifold $S_p \subseteq P$, called a *slice*, such that

i. S_p is G_p-invariant;

ii. there is an orthogonal decomposition $T_pO_p \oplus T_pS_p = T_pM$ for some submanifold $O_p \subseteq P$, that is, $\ker(d_q\pi) = T_qO_q$ for projection $\pi : M \to S_p$;

iii. for any $g \in G$ and $s \in S_p$ such that $g \cdot S_p$, we have $g \in G_p$.

Theorem 2.7.14. If $G \times P \to P$ is free and proper for a manifold P, then $G\backslash P$ has a unique smooth structure for which $\pi : P \to G\backslash P$ is a submersion.

Theorem 2.7.15. Given a proper and free right-action $P \times G \to P$ for a manifold P, $\pi : P \to P/G$ is a G-principal bundle. Moreover, any G-principal bundle is of this form.

Remark 2.7.16. G-principal bundles defined by left actions of G are equivalent to the principal bundles defined as the quotient using right-action as in Theorem 2.7.15.

Proof. Suppose $P \times G \to P$ is free and proper. For each $p \in P$, consider a slice $S_p \subseteq P$. Since the action is free, an orbit intersects S_p in at most one point by part iii. of Theorem 2.7.13. Shrinking the manifold S_p whenever necessary (for the sake of apply part ii. of Theorem 2.7.13), we see

$$\pi : S_p \to P/G = M$$

is a diffeomorphism onto an open $U_p = \pi(S_p) \subseteq M$, then we have a local trivialization, given by the diffeomorphism

$$\begin{aligned} \varphi_p : U_p \times G &\to \pi^{-1}(U_p) \\ (x, g) &\mapsto s \cdot g \end{aligned}$$

with $x = \pi(s)$, such that

$$\begin{array}{ccc} U_p \times G & \xrightarrow{\varphi_p} & \pi^{-1}(U_p) \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & M & \end{array}$$

Suppose $p, p' \in M$ are such that $U_p \cap U_{p'} \neq \varnothing$, then if $x \in U_p \cap U_{p'}$, we have $x = \pi(s) = \pi(s')$ for some $s \in S_p$ and $s' \in S_{p'}$. In particular, there exists some $g(x)$ such that $s' = s \cdot g(x)$. Now

$$\varphi_p(x, g(x)) = s \cdot g(x) = s' = \varphi_{p'}(x, e),$$

therefore

$$\varphi_p^{-1} \circ \varphi_{p'}(x, e) = (x, g(x)).$$

Note that each φ_p is G-equivariant, so

$$\varphi_p(x, gh) = \varphi_p(x, g)h,$$

and

$$\varphi_p^{-1} \circ \varphi_{p'}(x, h) = (x, g(x)h),$$

therefore the cocycles are given by

$$\begin{aligned} g_{pp'} : U_p \cap U_{p'} &\to G \\ x &\mapsto g(x). \end{aligned}$$

Now let $\xi = (P, \pi, M)$ be a G-principal bundle. Choose a trivialization $\{(U_\alpha, \varphi_\alpha)\}$, we have a right G-action on $\pi^{-1}(U_\varphi)$ as follows. For any $p \in \pi^{-1}(U_\alpha)$, we may define

$$p \cdot g = \varphi_\alpha^{-1}(x, \varphi_\alpha^x(p) \cdot g).$$

Here we take the notation

$$\begin{aligned}\varphi_\alpha : \pi^{-1}(U_\alpha) &\to U_\alpha \times G\\ p &\mapsto (\pi(p), \varphi_\alpha^x(p))\end{aligned}$$

for $x = \pi(p)$. Note that if $p \in \pi^{-1}(U_\beta)$, we get

$$(x, \varphi_\beta^x(p) \cdot g) = (x, g_{\alpha\beta}(x)\varphi_\alpha^x(p) \cdot g) = (x, g_{\alpha\beta}(x)(\varphi_\alpha^x(p) \cdot g)),$$

therefore the action is well-defined. Since

$$\begin{aligned}G \times G &\to G\\ (g,h) &\mapsto gh\end{aligned}$$

is free and proper, so is the action defined above. The orbits of this action are exactly the fibers of the projection $\pi : P \to M$. □

Remark 2.7.17. The G-action on a principal G-bundle is the unique one that makes trivializations G-equivariant maps.

Given a G-principal bundle $P \times G \to P$ and a left G-action $G \times F \to F$, then we define an action[7]

$$\begin{aligned}(P \times F) \times G &\to P \times F\\ (p, f) \cdot g &\mapsto (pg, g^{-1} \cdot f).\end{aligned}$$

This is a free and proper action, with

$$\begin{array}{ccc} E = (P \times F)/G & & [p, f] \\ \downarrow{\scriptstyle \pi_E} & & \downarrow \\ M = P/G & & [p] \end{array}$$

Exercise 2.7.18. $\xi_p := (E, \pi_E, M)$ is a fiber bundle with structure group G and fiber F.

