MATH 514 Notes

Jiantong Liu

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Contents

1 Cauchy's Formula and Applications

To start with the notation, set $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ for $z_i = x_i + iy_i$, then one should be familiar with the norm $||z||^2 = \sum |z_i|^2$. Let $U \subseteq \mathbb{C}^n$ be an open set, then $C^\infty(U)$ is the collection of smooth, i.e., C^∞-functions on U, and $C^\infty(\bar{U})$ is the collection of smooth functions on a neighborhood of $\bar{U}$. For $p \in \mathbb{C}^n$, we define the cotangent space to be the real vector space $T_{p,\mathbb{R}}\mathbb{C}^n = \operatorname{span}_\mathbb{R}\{dx_j, dy_k\}$ of dimension $2n$. There is then a dual notion $T_{p,\mathbb{R}}\mathbb{C}^n = \operatorname{span}_\mathbb{R}\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\}$.

Different from the real space, we have $T^\vee_{p,\mathbb{C}}\mathbb{C}^n = T^\vee_{p,\mathbb{R}} \otimes_\mathbb{R} \mathbb{C} = \operatorname{span}_\mathbb{C}\{dx_j, dy_k\}$ and $T_{p,\mathbb{C}}\mathbb{C}^n = \operatorname{span}_\mathbb{C}\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\}$ in the complex setting. This now creates new differentials $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$. In this setting, we can interpret $T^\vee_{p,\mathbb{C}}\mathbb{C}^n = \operatorname{span}_\mathbb{C}\{dz_j, d\bar{z}_j\}$ and $T_{p,\mathbb{C}}\mathbb{C}^n = \operatorname{span}_\mathbb{C}\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}\}$, where we can write the dual basis given by $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$. In a different formulation, we can write down

$$\begin{aligned} df &= \sum \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \\ &= \sum \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j. \end{aligned}$$

This allows us to understand differentiability in several complex variables. For now on, we only restrict to the case $\mathbb{C}$.

Definition 1.1. Consider $z \in \mathbb{C}$, an open subset $U \subseteq \mathbb{C}$, and a function $f \in C^\infty(U)$. We say f is holomorphic if $\frac{\partial f}{\partial \bar{z}} = 0$.

Remark 1.2. A holomorphic function f is equivalent to having f satisfying the Cauchy-Riemann equations, i.e., for $f = u + iv$, then $u_x = v_u$ and $u_y = -v_x$.

Definition 1.3. Consider $z \in \mathbb{C}$, an open subset $U \subseteq \mathbb{C}$, and a function $f \in C^\infty(U)$. We say f is analytic if for any $z_0 \in U$, there exists a neighborhood $z_0 \in V \subseteq U$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

that converges absolutely and uniformly in V.

Theorem 1.4. A function f is holomorphic if and only if it is analytic.

To prove this, we require Proposition 1.5.

Proposition 1.5 (Cauchy Integral Formula). Let $\Delta \subseteq \mathbb{C}$ be a disk, and say f is smooth in the boundary of the disk, i.e., $f \in C^\infty(\bar{\Delta})$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(w)dw}{w - z} + \frac{1}{2\pi i} \int_{\Delta} \frac{\frac{\partial f}{\partial \bar{w}}(w) dw \wedge d\bar{w}}{w - z}.$$

Proof of Theorem 1.4. ($\Rightarrow$): say $\frac{\partial f}{\partial \bar{z}} = 0$, then by Proposition 1.5, we know that

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(w)dw}{w - z}.$$

Using the identity

$$\frac{1}{w - z} = \frac{1}{w} \cdot \frac{1}{1 - \frac{z}{w}} = \sum \frac{z^n}{w^{n+1}},$$

say we work over the case where $z_0 = 0$,[1] therefore

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \sum \frac{f(w)dw}{w^{n+1}} z^n.$$

[1] For a general point z_0, refer to the textbook.

Since the geometric series convergences absolutely and uniformly on V, then by we may interchange the integral and the summation, then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(w)}{w^{n+1}} dw z^n \\ &= \sum \left(\frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(w)}{w^{n+1}} dw \right) z^n \end{aligned}$$

which gives rise to a power series expansion, as desired.

($\Leftarrow$): consider $f(z) = \sum a_n (z-z_0)^n$, then we write it as a limit of partial sums

$$\begin{aligned} f(z) &= \sum a_n (z-z_0)^n \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} a_n (z-z_0)^n \\ &= \lim_{N\to 0} \frac{1}{2\pi i} \int_{\partial\Delta} \frac{\sum_{n=0}^{N} a_n (w-z_0)^n}{w-z} dw \\ &= \frac{1}{2\pi i} \int \frac{f(w)}{w-z} dw \end{aligned}$$

by uniform convergence of compact set. Since the function is of C^∞, then we may differentiate and get

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2\pi i} \int_{\partial\Delta} \frac{d}{d\bar{z}} \left(\frac{f(w)}{w-z} \right) dw \\ &= 0. \end{aligned}$$

□

Proof of Proposition 1.5. We define a 1-form

$$\eta = \frac{f(w)}{w-z} dw.$$

Note that this is not C^∞ at $w = z$, so we cannot apply Stokes' theorem yet. Therefore, we excise a disk $\Delta_\varepsilon = \Delta(z, \varepsilon)$ around z, so by applying Stokes' theorem on $\Delta \backslash \bar{\Delta}_\varepsilon$, then we have a C^∞ 1-form on a set that we may integrate, and we get

$$\begin{aligned} \int_{\Delta\backslash\bar{\Delta}_\varepsilon} d\eta &= \int_{\partial\Delta} \eta - \int_{\partial\Delta_\varepsilon} \eta \\ &= \int_{\Delta} \frac{f}{w-z} dw - \int_{\Delta_\varepsilon} \frac{f}{w-z} dw, \end{aligned}$$

and so

$$-\frac{1}{2\pi i} \int_{\Delta\backslash\bar{\Delta}_\varepsilon} \frac{\frac{\partial f}{\partial \bar{w}} dw \wedge d\bar{w}}{w-z} = \frac{1}{2\pi i} \int_{\Delta\backslash\bar{\Delta}_\varepsilon} \frac{\frac{\partial f}{\partial \bar{w}} d\bar{w} \wedge dw}{w-z} = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f}{w-z} dw - \frac{1}{2\pi i} \int_{\partial\Delta_\varepsilon} \frac{f}{w-z} dw.$$

Let us write $w = z + \varepsilon e^{i\theta}$, then

$$\frac{1}{2\pi i} \int_{\partial\Delta_\varepsilon} \frac{f}{w-z} dw = \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon e^{i\theta}) d\theta,$$

and as $\varepsilon \to 0$, we have

$$2\pi \int_0^{2\pi} \frac{f(z+\varepsilon e^{i\theta})}{w-z} d\theta \to f(z).$$

We also know that $dw \wedge d\bar{w} = -2idx \wedge dy = -2irdr \wedge d\theta$ by taking polar coordinates (r, θ) centered at z, therefore we have an estimation

$$\left| \frac{\frac{\partial f}{\partial \bar{w}} dw \wedge d\bar{w}}{w-z} \right| \leqslant C \left| \frac{rdr \wedge d\theta}{r} \right| = C|dr \wedge d\theta|$$

where C is a bound given by the smoothness therefore boundedness of $\frac{\partial f}{\partial \bar{w}}$ around the point z, therefore the expression is integrable around z, as desired. □

Here are some other results to know.

Theorem 1.6 (Identity Theorem)**.** Let f, g be holomorphic functions on a connected open set U, such that $f \equiv g$ on a non-empty open subset $V \subseteq U$, then $f \equiv g$.

Theorem 1.7 (Maximum Modulus Theorem)**.** Let f be a holomorphic function on an open set U, then $|f|$ attains no maximum value in U.

Lemma 1.8 ($\bar{\partial}$-Poincaré Lemma)**.** Let $g \in C^\infty(\bar{\Delta})$ and $z_0 \in \Delta$, then one can solve the equation $\frac{\partial f}{\partial \bar{z}} = g(z)$ in a smaller disk near z_0 for function $f \in C^\infty(\Delta)$.