Definition 2.7.19. ξ_p defined above is called an *associated bundle*.

Remark 2.7.20. We can say that a fiber bundle is just an associated bundle for some G-principal bundle, that we can always recover it in this description. This is true because of the cocycle conditions.

Example 2.7.21.

1. For the trivial principal bundle given by the projection $M \times G \to M$, it has an associated bundle isomorphic to $M \times F \to M$. This is because $(M \times (G \times F))/G \simeq M \times F$ via $(x, g, f) \cdot h = (x, gh, h^{-1}f)$, so the map is defined by
$$(x, g, f) \mapsto (x, gf).$$

2. Let G be a Lie group and $H \subseteq G$ be a closed subgroup, then $\pi : G \to G/H$ gives an H-principal bundle.

3. A covering space gives rise to a principal bundle as follows. The universal covering $\tilde{M}$ of M has a $\pi_1(M)$-action, therefore giving rise to a $\pi_1(M)$-principal bundle $\tilde{M} \to M$. Moreover, if $N \subseteq \pi_1(M)$ is a normal subgroup, we still get a map $\tilde{M}/N \to M$. Since $\tilde{M}/N$ has a $\pi_1(M)/N$-action, then we also get a $\pi_1(M)/N$-principal bundle.

   Giving a representation $\rho : \pi_1(M) \to \mathrm{GL}_r$ is equivalent to giving a linear action $\pi_1(M) \times \mathbb{R}^r \to \mathbb{R}^r$, then there is an associated bundle constructed by
$$\begin{array}{c} E = (\tilde{M} \times \mathbb{R}^r)/\pi_1(M) \\ \downarrow{\scriptstyle \pi} \\ M = \tilde{M}/\pi_1(M) \end{array}$$

   In fact, this is a vector bundle.

[7] At this point, it does not matter whether the action defined on the left or on the right.

4. Let M be a manifold of dimension m. We define
$$F(M) = \{(v_1, \dots, v_m) : \text{ basis of } T_xM, x \in M\}$$
as a submanifold of $(TM)^m$. Note that $F(M)$ has a GL_m action given by
$$(v_1, \dots, v_m)A = (a_1^i v_i, \dots, a_m^i v_i)$$
with $A = (a_j^i)$. This action is free and proper, then the projection
$$\begin{aligned} \pi : F(M) &\to M \\ (v_1, \dots, v_m) &\mapsto x \end{aligned}$$
gives rise to a GL_m-principal bundle. This is called a *frame bundle*. The defining representation in this case is
$$\mathrm{GL}_m \times \mathbb{R}^m \to \mathbb{R}^m,$$
then there is an associated bundle $\pi_E : E = (F(M) \times \mathbb{R}^m)/\mathrm{GL}_m \simeq TM \to M$, where the isomorphism is given by
$$\big[(v_1, \dots, v_m), (\lambda^1, \dots, \lambda^m)\big] \mapsto \lambda^i v_i$$
Consider the defining action of GL_r on $\mathbb{R}^r$, then the bundle $\xi_{\mathbb{R}^r} = F(\xi) \times_{\mathrm{GL}_r} \mathbb{R}^r \to M$ is isomorphic to ξ.

5. This is a phenomenon that works in general: we can get all bundles associated to M given by constructing upon frame bundles. For example,
   - GL_m acts on $\Lambda^k \mathbb{R}^m$, therefore giving associated bundle $E \simeq \Lambda^k TM$;
   - GL_m acts on $(\mathbb{R}^n)^*$ therefore giving associated bundle $E \simeq T^*M$.

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Remark 2.7.22. Any functorial construction with vector spaces induces a construction with vector bundles.

Proposition 2.7.23. Given a G-principal bundle $\xi = (P, \pi, M)$ with action $G \times F \to F$, there is a one-to-one correspondence between

- $\Gamma(\xi_F)$ and
- the G-equivariant sections $\tilde{s} : P \to F$.

Proof. Consider a bundle $E = P \times_G F \to M$, then a section $s : M \to E$ must takes the form $s(x) = [(p, \tilde{s}(p))]$ for $\tilde{s} : P \to F$ such that $\tilde{s}(pg) = g^{-1}\tilde{s}(p)$. ☐

Remark 2.7.24. A few remarks in the vein of obstruction theory.

1. In general, G-fiber bundles do not have sections.
2. A G-fiber bundle with contractible fiber F always has sections, and any two sections are homotopic.
3. A G-principal bundle has a section if and only if it is isomorphic to the trivial bundle.