Proof. We get to write

$$f(z) = \frac{1}{2\pi i} \int_\Delta \frac{g(w)}{w-z} dw \wedge d\bar{w}.$$

Now take neighborhoods of radius ε and 2ε around z in Δ, then we may choose $g_1, g_2 \in C^\infty(\bar{\Delta})$ so that $g = g_1 + g_2$ and $g_2|_{\Delta_\varepsilon} \equiv 0$ in the ε-disk, and $g_1|_{\Delta \setminus \bar{\Delta}_{2\varepsilon}} \equiv 0$ outside of the (2ε)-disk. Therefore, let us write $f = f_1 + f_2$ where

$$f_j = \frac{1}{2\pi i} \int_\Delta \frac{g_i(w)}{w-z} dw \wedge d\bar{w}.$$

Note that f_2 is well-defined and of C^∞, then we may compute

$$\frac{\partial f_2}{\partial \bar{z}} = \frac{1}{2\pi i} \int_\Delta \frac{\frac{\partial g_2}{\partial \bar{w}}}{w-z} dw \wedge d\bar{w} = 0$$

since the integrand is continuous and of C^∞. Also, since g_1 has compact support, then by changes of coordinates $u = w-z$ and into polar coordinates $u = re^{i\theta}$, we have

$$\begin{aligned} f_1 &= \frac{1}{2\pi i} \int_\Delta \frac{\frac{\partial g_1}{\partial \bar{w}}}{w-z} dw \wedge d\bar{w} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(w)}{w-z} dw \wedge d\bar{w} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(u+z)}{u} du \wedge d\bar{u} \\ &= -\frac{1}{\pi} \int_{\mathbb{C}} g_1(z+re^{i\theta}) e^{-\theta} dr \wedge d\theta \end{aligned}$$

which is C^∞ in z. We may compute

$$\frac{\partial f_1(z)}{\partial \bar{z}} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} g_1(z+re^{i\theta}) e^{-\theta} dr \wedge d\theta$$

$$= \frac{1}{2\pi i} \int_{\Delta} \frac{\frac{\partial}{\partial \bar{w}} g_1(w)}{w - z} dw \wedge d\bar{w}.$$

Since g_1 vanishes on $\partial\Delta$, then by Proposition 1.5, we have

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{\partial}{\partial \bar{z}} f_1(z) = g_1(z) = g(z).$$

□

2 Hartog's Theorem and Weierstrass Preparation Theorem

Let $U \subseteq \mathbb{C}^n$ be an open subset, and recall that

$$\begin{aligned} df &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \right) \\ &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial z_j} dz_j \right) + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \right) \\ &=: \partial f + \bar{\partial} f. \end{aligned}$$

Therefore, Lemma 1.8 says that $\bar{\partial} f = g d\bar{z}$ is locally solvable for $f \in C^\infty(\Delta)$.

Definition 2.1. We denote $\mathcal{O}(U)$ to be the ring of holomorphic functions in open subset $U \subseteq \mathbb{C}^n$.

We now need an analogue of holomorphic functions in $\mathbb{C}^n$.

Definition 2.2. f is holomorphic in $\mathbb{C}^n$ if $\bar{\partial} f = 0$, i.e., $\frac{\partial f}{\partial \bar{z}_j} = 0$ for all j, or that f is holomorphic in each variable separately.

Definition 2.3. f is analytic if for all $z_0 \in U$ in a neighborhood of z_0, we have a partial sum expression that converges to f absolutely and uniformly.

The following results essentially use the same proof as their one-variable analogues.

Proposition 2.4. f is analytic in $\mathbb{C}^n$ if and only if f is holomorphic in $\mathbb{C}^n$.

Proposition 2.5. For a function f holomorphic in open subset $U \subseteq \mathbb{C}^n$, both the maximum modulus principle and the identity theorem hold.

However, Hartog's theorem shows that the behavior in $\mathbb{C}^n$ for $n \geqslant 2$ is very different from that of $\mathbb{C}$.

Theorem 2.6 (Hartog)**.** Let $\Delta(r)$ and $\Delta(r')$ be polydisks (i.e., products of disks) of radius $r > r'$ respectively in $\mathbb{C}^n$ for $n \geqslant 2$. Suppose $f \in \mathcal{O}(\Delta(r) \backslash \bar{\Delta}(r'))$, then f extends uniquely to a holomorphic function $F \in \mathcal{O}(\Delta(r))$.

Proof. Set $z = (z_1, \ldots, z_n)$ and $z' = (z_1, \ldots, z_{n-1})$, then

$$F(z) = F(z', z_n) = \frac{1}{2\pi i} \int_{r' < |w| = r_1 < r} \frac{f(z', w)}{w - z_n} dw$$

is holomorphic in z_n, where z is a point such that $|z_n| < r_1$. We know $f(z', z_n)$ is holomorphic in z_n for $|z_n| < r$. If $|z_n| > r'$, then $F(z', z_n) = f(z)$ by Proposition 1.5. In the case where $|z_n| \leqslant r'$, we know F is holomorphic in both z' and z_n. Therefore, $F \in \mathcal{O}(\Delta(r_1))$ and agrees with f on the domain, therefore they agree on a non-empty open set, namely $\Delta(r_1)$ where $|z_j| > r'$ for some $1 \leqslant j \leqslant n-1$. By the identity theorem, $F \equiv f$ on $\Delta(r_1) \backslash \bar{\Delta}(r')$. Therefore, the function is unique. To complete the proof, take $r_1 \to r$. ☐

Corollary 2.7. Let $K \subseteq \Delta \subseteq \mathbb{C}^n$ be a compact subset, and $f \in \mathcal{O}(\Delta \backslash K)$, then f extends to a function $\mathcal{O}(\Delta)$ uniquely.

Corollary 2.8. Given a function $f \in \mathcal{O}(\Delta \backslash \{0\})$, f extends uniquely to a function $F \in \mathcal{O}(\Delta)$.

Recall that every analytic function has a unique local reprsentation, i.e., given a neighborhood $z_0 \in U \subseteq \mathbb{C}$ and a function $0 \not\equiv f \in \mathcal{O}(U)$, then near z_0, we can write f uniquely as $f(z) = (z - z_0)^n g(z)$ such that $g(z_0) \neq 0$. The analogue for holomorphic functions is the Weierstrass polynomials. Let us denote the coordinate of $\mathbb{C}^n$ to be (z, w) where $z \in \mathbb{C}^{n-1}$.

Definition 2.9. A Weierstrass polynomial in w of degree d is a function $w^d + a_1(z) w^{d-1} + \cdots + a_{d-1}(z) w + a_d(z)$ where $a_i(z)$'s are holomorphic in a neighborhood of 0, and $a_i = 0$ for all i.

Definition 2.10. Let f be a function that is holomorphic in a neighborhood of $0 \in \mathbb{C}^n$ for $n > 1$. We say f is regular in w if $f(0, w) \neq 0$.

Theorem 2.11 (Weierstrass Preparation Theorem)**.** Suppose f is holomorphic near 0 and regular in w, then there exists a holomorphic function $h(z,w)$ near 0 with $h(0) \neq 0$ and a Weierstrass polynomial $g(z,w)$ in w such that $f = gh$. Moreover, such decomposition is unique.

Remark 2.12. Near 0, note that the sets $\{f=0\} = \{g=0\}$ are equal.

Proof. Let us write $f(0,w)$ as a power series of the form $cw^d + O(w^{d+1})$. Moreover, there exists some r such that $|f(0,w)| \geqslant \delta$ for $|w| = r$, such that the only zero of $f(0,w)$ in $\Delta(r)$ is $w = 0$. By compactness in $|z'| < \varepsilon$, the set $|f(z',w)| \geqslant \frac{\delta}{2}$, then by taking small $\varepsilon > 0$, for fixed z' in the range, $f(z',w)$ has exactly d roots when we count with multiplicity.[2] Let us write the roots as $b_1(z'), \dots, b_d(z')$, then we claim that

$$g(z',w) = \prod_{i=1}^{d}(w - b_i(z'))$$

is a Weierstrass polynomial. By construction, $g(z',w) = 0$ if and only if $f(z',w) = 0$ for $|z'| < \varepsilon$ and $|w| < r$. Using the elementary symmetric functions

$$\sigma_j(b_k(z')) = \prod_{i_1 < \cdots < i_j} b_{i_j}$$

we rewrite

$$g(z',w) = w^d + \sum (-1)^i \sigma_i(b_j(z')) w^{d-i}.$$

Claim 2.13. We claim that $\sigma_i(b_i(z'))$ is holomorphic, and equals to 0 at $z' = 0$.

Subproof. Since $b_j(0) = 0$, then the function is zero at $z' = 0$. By a version of implicit function theorem, it is clear (but messy) that this is holomorphic. Instead, we apply Cauchy's formula and calculate the countour integral

$$s_j := \sum_{k=1}^{d} b_k^j = \frac{1}{2\pi i} \int\limits_{|w|=r} \frac{w^j \frac{\partial f(z',w)}{\partial w}}{f(z',w)} dw$$

is holomorphic in z' since $f(z',w) \neq 0$ everywhere in the specified domain. Finally, note that the symmetric functions s_i's and σ_j's are the same up to a change of basis, since they both give rise to a basis, i.e., $\sigma_1 = s_1$, $\sigma_2 = s_1^2 - 2s_2$, and so on, therefore σ_j is holomorphic in z'. ■

Finally, we find that $h(z',w) = \frac{f(z',w)}{g(z',w)}$ which has removable singularity of dimension 1, therefore it is well-defined for fixed z' and holomorphic in w. Moreover, it is holomorphic in z' as we write down

$$h(z',w) = \frac{1}{2\pi i} \int\limits_{|w|=r} \frac{h(z',u)}{u-w} du$$

we may differentiate. □

[2] Indeed, write down the power series expansion of $f(z')w$ and convince oneself that the terms of order higher than d would not matter.

3 Weierstrass Theorems and Corollaries

Corollary 3.1 (Riemann Expansion Theorem)**.** Let $f \not\equiv 0$ be holomorphic in a disk Δ and $g \in \mathcal{O}(\Delta \backslash \{z : f(z)0\})$ is holomorphic on Δ outside of the zeros, and that g is bounded, then g extends uniquely to $\tilde{g} \in \mathcal{O}(\Delta)$.

Proof. The uniqueness is clear by the identity theorem. For existence, without loss of generality, we change the coordinates so that f is regular in w near z_0, which is a point such that $f(z_0) = 0$. Apply the Riemann Expansion theorem in one variable w, we note the function is holomorphic on the variables by Cauchy's formula. □

Theorem 3.2 (Weierstrass Division Theorem)**.** Let f be holomorphic on a polydisk $\Delta(k)$ (any open set would work as well) that is regular in w at 0, and g is a Weierstrass polynomial of degree d in w (and should be shrunk whenever necessary). One can write $f = gh + r$ where h, r are holomorphic near 0, and r is a polynomial in w of degree less than d.

Proof. Write down h locally as

$$h(z', w) = \frac{1}{2\pi i} \int\limits_{|w|=k} \frac{f(z', u)}{g(z', u)} \frac{du}{u - w}$$

and then we get to bound g away from the origin for a small enough neighborhood. The contour integral is well-defined and therefore h is holomorphic by the usual arguments. We also define another holomorphic function

$$r(z', w) = f(z', w) - g(z', w)h(z', w),$$

so it suffices to show that r is a polynomial in w of degree less than d, which can be done by writing the expression above as a contour integral. □

Corollary 3.3 (Weak Nullstellensatz)**.** Suppose f is holomorphci in a neighborhood of 0 and is irreducible[3], and let h be holomorphic in a neighborhood of 0. Suppose the zeros of f are also the zeros of h, then there exists a holomorphic function g such that $h = gf$.

Remark 3.4. Suppose f is not irreducible, then Corollary 3.3 would not hold: take $f = w^3$ and $h = w$.

Proof. Without loss of generality, assume that f is a Weierstrass polynomial regular in w: locally, the Weierstrass polynomials have the same zeros as the original functions, as $h(0) \neq 0$ if and only if h is a unit in the UFD. Therefore, we have $h = fg + r$ for $\deg_w(r) < \deg_w(f) = d$. Note that f has d roots in w when z' is small, but h has the same d roots, therefore r also has those d roots. This forces $r = 0$ by the degree argument. □

Remark 3.5. The proof given above is incomplete: see the textbook for the omitted discriminant argument.

Definition 3.6. Let $U \subseteq \mathbb{C}^n$ be an open subset, then we say $V \subseteq U$ is an analytic variety in U if for all $z_0 \in U$, there exists some neighborhood $z_0 \in U' \subseteq U$ such that $V \cap U'$ is exactly the set of zeros of finitely many holomorphic functions on U'.

Definition 3.7. Let $V \subseteq \Delta$ in a disk be an analytic variety. We say V is irreducible if $V \neq V_1 \cup V_2$ is not the union of two proper analytic subvarieties of Δ.

Example 3.8. The zero locus $V = \{z : f(z) = 0, f \text{ irreducible}\}$ is an analytic variety that is also irreducible. The larger set $V = \{z : f(z) = 0\}$ is called an analytic hypersurface.

Example 3.9. The set $\{xy = 0\} \subseteq \mathbb{R}^2$ is not irreducible, since it can be written as $\{x = 0\} \cup \{y = 0\}$.

Remark 3.10.

- Let $\mathcal{O}_n$ be the ring of germs of holomorphic functions near 0 in $\mathbb{C}^n$. It is a UFD, a local ring, with maximal ideal $\mathfrak{m} = \{f : f(0) = 0\}$.
- Gauss's lemma: suppose R is a UFD, then the polynomial ring $R[z]$ is a UFD.
- Suppose $f, g \in R[z]$ are relatively prime, then there exists α, β such that $\alpha f + \beta g = r \in R$. We say r is the resultant of f and g.

[3]Note that the rings of germs of holomorphic functions in a neighborhood of 0 gives rise to a UFD.

4 Analytic Varities

Example 4.1. Consider a set of functions $\{f_1, \dots, f_k\} \in \mathcal{O}(U)$, then the set of common zeros $V = \{z \in U_i : f_i(z) = 0\ \forall i\} = Z(\{f_i\})$ is an analytic variety.

Definition 4.2. We say an open subset V contained in the disk Δ is irreducible at $p \in V$ if $V \cap U$ is irreducible for arbitrary small neighborhoods U of p.

Let us study the structures of analytic variety.

Remark 4.3.

a. Suppose $p \in V \subseteq U \subseteq \mathbb{C}^n$ is an analytic variety contained in some open subset U. Locally we may write $V = \bigcup_{i=1}^{n} V_i$ where each V_i is an analytic variety that is irreducible at p, such that $V_i \nsubseteq V_j$. Moreover, such decomposition is unique.

b. Let V be an analytic variety that is irreducible at $p = 0$. Locally, we can choose coordinates $(z_1, \dots, z_n)$, such that for some $k \leqslant n$, the map

$$\begin{aligned} \pi : \mathbb{C}^n &\to \mathbb{C}^k \\ z &\mapsto (z_1, \dots, z_k) \end{aligned}$$

exhibits V as a finite-sheeted cover of $0 \in \Delta \subseteq \mathbb{C}^k$ that is branched over an analytic hypersurface.

c. Suppose $V \subseteq \mathbb{C}^n$ is an analytic variety that is irreducible at $p = 0$, and does not contain $V(z_1, \dots, z_{n-1})$. The image $\pi(V)$ of the projection $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ is an analytic variety in a neighborhood of p.

Example 4.4. Suppose $V = Z(z_1 - z_2 z_3)$. If $z_2 = 0$, then $z_1 = 0$; if $z_2 \neq 0$, then we can solve for z_1 and/or z_3. Therefore, $\pi(V) = \{z_2 \neq 0\} \cup \{(0, 0, 0)\}$ is not an analytic variety. The only issue with this being we have included the origin. So what happened there? We check that $\pi^{-1}(\{0, 0, 0\}) = Z(z_1, z_2) \subseteq \mathbb{C}^3$, which is isomorphic to $\mathbb{C}$, which is not compact.

The example above motivates the proper mapping theorem.

Definition 4.5. A mapping $f : X \to Y$ of topological spaces is proper if for all compact subsets $K \subseteq Y$, $f^{-1}(K)$ is compact.

Therefore, Example 4.4 shows that π is not proper.

Theorem 4.6 (Proper Mapping Theorem). Suppose $f : U \to U'$ is a mapping of open sets in $\mathbb{C}^n$, where $U \supseteq V$ contains an analytic variety. If f is proper, then $f(V) \subseteq U'$ is also an analytic variety.

The proof uses the following result.

Lemma 4.7. Let $f \in \mathcal{O}_n$ be irreducible, then $V = Z(f)$ is irreducible as an analytic variety near 0.

Proof. Suppose $V = V_1 \cup V_2$ is a union of proper analytic varieties. Obviously we should assume $f \not\equiv 0$, then there exists some function $f_1 \in \mathcal{O}_n$ such that $f_1|_{V_1} \equiv 0$. Suppose all such functions f_1's are such that $f_1|_{V_2} \equiv 0$, then $V_2 \subseteq V_1$, contradiction, therefore we may choose f_1 such that $f_1|_{V_2} \not\equiv 0$. Similarly, there exists some function $f_2 \in \mathcal{O}_n$ such that $f_2|_{V_2} \equiv 0$ but $f_2|_{V_1} \not\equiv 0$. This shows that $f_1, f_2 \in \mathcal{O}_n$ are non-zero elements. But $f_1 f_2|_V \equiv 0$, so by Corollary 3.3, $f \mid f_1 f_2$, therefore $f \mid f_1$ or $f \mid f_2$. Without loss of generality, say $f \mid f - 1$, then $V \subseteq V_1$, contradiction. □

Proof Sketch of Remark 4.3.

a. For hypersurface $V = Z(f)$, say we can write $V = \bigcup_{i=1}^{n} V_i$, then there exists choices of irreducible polynomials $f_1, \dots, f_n$ such that $f = f_1 \cdots f_n$ and $V = \bigcup_{i=1}^{n} Z(f_i)$ where each $Z(f_i)$ is irreducible.

b. For irreducible hypersurface $V \subseteq U$, without loss of generality we can say $V = Z(g)$ for some irreducible Weierstrass polynomial $g \in \mathcal{O}_{n-1}[w]$. Therefore, the projection mapping is d-to-1 to an analytic hypersurface. The discriminant $\delta(z) \in \mathcal{O}_{n-1}$ is the resultant of g and $\frac{\partial g}{\partial w}$ so it eliminates w. Therefore, we know π is d-to-1 as a cover away from the branched points.