Definition 2.7.25. A morphism between principal bundles $\xi = (P, \pi, M)$ with structure group G and $\xi' = (P', \pi', M')$ with structure group G' relative to a Lie group homomorphism $\varphi : G \to G'$ is a φ-equivariant map
$$\begin{aligned} \Psi : P &\to P' \\ pg &\mapsto \Psi(p)\varphi(g) \end{aligned}$$
defined for all $p \in M$ and $g \in G$. Such morphism maps fibers to fibers, giving commutative squares
$$\begin{array}{ccc} P & \xrightarrow{\Psi} & P' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[\psi]{} & M' \end{array}$$

Remark 2.7.26.

1. If Ψ is an embedding, then we say P is a *principal subbundle* of P'.
2. If Ψ is an embedding and $\psi = \mathrm{id}$, then P is a *reduction* of the structure group G' to $H = \varphi(G) \subseteq G'$.

Definition 2.7.27. A *G-structure* on a manifold M is a reduction of frame bundle $F(M)$ to a closed subgroup $H \subseteq \mathrm{GL}_m$.

Example 2.7.28. Given a Riemannian manifold (M, g), we obtain a reduction of the frame bundle $F(M)$ to the orthogonal frame bundle

$$O(m) = \{(v_,1 \ldots, v_n) : \text{ orthonormal basis of } T_xM,\ \forall x \in M\} \subseteq F(M).$$

This bundle has an O_n-action, therefore becoming an O_n-principal bundle over M. A G-structure $P \subseteq F(M)$ is *integrable* if it has local sections consisting of connecting vector fields.

Reduction	G-structure	Integrable G-structure
$\mathrm{O}_n \subseteq \mathrm{GL}_n$	Riemannian structure	Flat Riemannian structure
$\mathrm{Sp}_n \subseteq \mathrm{GL}_{2n}$	Almost symplectic structure $\omega \in \Omega^2(M)$, $\omega^{\wedge n}$ non-vanishing	Symplectic structure $\omega \in \Omega^2(M)$, $\omega^{\wedge n}$ nowhere vanishing, $d\omega = 0$
$\mathrm{GL}_n(\mathbb{C}) \subseteq \mathrm{GL}_{2n}$	Almost complex structure[8] $\mathrm{j} : TM \to TM$, $\mathrm{j}^2 = -I$	Complex structure[9] Nijenhuis tensor[10] $N_{\mathrm{j}} = 0$
$\mathrm{U}(n) = \mathrm{Sp}_{2n} \cap \mathrm{O}_n(\mathbb{C}) \subseteq \mathrm{GL}_{2n}$	Almost Hermitian	Kähler structure

We will now demonstrate properties of the pullback on fiber bundles.

Definition 2.7.29. Given a G-bundle $P \to M$ with $\psi : M \to M$, the *pullback* $\psi^* P = N \times_G P = \{(x, p) : \psi(x) = \pi(p)\} \to N$ is a bundle determined by $(x, p) \mapsto x$. $\psi^* P$ admits a right G-action, given by $(x, p) \cdot g = (x, p \cdot g)$.

Remark 2.7.30.

1. It is characterized by the usual universal property of pullbacks.
2. If $\psi_0, \psi_1 : N \to M$ are homotopic, then $\psi_0^* P \simeq \psi_1^* P$.
3. If we have an action $G \times F \to F$, then there is a pullback

$$\begin{array}{ccc} E = P \times_G F & \longleftarrow & \psi^* P \times_G F = \psi^* E \\ \downarrow & & \downarrow \\ M & \xleftarrow[\psi]{} & N \end{array}$$

   giving an associated bundle.

4. There is a universal G-principal bundle $EG \to BG$ into the classifying space BG, with the property that for any G-principal bundle $P \to M$, there exists a (unique up to homotopy class/isomorphism class) map $\psi : M \to BG$ such that $P \simeq \psi^* EG$. In particular, given a fixed manifold M, there is a one-to-one correspondence between
   - isomorphism classes of G-principal bundles over M, and
   - homotopy classes $[M, BG]$ of maps.

   However, BG is infinite-dimensional in general, so for this to work we may look at it as a limiting construction.

[8] Fibers of manifold has complex structure.
[9] Manifold has global holomorphic structure, suitable with transition maps.
[10] See Newlander–Nirenberg theorem.

2.8 Principal Connections

Definition 2.8.1. A connection on a G-principal bundle $\xi = (P, \pi, M)$, or a principal connection, is a distribution[11] $H \subseteq TP$ that is

1. horizontal, i.e., $H_p \oplus \ker(d_p\pi) = T_pP$, and
2. G-invariant, i.e., $H_{pg} = d_pR_g(H_p)$ for any $p \in P$ and $g \in G$.

Remark 2.8.2.