□

5 Complex Manifolds and Tangent Spaces

Definition 5.1. A complex manifold M of dimension n is a differentiable (C^∞-)manifold of dimension $2n$.

Example 5.2. $\mathbb{P}^1 = \{[z_0 : z_1] \in \mathbb{C}^2 \backslash \{(0,0)\}\}/\sim$ over quotient topology where $\sim$ is the equivalence relation generated by scalar multiplication. It is a complex manifold when we set $U_0 = \{z_0 \neq 0\}$ and $U_1 = \{z_1 \neq 0\}$. We have homeomorphisms φ_0 on U_0 mapping $z_0 \to \frac{z_1}{z_0}$ and φ_1 on U_1 mapping $z_1 \to \frac{z_0}{z_1}$. Finally, $\varphi_0 \varphi_1^{-1}(z) = z^{-1}$ which is homeomorphic since $0 \notin U_0 \cap U_1$. Note that $\mathbb{P}^1$ is the compactification of the complex plane, hence it is homeomorphic to S^2.

Example 5.3. Using the same idea, $\mathbb{P}^n$ is a complex manifold given by open covers $U_j = \{z_j \neq 0\}$ and maps

$$\begin{aligned} \varphi_i : U_i &\to \mathbb{C}^n \\ z &\mapsto \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i}\right). \end{aligned}$$

Example 5.4. Let M be a complex manifold, then any open subset $U \subseteq M$ is also a complex manifold given by the restriction.

Example 5.5. $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}_\tau)$ is a complex surface of genus 1. The manifold structure is given by the identity map as the coordinates.

Definition 5.6. A map $f : M \to \mathbb{C}$ of complex manifold M is holomorphic if $f \circ \varphi_\alpha^{-1} : U_\alpha \to \mathbb{C}$ is holomorphic for all $(U_\alpha, \varphi_\alpha)$ from the manifold structure of M.

A map $f : M \to N$ of complex manifolds is holomorphic if it is holomorphic in all local coordinates.

Example 5.7. The map

$$\begin{aligned} f : \mathbb{P}^1 &\to \mathbb{P}^2 \\ (u, v) &\mapsto (u^2 - v^2, 2uv, u^2 + v^2) \end{aligned}$$

is holomorphic. (Note that this requires the components to be homogeneous polynomials of the same degree.) The restriction $f|_{U_0}(z = \frac{z_1}{z_0}) = (1 - z^2, 2z, z^2 + 1)$. Whenever $z \neq 0$, over the pair (U_1, φ_1), the local coordinates look like $\left(\frac{1-z^2}{2z}, \frac{z^2+1}{2z}\right)$.

More generally, consider $F(z_0, \dots, z_n)$ given by homogeneous polynomial of degree d. The zero set $Z(f) \subseteq \mathbb{P}^n$ is a well-defined set in the projective space. We will see that $Z(F)$ is a complex manifold of dimension-$(n-1)$ if the only solution to $\left\{\frac{\partial F}{\partial z_i} = 0\right\}$ is 0.

Definition 5.8. Let $p \in M$ be a complex manifold with a point. We denote $C^\infty(M)_p$ to be the ring of germs of C^∞-functions near p. (This is a intrinsic property of the manifold without any choice of local coordinates.) The real tangent space of M is the set $T_{\mathbb{R},p}(M)$ is the set of derivations $C^\infty(M)_p \to \mathbb{R}$ at p. By Taylor approximation, in coordinates we have a basis $\left\{\left.\frac{\partial}{\partial x_j}\right|_p, \left.\frac{\partial}{\partial y_j}\right|_p\right\}$ of real dimension $2n$. By choosing different coordinates, we have a change of basis matrix given by the Jacobian.

If we complexify, we get a complex tangent space $T_{\mathbb{C},p}M = T_{\mathbb{R},p}M \otimes_\mathbb{R} \mathbb{C}$ with the basis $\left\{\left.\frac{\partial}{\partial z}\right|_p, \left.\frac{\partial}{\partial \bar{z}}\right|_p\right\}$ but over $\mathbb{C}$, i.e., with complex dimension $2n$. Separating the basis, we have a subspace $T'_p M \subseteq T_{\mathbb{C},p}M$ given by $\operatorname{span}(\left\{\frac{\partial}{\partial z_j}\right\})$ which annihilates anti-holomorphic functions, and similarly $T''_p M \subseteq T_{\mathbb{C},p}M$ given by $\operatorname{span}(\left\{\frac{\partial}{\partial \bar{z}_j}\right\})$ which annihilates holomorphic functions.

By setting $z_j = x_j + iy_j$ and $w_j = u_j + iv_j$, then the Jacobian looks like

$$J_\mathbb{R}(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

for $(u, v) = f(x, y)$ and $w = f(z)$. The complex Jacobian $J_{\mathbb{C}}(f)$ is the same matrix but in the complex basis. With respect to the basis of elements $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$, this should just be

$$\begin{pmatrix} \frac{\partial w}{\partial z} & 0 \\ 0 & \frac{\partial w}{\partial \bar{z}} \end{pmatrix} = \begin{pmatrix} \mathfrak{J}(f) & 0 \\ 0 & \bar{\mathfrak{J}}(f) \end{pmatrix}$$

where $\mathfrak{J}(f) = \frac{\partial z}{\partial w}$ is the holomorphic Jacobian. Therefore,

$$\det(J_{\mathbb{C}}(f)) = |\det(\mathfrak{J}(f))|^2 \geqslant 0.$$

More generally, say f is a holomorphic map of complex manifolds of the same dimension, then

- such holomorphic maps preserve orientation, and
- M is orientable, and is canonically oriented.

The choice we make is $\left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge z_n \wedge d\bar{z}_n = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$. The orientability gives global obstructions for differentiable manifolds to obtain complex structure.

Theorem 5.9 (Inverse Function Theorem). Let $F : M \to N$ be a holomorphic function of complex manifolds of the same dimension, with $p \in M$, and that $\det(\mathfrak{J}(F))(p) \neq 0$, then there exists $p \in U \subseteq M$ and $F(p) \in V \subseteq N$ such that $F|_U\, LU \to V$ has a holomorphic inverse.

Proof. Without loss of generality, take $M = N = \mathbb{C}^n$. Since the Jacobian is given in the form of a block matrix, then the real Jacobian is also invertible, so by the inverse function theorem in the C^∞ context, there exists neighborhoods U and V such that there is a C^∞-inverse $G : V \to U$ with $w \in V$ mapped to $z \in U$. We need to show that G is holomorphic as well. Indeed, we know $Z = G(F(Z))$, so we want to take $\frac{\partial}{\partial \bar{z}_j}$. In local coordinates $F = (F_1, \ldots, F_n)$ and $G = (G_1, \ldots, G_n)$, so taking the kth component of $Z = G(F(Z))$, we have $z_k = G_k(F(z))$, and by differentiating we have $0 = \sum_{\ell} \frac{\partial G_k}{\partial w_\ell} \frac{\partial F_\ell}{\partial \bar{z}_j} + \frac{\partial G_k}{\partial \bar{w}_\ell} \frac{\partial \bar{F}_\ell}{\partial \bar{z}_j}$. Since F is holomorphic, so the first term is zero. Therefore, we have

$$0 = \frac{\partial G_k}{\partial \bar{w}_\ell} \bar{\mathfrak{J}}(F),$$

and since $\bar{\mathfrak{J}}(F)$ is invertible, then $\frac{\partial G_k}{\partial \bar{w}_\ell} = 0$, hence G_k is holomorphic, and therefore G has to be holomorphic as well. ☐

Theorem 5.10 (Implicit Function Theorem). Given $p \in U \subseteq \mathbb{C}^n$, and let $V \subseteq U$ be an analytic hypersurface as the zeros of a collection $\{f_j\}_{1 \leqslant j \leqslant k}$ of functions $f_j \in \mathcal{O}(U)$. Suppose the rank of the Jacobian matrix $\mathfrak{J}(f)(p)$ evaluated at p is $k \leqslant n$, then after an explicit coordinate change with $\det\left(\left(\frac{\partial f}{\partial z_j}\right)_{1 \leqslant j \leqslant k}\right)(p) \neq 0$, there exists germs $g_1, \ldots, g_k \in \mathcal{O}_{n-k}$ such that $\{f_j(z_1, \ldots, z_n) = 0\ \forall j\}$ near p if and only if $Z_i = g_i(Z_{k+1, \ldots, Z_n})$.