- The component $V_p = \ker(d_p\pi)$ is usually called vertical distribution.
- $d_p\pi : J_pT_{\pi(p)}M$ is a linear isomorphism.
- For this reason, we get a map
$$\begin{aligned} h : \pi^*TM &\to TP \\ (p, v) &\mapsto h(p, v) \end{aligned}$$
whose image is exactly H, where $h(p, v)$ is the unique vector in T_pP belonging to H_p such that $d_p\pi(h(p, v)) = v$. Therefore, h is called the horizontal lift.

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Example 2.8.3. Suppose we have a vector bundle $\xi = (E, p, M)$ with a linear connection ∇, and let $F(\xi)$ be the bundle of frames over M given by GL_r-action. An element $u \in F(\xi)$ is a frame $u = (v_1, \ldots, v_r) \in F(\xi)_x$, for any $\gamma : I \to M$ with $\gamma(0) = x$, doing parallel transport gives $u(t) = (\tau_{\gamma(t)}v_1, \ldots, \tau_{\gamma(t)}v_r)$, then the principal connection H_u is defined by
$$\{\dot{u}(0)\}$$
for the information provided above. One can show that this satisfies the definition on local trivialization.

As usual, we want to determine connection 1-form and curvature 2-form. Given a principal connection $H \subseteq TG$, the connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ is defined by
$$\omega(v) = \xi$$
as follows: an element $v \in T_uP$ is given by an orthogonal decomposition
$$v = v^H + v^V \in H_u \oplus V_u,$$
but each $\xi \in \mathfrak{g}$ corresponds to $\xi_P \in \mathfrak{X}(P)$, the infinitesimal generator corresponding to the Lie algebra $\mathfrak{g}$, with
$$(\xi_P)_u = \left.\frac{d}{dt}u\exp(t\xi)\right|_{t=0}$$
therefore we have note that the fiber $V_u \simeq \mathfrak{g}$ by the assignment given above. We then conclude that $v^V = (\xi_P)_u$, which defines our assignment above.

Remark 2.8.4. Note that $H = \ker(\omega)$.

Proposition 2.8.5. The following are a few properties of the connection 1-form ω.

i. ω is horizontal: $\omega(\xi_P) = \xi$ for any $\xi \in \mathfrak{g}$, and this condition implies that the kernel of the form must be transversal for dimension reasons.

ii. ω is G-equivariant: $(R_g)^*\omega = \mathrm{Ad}_{g^{-1}} \cdot\omega$ for all $g \in G$.

[11] This is a smooth varying family of subspaces of vector bundles.

Proof. We will prove property ii. Take $v \in T_u P$, then we may write $v = v^H + (\xi_P)_u$ and write $\omega(v) = \xi$. We take

$$\begin{aligned}(R_g^*\omega)(v) &= \omega_{ug}(dR_g(v)) \\ &= \operatorname{Ad}_{g^{-1}} \xi \\ &= H_{ug} + (\operatorname{Ad}_{g^{-1}} \xi)_p\big|_{ug} \\ &= H_{ug} + \frac{d}{dt} ug \exp(\operatorname{Ad}_{g^{-1}} \xi)\bigg|_{t=0} \\ &= H_{ug} + \frac{d}{dt} ugg^{-1} \exp(t\xi)g\bigg|_{t=0} \\ &= H_{ug} + \frac{d}{dt}(u\exp(t\xi)g)\bigg|_{t=0} \\ &= H_{ug} + dR_g \frac{d}{dt} u\exp(t\xi)\bigg|_{t=0} \\ &= dR_g(v^H) + dR_g(\xi_P)_u \\ &= dR_g(v)\end{aligned}$$

given by the orthogonal decomposition $dR_g(v^H) \in H_{ug}$ and $(\operatorname{Ad}_{g^{-1}} \xi)_p\big|_u \in V_{ug}$. ☐

From this, we conclude that

Proposition 2.8.6. Fixing a G-principal bundle, there is a one-to-one correspondence between

- principal connections $H \subseteq TP$ and
- elements $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying Proposition 2.8.5.

Corollary 2.8.7. Any G-principal bundle has connections.

Definition 2.8.8. Given a G-principal bundle with connection ∇, the exterior covariant derivative D is defined by

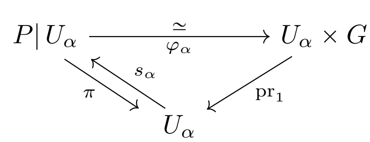
$$\begin{aligned} D : \Omega^*(P; \mathfrak{g}) &\to \Omega^{*+1}(P; \mathfrak{g}) \\ (v_0, \dots, v_k) &\mapsto (D\omega)(v_0, \dots, v_k) := d\omega(\operatorname{hor}(v_0), \dots, \operatorname{hor}(v_k))\end{aligned}$$

where $\operatorname{hor}(v)$ is the horizontal component of a vector v.

Remark 2.8.9. In general, $D^2 \neq 0$.

Definition 2.8.10. The curvature 2-form is defined by $\Omega := D\omega \in \Omega^2(P, \mathfrak{g})$.

To describe this locally, we fix local sections $s_\alpha : U_\alpha \to P$ of $\pi : P \to M$. Define $\omega_\alpha = s_\alpha^*\omega \in \Omega(U_\alpha, \mathfrak{g})$ and $\Omega_\alpha = s_\alpha^*\Omega \in \Omega^2(U_\alpha, \mathfrak{g})$. On intersection $U_\alpha \cap U_\beta$, we have $s_\beta(x) = s_\alpha(x)g_{\alpha\beta}(x)$ for $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ gives the G-cocycle associated with trivialization



for

$$(\varphi_\alpha^{-1})(x, g) = s_\alpha(x)g.$$

From this,

- $\omega_\beta = \operatorname{Ad}_{g_{\alpha\beta}^{-1}} \omega_\alpha + g_{\alpha\beta}^*\omega_{\mathrm{MC}}$, where $\omega_{\mathrm{MC}} := d_g L_{g^{-1}}(v) \in \Omega^1(G, \mathfrak{g})$ is the left Maurer-Cartan form, characterized by the fact that it is the unique form that valuates as identity at identity.

- $\Omega_\beta = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \omega_\beta$.

Example 2.8.11. Going back to the example $\xi = (E, p, M)$ of a vector bundle with linear connection ∇, we get a frame bundle $F(\xi) \to M$ with principal connection. Choose local sections $s_\alpha : U_\alpha \to F(\xi)$ defined by $s_\alpha(x) = (s_1^\alpha(x), \ldots, s_r^\alpha(x))$ for $s_i^r : U_\alpha \to \xi$, then we get $\nabla_X s_i^\alpha = \omega_a^b(X) s_b^\alpha$ where $\omega = [\omega_a^b] \in \Omega^1(U_\alpha; \mathfrak{gl}_r)$. Similarly, we have $R^\nabla(X,Y)s_a^\alpha = \Omega_a^b(X,Y)s_b^\alpha$ where $\Omega_\alpha = [\Omega_a^b] \in \Omega^2(U_\alpha; \mathfrak{gl}_r)$. In this context, expressing $s_\beta(x) = s_\alpha(x)A$ gives

$$\omega_\beta = A_{\alpha\beta}^{-1} \omega_\alpha A_{\alpha\beta} + A_{\alpha\beta}^{-1} dA_{\alpha\beta},$$

where the pullback $A_{\alpha\beta}^* \omega_{\mathrm{MC}} \in \Omega^1(\mathrm{GL}_r; \mathfrak{gl}_r)$ of Maurer-Cartan form is just the second term $A_{\alpha\beta}^{-1} dA_{\alpha\beta}$.

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From this, we recover the usual properties like

- Cartan's structure equations $\Omega = d\omega = \frac{1}{2}[\omega, \omega]$, and
- Bianchi's identity: $D\Omega = 0$.

Remark 2.8.12. If $\eta_1, \eta_2 \in \Omega^1(P, \mathfrak{g})$, then $[\eta_1, \eta_2] \in \Omega^2(P, \mathfrak{g})$ is defined via

$$[\eta_1, \eta_2](X,Y) = [\eta_1(X), \eta_2(Y)] - [\eta_1(Y), \eta_1(X)].$$

Note that this is not skew-symmetric in X and Y. For X and Y, if one is vertical and one is horizontal, we note $d\omega = 0$. We may assume one is the infinitesimal generator $X = \xi_P$, so $\omega(X) = \xi$, and $\omega(Y) = 0$, therefore $[\omega, \omega](X,Y) = 0$. One can show that $\Omega(X,Y) = 0$ by definition.

Definition 2.8.13. If $X \in \mathfrak{X}(M)$, then there exists a unique vector field $\tilde{X} \in \mathfrak{X}(P)$ that is horizontal: $\tilde{X}_u \in H_u$, or equivalently $\omega(\tilde{X}) = 0$, and the projection $\pi_*(\tilde{X}) = X$. Such $\tilde{X}$ is unique, and is called the *horizontal lift* of X.

Proposition 2.8.14. For any vector fields $X, Y \in \mathfrak{X}(M)$, we have

$$(\widetilde{[X,Y]} - [\tilde{X}, \tilde{Y}])_u = \xi_P|_u$$

where $\xi = \Omega(\tilde{X}, \tilde{Y})_u$.

This gives an interpretation of what the curvature is. In particular, if $\xi = 0$, this says that the bracket of horizontal vector fields is zero.