Proof. Note that g_i exists in C^∞-functions at p. One can show that it is holomorphic, c.f., the textbook. ☐

Definition 5.11. A subset $S \subseteq M$ of a complex manifold M of dimension n is a submanifold of M if, equivalently,

a. $\forall p \in S$, there exists a neighborhood $p \in U \subseteq M$ with holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(U)$ such that the rank of $J(f_1, \ldots, f_k)_p$ is k, and $Z(f_1, \ldots, f_k) = S \cap U$;

b. $\forall p \in S$, there exists neighborhoods $p \in U \subseteq S$ and $V \subseteq \mathbb{C}^{n-k}$ with $g_1, \ldots, g_k \in \mathcal{O}(V)$, such that $U = g(V)$ for $g = (g_1, \ldots, g_k)$.

Remark 5.12. S is a complex manifold of dimension $n - k$, where g_j^{-1}'s are local coordinates.

Definition 5.13. A subset $V \subseteq M$ is an analytic subvariety of M if for every $p \in M$, there exists neighborhood $p \in U \subseteq M$ and $f_1, \ldots, f_k \in \mathcal{O}(U)$ such that $Z(f) = V \cap U$.

Set $V^* \subseteq V$ to be the set of points $p \in V$ such that V is a complex manifold near p, then its complement $V_S = V \backslash V^*$ is the singular locus, the set of singular points of V.

Remark 5.14. V^* is a complex manifold.

Theorem 5.15. V_S is a proper analytic subvariety of M, and $V_S \subsetneq V$, i.e., V^* is non-empty. In fact, V_S is contained in some analytic variety that is a proper subset of V.

The textbook proves that $V_S \subseteq W \subsetneq V$ is contained in a proper analytic subvariety W.

Definition 5.16. We say a variety $V \subseteq M$ is an analytic hypersurface if locally $V = Z(f)$ locally for $f \in \mathcal{O}(U)$.

Conventionally, we want f to be square-free.

Definition 5.17. If V is irreducible, then V^* is a complex manifold, and $\dim(V) := \dim(V^*)$.

Proposition 5.18. A variety V is irreducible if and only if V^* is connected.

Proof of (⇐). Suppose that $V = V_1 \cup V_2$ is a union of two proper subvarieties. (Without loss of generality, we assume that V_1 and V_2 do not contain the same irreducible component of V.) Near a point $p \in V_1 \cap V_2$, it is locally (homeomorphic to) two Euclidean spaces with respect to both directions (or worse[4]),[5] but their union would not be a complex manifold, therefore $V_1 \cap V_2 \subseteq V_S$. □

Definition 5.19. Given a point $p \in V$ of an irreducible analytic variety, the multiplicity $\mathrm{mult}_p(V)$ of V at p measures the behavior of the singularity. Locally, this is defined at $p \in V$ as the $(d-1)$-covering $V \to \Delta_k$ to be multiplicity d. Moreover, if V is a hypersurface $V = Z(f)$, where we assume $f \in \mathcal{O}_p$ to be square-free, then $\mathrm{mult}_p(V) = \mathrm{ord}_p(f)$ is the degree of vanishing of f.

Remark 5.20. $\mathrm{mult}_p(V) = 1$ if and only if $p \in V^*$.

[4]In that case, we have branched covers over the Euclidean disks, then what we need is that the union quotient by intersection is not locally a disk.
[5]There cannot be an open neighborhood that is contained in the intersection, according to the identity theorem and the connectedness.

6 Cohomology

Let M be a complex manifold, and set $A^p(M;\mathbb{R})$ be the set of $\mathbb{R}$-valued differential p-forms, and $Z^p(M;\mathbb{R}) \subseteq A^p(M;\mathbb{R})$ is the set of closed p-forms, i.e., p-forms $w \in A^p$ such that $dw = 0$. Therefore, $dA^{p-1}(M;\mathbb{R}) \subseteq Z^p(M;\mathbb{R})$. Therefore, $d^2 = 0$. (Eventually, everything here should be thought of as C^∞, but it is not necessary.)

In the C^∞ case, given local coordinates $(x_1, \dots, x_k)$, we can write $w = \sum\limits_{|I|=p} f_I(x) dx_I$ where $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

Definition 6.1. The de Rham cohomology is defined by $H^p_{\text{dR}}(M;\mathbb{R}) = Z^p/dA^{p-1}$.

Remark 6.2. As long as M has finite dimension, then the de Rham cohomology $H^p_{\text{dR}}(M;\mathbb{R})$ coincides with the cohomology $H^p(M;\mathbb{R})$.

Remark 6.3. We can upgrade this to the complex-valued differential forms, where all constructions are analogous, i.e., the closed forms over the exact forms. In particular,

$$H^p_{\text{dR}}(M) = Z^p(M)/dA^{p-1}(M) \cong H^p_{\text{dR}}(M;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Suppose M is complex and of C^∞. The k-forms can be described as $A^k(M) = \{w : \forall p \in M, w(p) \in \bigwedge^k T^*_{\mathbb{C},p} M\}$. Therefore, the tangent space splits into a holomorphic part and a non-holomorphic part.

$$T^*_{\mathbb{C},p} M = T'^*_p M \oplus T''^*_p M$$

with respect to dz_i's and $d\bar{z}_i$'s respectively. Now we have a decomposition of exterior powers

$$\Lambda^k T^*_{\mathbb{C},p} M = \bigoplus_{p+q=k} \bigwedge^p T'^*_p M \otimes \bigwedge^q T''_p M.$$

We may then define

$$A^{p,q}(M) = \{w \in A^{p+q}(M) : \forall p \text{ such that } w(p) \in \bigwedge^p T'_p \otimes \bigwedge^q T''_p\}.$$

In the coordinates of z_j's, this means

$$w = \sum_{\substack{|I|=p \\ |J|=q}} f_{IJ}(z) dz_I \wedge d\bar{z}_J.$$

This gives projections

$$\pi^{p,q} : A^{p+q}(M) \to A^{p,q}(M)$$

that is independent of the choice of coordinates. Since $d : A^k(M) \to A^{k+1}(M)$, it can be restricted to $d : A^{p,q}(M) \to A^{p+q+1}(M)$. Note that it is well-defined and compatible, therefore we have globally defined functions

$$\partial = \prod^{p+1,q} \circ d : A^{p,q}(M) \to A^{p+1,q}(M)$$

and

$$\bar{\partial} = \prod^{p,q+1} \circ d : A^{p,q}(M) \to A^{p,q+1}(M).$$

Therefore, $d = \partial + \bar{\partial}$. Since $d^2 = 0$, by expansion we see that each term has to be zero after we grouped them by degrees. That is, $\partial^2 = \bar{\partial}^2 = 0$, and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. Therefore, there is now a notion of

$$Z^{p,q}_{\bar{\partial}}(M) = \{w \in A^{p,q}(M) : \bar{\partial} w = 0\}$$

such that

$$\bar{\partial} A^{p,q-1}(M) \subseteq Z^{p,q}_{\bar{\partial}}(M).$$

The Dolbeault cohomology is therefore defined by

$$H^{p,q}_{\bar{\partial}}(M) = Z^{p,q}_{\bar{\partial}}(M)/\bar{\partial} A^{p,q-1}(M).$$

This is not a topological invariant anymore, i.e., it depends on the complex manifold structure.

Suppose $f : M \to N$ is a holomorphic map of complex manifolds, then $f_* : T'_p M \to T'_{f(p)}(M)$ is defined on holomorphic tangent spaces, and similarly $f^* : T'^*_{f(p)} N \to T'^*_p M$. Therefore, this induces a pullback $f^* : A^{p,q}(N) \to A^{p,q}(M)$. By calculation, it satisfies $\bar{\partial} f^* = f^* \bar{\partial}$. Similarly, there is $f^* : Z^{p,q}_{\bar{\partial}}(N) \to Z^{p,q}_{\bar{\partial}}(M)$ which induces a pullback

$$f^* : H^{p,q}_{\bar{\partial}}(N) \to H^{p,q}_{\bar{\partial}}(M)$$

of Dolbeault cohomology.

Lemma 6.4 ($\bar{\partial}$-Poincaré Lemma)**.** Let Δ be a polydisk, then $H^{p,q}_{\bar{\partial}}(\Delta) = 0$ for all $q \geqslant 1$.

Remark 6.5. We have seen this for $p + q = n = 1$. The set $Z^{0,1}_{\bar{\partial}}(\Delta)$ is exactly the $\bar{\partial}$-closed forms. This is exactly the exact forms.

Proof. Take $s < r$. We should prove this for restriction to $\Delta(s)$, and use the limiting process $s_1 \to r$ as outlined in the textbook. Now we reduce to the case where $p = 0$. For $\varphi \in A^{p,q}(\Delta(s))$, we can write $\varphi = \sum_{|I|=p} dz_I \wedge \varphi_I$, where

$$\varphi_I = \sum_{|J|=q} \varphi_{IJ}(z) d\bar{z}_j \in A^{0,q}(\Delta),$$

therefore

$$\varphi = \sum_{\substack{|I|=p \\ |J|=q}} \varphi_{IJ}(z) dz_I \wedge d\bar{z}_J.$$

Similarly, if $\eta = \sum dz_I \wedge \eta_I$ for $\eta_I \in A^{0,q-1}(\Delta)$, then

$$\bar{\partial}\eta = \sum (-1)^p dz_I \wedge \bar{\partial}\eta_I$$

which is φ if and only if $\bar{\partial}\eta_I = \varphi_I$ for all I.

We may now proceed by induction. Assume $\varphi \in A^{0,q}(\Delta)$ with $\bar{\partial}\varphi = 0$, and note that this involves $d\bar{z}_1, \ldots, d\bar{z}_k$ but not $d\bar{z}_{k+1}, \ldots, d\bar{z}_n$. We claim that there exists $\eta \in A^{0,q-1}(\Delta)$ such that $\varphi - \bar{\partial}\eta$ involves $d\bar{z}_1, \ldots, d\bar{z}_{k-1}$ but not $d\bar{z}_k, \ldots, d\bar{z}_n$. It suffices to prove the $\bar{\partial}$-Poincaré lemma eventually for $\varphi - \bar{\partial}(\eta + \eta' + \cdots) = 0$. Now $\varphi = \sum_{|I|=q} \Phi_I d\bar{z}_I$, now either $k \in I$ or $k \notin I$, so set $\varphi = \varphi_1 + \varphi_2$, where

$$\varphi_1 = \sum_{\substack{|I|=q \\ k \in I}} \varphi_I d\bar{z}_I,$$

and

$$\varphi_2 = \sum_{\substack{|I|=q \\ k \in I}} \varphi_I d\bar{z}_I \wedge \sum \varphi_I dz_{I \setminus \{k\}} = \sum_{\substack{|I|=:q \\ k \in I}} \varphi_I d\bar{z}_I \wedge \varphi'_2.$$

Since the form is $\bar{\partial}$-closed, then $\bar{\partial}\varphi = 0$. Therefore, $\bar{\partial}\varphi_1 + \bar{\partial}\varphi'_2 \wedge d\bar{z}_k = 0$, but the first term does not contain $d\bar{z}_k \wedge d\bar{z}_\ell$ for any $\ell > k$, therefore for any I such that $k \in I$, we have $\frac{\partial \varphi_I}{\partial \bar{z}_\ell} = 0$. Now we get

$$\eta_I = \frac{1}{2\pi i} \int_{|w|=s} \varphi_I(z_1, \ldots, w_k, \ldots, z_n) \frac{dw_k \wedge d\bar{w}_k}{w_k - z_k},$$

hence

$$\frac{\partial}{\partial \bar{z}_k} \eta_I = \varphi_I$$

and for $\ell > k$, we know

$$\frac{\partial}{\partial \bar{z}_\ell} \eta_I = 0.$$

Therefore, $\varphi - \bar{\partial}\eta$ involves $d\bar{z}_1, \ldots, d\bar{z}_{k-1}$, as desired. □

7 Hermitian Metric

Let M be a complex manifold.

Definition 7.1. A Hermitian metric on M is, for all $p \in M$, a Hermitian inner product

$$\langle -, - \rangle : T'_p M \times \overline{T'_p M} \to \mathbb{C}$$

that is positive definite and varies in a C^∞-manner on p. That is, locally, we can choose $\{dz_i\}$ as a basis for $T'^*_p M$, therefore $ds^2 = \sum\limits_{i,j} h_{ij}(z) z_i \otimes d\bar{z}_j$, where each $h_{ij}(z)$ is positive definite Hermitian matrix, and h_{ij} is C^∞.

Remark 7.2. $\overline{\langle X, Y \rangle} = \langle Y, X \rangle$.

Now set $g = \mathrm{Re}(\langle -, - \rangle)$, then $g(X,Y) = g(Y,X)$ and it is positive definite. Therefore, we have a Riemannian metric

$$g : T'_p M \times T'_p M = T_{\mathbb{R},p} M \times T_{\mathbb{R},p} M \to \mathbb{R}.$$

As for $w = -\frac{1}{2}\,\mathrm{Im}(ds^2)$, we have an anti-symmetric relation $w(\langle X, Y \rangle) = -w(\langle Y, X \rangle)$, and therefore w is a 2-form.

Example 7.3. Let $\mathbb{C}^n$ be equipped with the Euclidean metric, then

$$\begin{aligned} ds^2 &= \sum dz_i \otimes d\bar{z}_j \\ &= \sum (dx_i + i dy_i) \otimes (dx_j - i dy_j) \\ &= \sum (dx_i \otimes dx_j + dy_i \otimes dy_j) + i \sum (dy_i \otimes dx_j - dx_i \otimes dy_j) \end{aligned}$$

where the first term corresponds to g, and is the Euclidean Riemannian metric in this case; the second term corresponds to $w = \sum dx_i \wedge dy_i = \frac{i}{2} \sum dz_i \wedge d\bar{z}_j$, and is a $(1,1)$-form.

Definition 7.4. A coframe for M on an open set U is a collection $\varphi_1, \ldots, \varphi_n$ of C^∞-$(1,0)$ form so that for all $p \in U$, $\varphi_1(p), \ldots, \varphi_n(p)$ are a basis for $T'^*_p M$. For $\varphi_j = \alpha_j + i\beta_j$, we can apply Gram-Schmidt process that gives ds^2 the coframe $\varphi_1, \ldots, \varphi_n$ such that $ds^2 = \sum \varphi_i \otimes \bar{\varphi}_i$ as $(1,1)$-form. Therefore, $g = \sum (\alpha_i \otimes \alpha_i) + (\beta_i \otimes \beta_i)$, and $w = \frac{i}{2} \sum \varphi_i \wedge \bar{\varphi}_i$.

Example 7.5. If we consider the lattice $\Lambda \cong \mathbb{Z}^{2n} \hookrightarrow \mathbb{C}^n$, we can take a compact manifold $M = \mathbb{C}^n / \Lambda$ that is homeomorphic to $(S^1)^{2n}$. Now $\varphi_j = dz_j$ is a global $(1,0)$-form. This gives global Hermitian metric $ds^2 = \sum dz_j \otimes d\bar{z}_j$ as an Euclidean metric. For ds^2, we choose coordinates $\sum h_{ij}(z) dz_i \otimes d\bar{z}_j$, then

$$w = \frac{1}{2} \sum h_{ij}(z) dz_i \wedge d\bar{z}_j.$$

Remark 7.6. One can actually retrieve ds^2 from a construction of w. Therefore, these information are in correspondence.

Definition 7.7. We say $w \in A^{1,1}(M)$ is positive if writing $w = \frac{1}{2} \sum h_{ij}(z) dz_i \wedge d\bar{z}_j$ locally gives a positive definite Hermitian matrix $h_{ij}(z)$ for all choices i, j.

Definition 7.8. In the Riemannian sense, we define the volume form as

$$\frac{1}{n!} w^{\wedge n} = \frac{1}{n!} w^n \in A^{n,n}(M).$$

Now $\mathrm{vol}(M) = \frac{1}{n!} \int\limits_M \omega^n$. For any submanifold $S \subseteq M$ of dimension n that contains p, there is an inclusion

$$T'_p(S) \hookrightarrow T'_p(M)$$

which gives a restriction of positive-definite Hermitian metric to $T'_p(S)$.

Theorem 7.9 (Wirtingen)**.** Given a $(1,1)$-form ω_M, we may compute the $(1,1)$-form ω_S of S, by restriction $\omega_M|_S = \omega_S$, then

$$\mathrm{vol}(S) = \frac{1}{n!} \int\limits_S \omega^k.$$

Proof. We can prove that $\omega_M|_S = \omega_S$ by performing Gram-Schmidt on T'_pS first. Take the coframes $\varphi_{k+1}, \dots, \varphi_n$ that vanishes on T'_pS, then we can extend it to $\varphi_1, \dots, \varphi_n$ on T'_pM. Now

$$\omega = \frac{i}{2}\sum_{i=1}^{n} \varphi_i \wedge \bar{\varphi}_i,$$

then $\omega|_S = \frac{i}{2}\sum_{i=1}^{k} \varphi_i \wedge \bar{\varphi}_i = \omega_S$. □

Example 7.10. Consider the Fubini-Study metric, a Hermitian metric on $\mathbb{P}^n$. We may construct $(1,1)$-forms, check that they are positive, then changing wedges to tensors gives a Hermitian metric. To construct these forms locally, let $U \subseteq \mathbb{P}^n$ be an open subset, then we have $\pi : \mathbb{C}^{n-1}\backslash\{0\} \to \mathbb{P}^n$, and we can find a section z with $\pi z = 0$, that restricts to $z^{\text{hol}} : U \to \mathbb{C}^{n+1}\backslash\{0\}$. Now $\omega_U = \frac{i}{2\pi}\partial\bar{\partial}\log(||z||^2)$. To see that this is a real operator, note that $\partial\bar{\partial}$ is an imaginary operator. Let $U_0 \cong \mathbb{C}^n$ be the chart with $w_0 = 0$, then

$$\begin{aligned}\omega_{U_0} &= \frac{i}{2\pi}\partial\bar{\partial}\log(1 + \sum_{i=1}^{n}|w_i|^2)\\ &= \frac{i}{2\pi}\partial\left(\frac{\sum w_i d\bar{w}_i}{(1+||w||^2)}\right)\\ &= \frac{i}{2\pi}\frac{(1+||w||^2)\sum dw_i \wedge d\bar{w}_i - (\sum w_i d\bar{w}_i)(\sum \bar{w}_i dw_i)}{(1+||w||^2)^2}.\end{aligned}$$

At $w = 0 = (1, 0, \dots, 0)$, this is $\frac{i}{2\pi}\sum dw_i \wedge d\bar{w}_i$, so this is a positive $(1,1)$-form in $A^{1,1}(U_0)$.