Proof. We have horizontal lift $\pi_*(\widetilde{[X,Y]}) = [X,Y]$, and projection $\pi_*([\tilde{X}, \tilde{Y}]) = [\pi_* \tilde{X}, \pi_* \tilde{Y}] = [X,Y]$. But since they have the same horizontal portion, their difference must be vertical. We compute

$$\begin{aligned}
\Omega(\tilde{X}, \tilde{Y}) &= d\omega(\tilde{X}, \tilde{Y}) \\
&= \tilde{X}(\omega(\tilde{Y})) - \tilde{Y}(\omega(\tilde{X})) - \tilde{\omega}([\tilde{X}, \tilde{Y}]) \\
&= 0 - 0 - \omega([\tilde{X}, \tilde{Y}]) \\
&= -\omega([\tilde{X}, \tilde{Y}])
\end{aligned}$$

□

Corollary 2.8.15. A connection is flat, i.e., $\Omega \equiv 0$, if and only if $H \subseteq TP$ is involutive distribution.

We can do the same thing for curves.

Definition 2.8.16. Given a curve $c : [0,1] \to M$ on the base, its *horizontal lift* $\tilde{c} : [0,1] \to P$ through a point $u_0 \in P_{c(0)}$ is a curve with $\pi(\tilde{c}(t)) = c(t)$ such that

$$\begin{cases} \dot{\tilde{c}}(t) \in H_{c(t)} \\ c(0) = u_0 \end{cases}$$

One natural thing to ask is why such lifts always exist.

Remark 2.8.17. If we think of this as a map from $[0,1]$, we can pullback c to $[0,1]$, since the base is contractible, then the principal bundle is trivial, therefore it comes down to such horizontal lift exists for trivial principal bundle (with non-trivial connection).

Lemma 2.8.18. Given a curve $c : [0,1] \to M$ with a point $u_0 \in P_{c(0)}$, a horizontal lift exists.

Proof. Since the bundle is locally trivial, we can choose a non-horizontal curve $v : I \to P$ that goes through u_0 and projects horizontally to c, i.e., $v(0) = u_0$ and $\pi(v(t)) = c(t)$. We modify this curve by group action so that the new curve $\tilde{c}$ acts horizontally, i.e., $\tilde{c}(t) = v(t)g(t)$ satisfying the given conditions. We have

$$\begin{aligned}
\omega(\dot{\tilde{c}}(t)) &= \omega(\dot{v}(t)g(t) + v(t)\dot{g}(t)) \\
&= \omega(\dot{v}(t)g(t) + v(t)g(t)g^{-1}(t)\dot{g}(t)) \\
&= \mathrm{Ad}_{g^{-1}(t)}\,\omega(\dot{v}(t)) + g^{-1}(t)\dot{g}(t) \\
&= \mathrm{Ad}_{g^{-1}(t)}\,\omega(\dot{v}(t)) + \omega_{\mathrm{MC}}(\dot{g}(t)) \\
&= 0.
\end{aligned}$$

It suffices to show that

$$\begin{cases} \dot{g}(t) = g(t)A(t) \\ g(0) = e \end{cases}$$

has solution between 0 and 1, where

$$A(t) = -\mathrm{Ad}_{g^{-1}(t)}\,\omega(\dot{v}(t)).$$

Being a time-dependent linear ODE, it always has a solution in the time interval. □

Remark 2.8.19. Suppose $\pi : E \to M$ is a surjective submersion. An *Ehresmann connection* for $\pi : E \to M$ is a distribution $H \subseteq TE$ such that $H \oplus \ker(d\pi) = TE$. The condition may say that the horizontal lift which depends on time as well as point in the fiber. In this language, we have shown that the connection of principal bundle is complete.

Definition 2.8.20. Given a curve $c : I \to M$, the *parallel transport* along c is defined by

$$\begin{aligned}
\tau_t^c : P_{c(0)} &\to P_{c(t)} \\
u &\mapsto \tilde{c}(t)
\end{aligned}$$

where $\tilde{c}$ is the horizontal lift of c through the point u.

Proposition 2.8.21. A few properties of this parallel transport.

1. Parallel transport is equivariant, i.e., $\tau_t^c(ug) = \tau_t^c(u)g$ for any g.
2. $\tau_1^c : P_{c(0)} \to P_{c(1)}$ has an inverse $\tau_1^{\bar{c}} : P_{c(1)} \to P_{c(0)}$ with $\bar{c}(t) = c(1-t)$.
3. Given two curves $c_1, c_2 : [0,1] \to M$ such that $c_1(0) = c_2(1)$, then the parallel transport of the concatenation $c_1 \cdot c_2$ is the composition of parallel transports:
$$\tau_1^{c_1\cdot c_2} = \tau_1^{c_2} \circ \tau_1^{c_2}.$$

Remark 2.8.22. We note that the concatentation of two smooth curves may not be smooth, and we did not define parallel transport over such curves. However, to get around this, we can extend the definition to piecewise smooth curves, or we can show that the parallel transport is invariant under reparametrization.

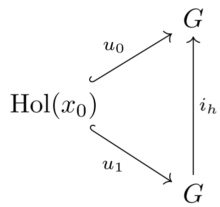
Definition 2.8.23. The *holonomy group* $\mathrm{Hol}(x_0)$ of the connection based at a point x_0 is the set of parallel transports τ_1^c along a curve $c : I \to M$ that are loops based at x_0. This is a subgroup of the diffeomorphism group $\mathrm{Diff}(P_{x_0})$.

Remark 2.8.24. If we choose a point $u_0 \in P_{x_0}$ in the fiber, one can identify this group as a Lie subgroup into structure group G via

$$\begin{aligned}\mathrm{Hol}(x_0) &\hookrightarrow G\\ \tau_1^c &\mapsto g\end{aligned}$$

via unique g such that $\tau_1^c(u_0) = u_0 g$. If we choose a different point $\bar{u}_0 \in P_{x_0}$, the Lie subgroups we get are conjugates. That is, $\bar{u}_0 = u_0 h$ for some h, so

$$\tau_1^c(\bar{u}_0) = \tau_1^c(u_0 h) = \tau_1^c(u_0)h = u_0 g h h^{-1} g h = \bar{u}_0 h^{-1} g h.$$



We will now make an identification of the Lie algebra on the Lie subgroup structure.

Theorem 2.8.25 (Ambrose-Singer)**.** Let $\pi : P \to M$ be a G-principal bundle with a connection. Fixing a point $u_0 \in P_{x_0}$ in the fiber, the image of the embedding $\mathrm{Hol}(x_0) \hookrightarrow G$ has a Lie algebra

$$\{\Omega_u(v_1, v_2) : u \in P \text{ any point that can be joined to } u_0 \text{ by horizontal curves}, v_1.v_2 \in H_u\}.$$

Remark 2.8.26. Given a flat connection, the Lie group is obtained as a homomorphism of the fundamental group to G, therefore it is discrete. The horizontal lifts are given by covering foliations. We know a holonomy allows variation of the loop a point along its fiber (interpretative as a path), while the curvature measures the variation allowed.

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Let $\xi = (P, \pi, M)$ be a G-prioncipal bundle. Fix a connection with curvature $\Omega \in \Omega^2(P, \mathfrak{g})$, then we have defined

$$\mathrm{CW}(\xi) : I^k(G) \to \Omega^{2k}(M)$$

where $I^k(G)$ is the set of Ad-invariant $P : \mathfrak{g} \to \mathbb{R}$ of degree k, such that $\mathrm{CW}(\xi) = P(\Omega^k)$. This is a closed form by the Bianchi's identity, is dependent on the connection. However, the cohomology class is independent of the choice of connection

$$\mathrm{CW}(\xi) : I^k(G) \to H^{2k}(M).$$

We saw that

- in the case where $G = \mathrm{GL}_n(\mathbb{R})$, $\det(\lambda I - x) = \sum \sigma_k(x)x^k$ gives rise to the Pontryagin classes;
- similarly, we defined Chern classes for $G = \mathrm{GL}_n(\mathbb{C})$;
- for $G = \mathrm{SO}(2m)$, wehave the Pfaffian form $\mathrm{Pf} \in I^m(\mathfrak{so}(2m))$, and we used it to define the Euler class.

In the case of torsion, we need G-structures. Pick a basis on $\mathbb{R}$-vector space V, this is equivalent to choosing a linear isomorphism $u : \mathbb{R}^n \to V$, then we may define a frame bundle

$$\pi : F(M) = \{u : \mathbb{R}^n : T_x M : x \in M\} \to M$$

has a right GL_m-action, thinking of GL_m as linear isomorphisms $g : \mathbb{R}^n \to \mathbb{R}^n$ acting by precomposition.

Definition 2.8.27. The *tautological form* or *solder form* of $F(M)$ is $\theta \in \Omega^1(F(M), \mathbb{R}^n)$, defined by

$$\theta_u(v) = u^{-1}(d\pi(v)).$$

Proposition 2.8.28. Here are some properties of the tautological form.

i. It is horizontal: $\theta(v) = 0$ if and only if $v \in \ker(d\pi)$.

ii. It is G-equivariant: $(R_g)^*\theta = g^{-1} \cdot \theta(v)$.