Let us pick a different section $\tilde{Z}$ on U, then by construction $\tilde{Z} = fZ$ where $f \in \mathcal{O}(U)$. Now

$$\begin{aligned}\frac{i}{2\pi}\partial\bar{\partial}\log||\tilde{Z}||^2 &= \frac{i}{2\pi}\partial\bar{\partial}(\log||Z|| + \log(f) + \log(\bar{f}))\\ &= \frac{i}{2\pi}(\partial\bar{\partial}\log||Z||^2 + \partial\bar{\partial}\log(\bar{f}) - \bar{\partial}\partial\log(\bar{f}))\\ &= \frac{i}{2\pi}\partial\bar{\partial}\log||Z||^2,\end{aligned}$$

which is independent of Z. Hence, this defines a global $(1,1)$-form in $A^{1,1}(\mathbb{P}^n)$.

For $A \in \mathrm{GL}(n+1)$, we have φ_A as an isomorphism on $\mathbb{P}^n$ by left multiplication. However, this does not preserve the norm. Instead, we require $U(n+1) \subseteq \mathrm{GL}(n+1)$.

For $A \in U(n+1)$, we note $\varphi_A^*\omega = \omega$ since φ_A lifts to $\mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}^{n+1}\backslash\{0\}$ which preserves $||Z||^2$. Now $U(n+1)$ acts on $\mathbb{P}^n$ transitively. For $(p_0, \dots, p_n) \in \mathbb{P}^n$, we choose $A \in U(n+1)$ so that $\varphi_A(1, 0, \dots, 0) = (p_0, \dots, p_n) = p$. Therefore, $\omega_p = (\varphi_A^*\omega_p) = \omega_{p_0}$ is positive.

Let $V \subseteq M$ be an analytic subvariety and let $\varphi \in A^*(M)$. For $k = \dim(V)$, we may define $\int_V \varphi = \int_{V^*} \varphi$ as a complex manifold. Therefore,

$$\mathrm{vol}(V) = \frac{1}{k!}\int_V \omega^k = \frac{1}{k!}\int_{V^*} \omega^k = \mathrm{vol}(V^*).$$

Note that this may not be a finite volume.

Proposition 7.11. V^* has finite volume in compact neighborhoods.

Proof. Assume $V^* \subseteq \Delta_n$ inside a disk, then locally we have a projection $\pi : V^* \to \Delta_k$ as a d-sheeted branch cover. We then want to show that the volume form of any metric on $\bar{\Delta}_k$ is bounded above by some constant multiplied by the Euclidean volume. Without loss of generality, we reduce to the case of a Euclidean metric, so the volume of V^* is at most $Cd\,\mathrm{vol}(\bar{\Delta}_k)$, which is finite. □

This is the argument from the book. However, it is incomplete: we cannot really use compactness here since we made changes of variables, which may allow the volume to blow-up. Let us take a detour in hypersurfaces. Set $k = n-1$, then consider a Weierstrass polynomial say $w^2 - 2wz + \sin(z) = 0$. By solving for w, we note that it has some power series expansion, then w is a Puiseux series. In general, there is a Puiseux series for any Weierstrass polynomial, and this should help with the proof above.

Corollary 7.12. Suppose $\varphi \in A^{2k}(\bar{U})$ where $\bar{U}$ is the compact closure of U, then $|\int_{V^* \cap U} \varphi| < \infty$.

Corollary 7.13 (Stokes)**.** Given compactly-supported $\varphi \in A^*(M)$, then $\int_V d\varphi = 0$.

Example 7.14. Given a tubular neighborhood T_ε of V_s of a manifold V, we find

$$\begin{aligned}\int_V d\varphi = \int_{V \setminus V_s} d\varphi &= \lim_{V \setminus \bar{T}_\varepsilon} \int d\varphi \\ &= \lim \int_{\partial T_\varepsilon} \varphi \\ &\to 0.\end{aligned}$$

8 Presheaves and Sheaves

Definition 8.1. Let X be a topological space. We define a presheaf $\mathcal{F}$ of abelian groups on X such that 1) every open subsets $U \subseteq X$ is assigned to an abelian group $\mathcal{F}(U)$, and that 2) for any subset relation $V \subseteq U$, there is an associated restriction $r_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ as a homomorphism of abelian groups such that $r_{U,U} = \mathrm{id}_U$, and for any nested inclusion $W \subseteq V \subseteq U$ of open subsets, we have $r_{V,W} \circ r_{U,V} = r_{U,W}$, and finally 3) $\mathcal{F}(\varnothing) = 0$.

We say $\mathcal{F}(U)$ is the section of $\mathcal{F}$ over U.

Example 8.2. Suppose $E \to V$ is a vector bundle over X. Then we say $E(U)$ is the group of sections of E over U.

Example 8.3. The set $\mathcal{C}_X$ of presheaves of continuous $\mathbb{R}$-valued functions on X is a presheaf.

Example 8.4. The set $\mathbb{Z}^{\mathrm{pre}}$ of constant functions $\mathbb{Z}$ is a presheaf, where it assigns each open subset U a set of constant function $U \to \mathbb{Z}$.

Let X be a complex manifold and $\mathcal{O}_X^\times$ be its set of nowhere vanishing holomorphic functions on X.

Definition 8.5. A presheaf $\mathcal{F}$ is a sheaf is

a. if $U = \bigcup_{\alpha \in I} U_\alpha$, $s_\alpha \in \mathcal{F}(U_\alpha)$, and $r_{U_\alpha, U_\alpha \cap U_\beta} s_\alpha = r_{U_\beta, U_\alpha \cap U_\beta}$ for all α, β, then there exists $s \in \mathcal{F}(U)$ such that $s_\alpha = r_{U,U_\alpha} s$ for all α;

b. if $s \in \mathcal{F}(U)$ and $0 = r_{U,U_\alpha} s \in \mathcal{F}(U_\alpha)$ for all α, then $s = 0$.

We write the shorthand $s|_V = r_{U,V}(s)$. Among the examples above, only the constant presheaf $\mathbb{Z}^{\mathrm{pre}}$ is not sheaf. Suppose we write it as two disconnected subsets so that the presheaf is constant on each of them, then it does not satisfy condition a. above. This means we cannot glue them together as a sheaf. Instead, we write $\mathbb{Z}$ to be the sheaf of locally constant $\mathbb{Z}$-valued functions.

Definition 8.6. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves. A morphism of presheaves $f : \mathcal{F} \to \mathcal{G}$ is a homomorphism $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open subset U that is compatible with restrictions $V \subseteq U$.

Example 8.7. Let M be a complex manifold, and let $\mathcal{O}$ be the sheaf of holomorphic functions. Then $\exp : \mathcal{O} \to \mathcal{O}^\times$ is a morphism of presheaves where $\exp(U) : \mathcal{O}(U) \to \mathcal{O}^\times(U)$ defined by $f \mapsto e^{2\pi i f}$.

Definition 8.8. A morphism of sheaves is a morphism of the underlying presheaves.

Proposition 8.9. For every presheaf $\mathcal{F}$, there exists a unique sheaf $\mathcal{F}^+$ and a morphism $\mathcal{F} \to \mathcal{F}^+$ such that for all sheaves $\mathcal{G}$ and morphisms $\mathcal{F} \to \mathcal{G}$, there exists a unique map $\mathcal{F}^+ \to \mathcal{G}$ such that the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\psi} & \mathcal{F}^+ \\ {\scriptstyle \rho}\big\downarrow & \swarrow & \\ \mathcal{G} & & \end{array}$$

That is, $\rho = \sigma\psi$.