Proof. i. This is obvious.

ii. We have

$$\begin{aligned}((R_g)^*\theta)_u(v) &= \theta_{ug}(d_u R_g \cdot v) \\ &= (ug)^{-1}(d\pi(d_u R_g \cdot v)) \\ &= g^{-1} \cdot (u^{-1}(d\pi(v))) \\ &= g^{-1} \cdot \theta_u(v).\end{aligned}$$

□

These properties actually characterize the solder form. If $\varphi : M \to M$ is a diffeomorphism, then there is a lift

$$\begin{array}{ccc} F(M) & \xrightarrow{\tilde{\varphi}} & F(M) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\varphi]{} & M \end{array}$$

given by $\tilde{\varphi}(u) = d\varphi \circ u$. Since this map is GL_n-equivariant, then it is an automorphism of $F(M)$, and the pullback

$$(\tilde{\varphi})^*\theta = \theta.$$

Proposition 2.8.29. An automorphism

$$\begin{array}{ccc} F(M) & \xrightarrow{\tilde{\varphi}} & F(M) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\varphi]{} & M \end{array}$$

is of the form $\Phi = \tilde{\varphi}$ if and only if $\Phi^*\theta = \theta$.

Proof.

($\Rightarrow$) This is obvious.

($\Leftarrow$) We have $\Phi \circ \tilde{\varphi}^{-1} \in \mathrm{Aut}(F(M))$ preserving θ and covering id_M. Given $\Phi \in \mathrm{Aut}(F(M))$ has the same property, we claim that $\Phi = \mathrm{id}$. To see this, since $\Phi(u) = ug(x)$ and $\pi(u) = x$, so this is incarnated by $\Phi = R_g$ for some $g : M \to G$, therefore

$$\Phi^*\theta = (R_g)^*\theta = g^{-1}\theta = \theta.$$

This is usually true for fixed g, but even if g is varying this is true by the definition of the solder form. We conclude that $g(x) = e$ for any x, therefore $\Phi = \mathrm{id}$.

□

Given a fixed closed subgroup $G \subseteq \mathrm{GL}_n$ and a G-structure $i : P \hookrightarrow F(M)$, then we may define $\theta_p = i^*\theta$. This satisfies Proposition 2.8.28 as well.

Theorem 2.8.30. Given a G-principal bundle $\pi : P \to M$ with a 1-form $\theta_p \in \Omega^1(P, \mathbb{R}^n)$ which is a fiberwise surjection satisfying Proposition 2.8.28, then there is a canonical embedding $i : O \hookrightarrow F(M)$ that is G-equivariant, i.e., P becomes a G-structure, and $\theta_p = i^*\theta$.

Proof. Fix $u \in P$, we construct i as follows. We need to construct a map $i(u)$ such that the following diagram

$$\begin{array}{ccc} T_uP & \xrightarrow{\theta} & \mathbb{R}^n \\ {\scriptstyle d\pi}\downarrow & \swarrow{\scriptstyle i(u)} & \\ T_xM & & \end{array}$$

commutes. In particular, θ descends to an isomorphism in $i(u)$. We define i by choosing such $i(u)$ for each $u \in P$. $\square$

Definition 2.8.31. Fix a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ on a G-structure (P, θ), then the *torsion* 2*-form* is the covariant derivative

$$\Theta = D\theta \in \Omega^2(P, \mathbb{R}^n).$$

Proposition 2.8.32 (Cartan's Structure Equation)**.** indexCartan's structural equations We have

$$\begin{cases} d\theta &= -\omega \wedge \theta + \Theta \in \Omega^2(P, \mathbb{R}^n) \\ d\omega &= -\omega \wedge \omega + \Omega \in \Omega^2(P, \mathfrak{g}) \end{cases}$$

For $\mathfrak{g} \subseteq \mathfrak{gl}_m(\mathbb{R})$, we define

$$(\omega \wedge \theta)(v_1, v_2) = \omega(v_1) \cdot \theta(v_2) - \omega(v_2) \cdot \theta(v_1).$$

$$\omega \wedge \omega(v_1, v_2) = \omega(v_1) \cdot \omega(v_2) - \omega(v_2) \cdot \omega(v_1).$$

These recover the same formula over tangent spaces.

Cover M by $\{U_\alpha\}$ with sections $s_\alpha : U_\alpha \to P$, then

$$s_\alpha\theta = (\theta_\alpha^1, \dots, \theta_\alpha^m) \in \Omega^1(U_\alpha, \mathbb{R}^n)$$

is a vector-valued form on U_α. This then becomes a local coframe on U_α. We have seen that the pullback

$$s_\alpha^*\omega = (\omega_i^j) \in \Omega^1(U_\alpha, \mathfrak{gl}_n),$$

and similarly, we can calculate the pullback of local torsion 2-form

$$s_\alpha^*\Theta = (\Theta^1, \dots, \Theta^n) \in \Omega^2(U_\alpha, \mathbb{R}^n)$$

and compare it with the usual local curvature 2-form.

$$s_\alpha^*\Omega = (\Omega_i^j) \in \Omega^2(U_\alpha, \mathfrak{gl}_n)$$

The punchline of all of this being, the pullback of Cartan's structural equations above gives the usual Cartan's structural equations.

Remark 2.8.33. The total space of a G-structure is parallelizable: this is very different from the overall behavior of the general vector bundles.

End of Lecture 37

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