Proof. Uniqueness follows from diagram chasing. There are unique maps $\sigma : \mathcal{F}^+ \to \tilde{\mathcal{F}}^+$ and $\tilde{\sigma} : \tilde{\mathcal{F}}^+ \to \mathcal{F}^+$, then so is the identity map. To prove existence, we build the sheaf as follows. For any open subset U, we define

$$\mathcal{F}^+(U) = \{U = \bigcup_{\alpha \in I} U_\alpha, s_\alpha \in \mathcal{F}(U_\alpha) : s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}}\}.$$

□

Remark 8.10. The direct sum of sheaves is still a sheaf.

Definition 8.11. The kernel of a morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is defined by $(\ker(f))(U) = \ker(\mathcal{F}(U) \to \mathcal{G}(U))$. Therefore, the kernel is a presheaf. Moreover, since $\mathcal{F}$ is a sheaf, then so is the kernel. Hence, the kernel of $\mathcal{F}$ is a subsheaf.

Example 8.12. The kernel $\ker(\exp)$ is defined by $\ker(\exp)(U) = \{f \in \mathcal{O}(U) : e^{2\pi i f} = 1\} = \mathbb{Z}(U)$.

The cokernel is defined similarly, but the difference being, the cokernel of a morphism of sheaves is not a sheaf.

Example 8.13. Consider the cokernel $\operatorname{coker}(\exp)$ of $X = \mathbb{C}^*$. For $z \in \mathcal{O}^\times(\mathbb{C}^*)$, $0 \neq z \in \operatorname{coker}(\exp)^{\text{pre}}(\mathbb{C}^*)$. Now $z|_{U_\alpha} = \exp(2\pi i \ln(z)) = 0$ in the cokernel. Therefore, we have a non-zero element that is zero once we restrict it to an open subset.

Definition 8.14. The cokernel of a morphism of sheaves is the associated sheaf of the cokernel presheaf.

Example 8.15. The cokernel of $\exp$ is 0.

Definition 8.16. We say a morphism of sheaves is surjective if its cokernel is 0.

Definition 8.17. The image sheaf of a morphism of sheaves is the sheafification of the image presheaf, which is also defined locally.

Definition 8.18. Given morphisms

$$\mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{G}$$

of sheaves, we say the sequence is exact at $\mathcal{F}$ if $\operatorname{im}(f) = \ker(g)$. A short exact sequence of sheaves is a sequence that is exact at $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$.

Example 8.19.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \longrightarrow 0$$

is a short exact sequence of complex manifolds. For C^∞-manifold M, there is an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow a_M^0 \xrightarrow{d} a_M^1 \longrightarrow a_M^2 \longrightarrow \cdots$$

of k-forms. Similarly, for any complex manifold M, there exist a sequence

$$0 \longrightarrow \mathcal{O}_M \longrightarrow a_M^{0,0} \xrightarrow{\bar{\partial}^M} a_M^{0,1} \xrightarrow{\bar{\partial}^M} a_M^{0,2} \longrightarrow \cdots$$

Example 8.20. Let $\mathfrak{M}$ be the sheaf of meromorphic functions on M. A section $s \in \mathcal{M}(U)$ is of the form

$$\left\{ \frac{f_\alpha}{g_\alpha} : f_\alpha, g_\alpha \in \mathcal{O}(U_\alpha), g_\alpha \not\equiv 0 \right\}.$$

Example 8.21. There exists an exact sequence associated to the quotient sheaf

$$0 \longrightarrow \mathcal{A} \xrightarrow{i} \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A} \longrightarrow 0$$

given by the cokernel.

Example 8.22. There exists an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{O} \longrightarrow 0$$

where $\mathcal{M}/\mathcal{O}$ is the sheaf of principal parts, that is, the power series with degree at most -1. Given Laurent tails in $\mathcal{M}/\mathcal{O}$, we may ask whether we can find a section back in $\mathcal{M}$, which turns into a question of cohomology.

9 Sheaf Cohomology

To define sheaf cohomology, let us first define Čech cohomology.

Definition 9.1. Let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open cover of X, then we define the Čech cochain complexes as $C^n(\mathcal{U},\mathcal{F}) = \prod_{\alpha_0\neq\cdots\neq\alpha_q} \mathcal{F}(U_{\alpha_0,\dots,\alpha_q})$, with coboundary map

$$\begin{aligned}\delta : C^q(\mathcal{U},\mathcal{F}) &\to C^{q+1}(\mathcal{U},\mathcal{F})\\ \sigma &\mapsto \delta\sigma\end{aligned}$$

where $(\delta\sigma)_{\alpha_0\cdots\alpha_{q+1}} = \sum(-1)^i\ \sigma_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_{q+1}}\big|_{U_{\alpha_0\cdots\alpha_{q+1}}}$. Here $U_{\alpha_0\cdots\alpha_n} = \bigcap_{i=0}^{q} U_{\alpha_i}$.

Exercise 9.2. $\delta^2 = 0$.

For instance, for $\sigma \in C^0$, we know $(\delta^2\sigma)_{\alpha_0\alpha_1\alpha_2} = (\delta(\delta\sigma))_{\alpha_0\alpha_1\alpha_2} = (\delta\sigma)_{\alpha_1\alpha_2} - (\delta\sigma)_{\alpha_0\alpha_2} + (\delta\sigma)_{\alpha_0\alpha_1} = \sigma_{\alpha_2} - \sigma_{\alpha_1} - \sigma_{\alpha_2} + \sigma_{\alpha_1} + \sigma_{\alpha_1} - \sigma_{\alpha_0} = 0$.

Definition 9.3. Čech cohomology is defined as $\check{H}^q(\mathcal{U},\mathcal{F}) = Z^q(\mathcal{U},\mathcal{F})/\delta C^{q-1}(\mathcal{U},\mathcal{F})$ with respect to an open cover.

Example 9.4. Let $X = \mathbb{P}^1$ and $\mathcal{U} = \{U_0, U_1\}$ be the standard cover. Then $C^0(\mathcal{U},\mathcal{F}) = \mathcal{O}(U_0)\times\mathcal{O}(U_1)$ and $C^1(\mathcal{U},\mathcal{F}) = \mathcal{O}(U_{01})$. Let $\delta : \mathcal{O}(U_0)\times\mathcal{O}(U_1) \to \mathcal{O}(U_{01})$. For $z \in U_0$ and $w \in U_1$, we take $w = \frac{1}{z}$, then $\delta(\sigma_0,\sigma_1) = \sigma_1 - \sigma_0|_{U_{01}} = \sigma_0(z) - \sigma_1(\frac{1}{z})$, so we have a difference of two Laurent series expansions that is zero in the cocycle, therefore this forces all coefficients to be zero, therefore $Z^0(\mathcal{U},\mathcal{F}) = \{(C,C) : C \in \mathbb{C}\} \cong \mathbb{C}$, hence $H^0(U,\mathcal{F}) = \mathbb{C}$. Now $Z^1(\mathcal{U},\mathcal{F}) = C^1(\mathcal{U},\mathcal{F})$ in the chain complex

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow 0$$

Definition 9.5. Let $\mathcal{V}$ and $\mathcal{U}$ be open covers. We say $\mathcal{V} \lhd \mathcal{U}$ is a refinement of $\mathcal{U}$ if for all V_j's in $\mathcal{V}$, there exists some index i such that $V_i \subseteq U_j$. Note that this is not a canonical choice.

Given a refinement $\mathcal{V} \lhd \mathcal{U}$ assigning $\rho : J \to I$ on index sets, we want to find a map $\psi : H^q(\mathcal{U},\mathcal{F}) \to H^q(\mathcal{V},\mathcal{F})$. For any $[\sigma] \in H^q(\mathcal{U},\mathcal{F})$, we take $\sigma \in Z^q(\mathcal{U},\mathcal{F})$ with $\delta\sigma = 0$, then say σ is mapped to $\gamma \in Z^q(\mathcal{V},\mathcal{F})$, so we want it to correspond to an assignment $\psi([\sigma]) = [\gamma]$. Let

$$\gamma_{j_0\cdots j_q} = \sigma_{\rho(j_0)\cdots\rho(j_q)}\big|_{V_{j_0\cdots j_q}} \in \mathcal{F}(U_{\rho(j_0)\cdots\rho(j_q)}) \supseteq V_{j_0\cdots j_q}.$$

Exercise 9.6. ψ is independent of choice of σ.

Remark 9.7. If $\delta\sigma = 0$, then σ is skew in indices.

We now have a diagram

$$\begin{array}{ccccc} C^0(\mathcal{U},\mathcal{F}) & \xrightarrow{\delta} & C^1(\mathcal{U},\mathcal{F}) & \xrightarrow{\delta} & C^2(\mathcal{U},\mathcal{F}) \\ \psi\downarrow & & \downarrow\psi & & \downarrow\psi \\ C^0(\mathcal{V},\mathcal{F}) & \xrightarrow[\delta]{} & C^1\mathcal{V},\mathcal{F}) & \xrightarrow[\delta]{} & C^2(\mathcal{V},\mathcal{F}) \end{array}$$

that commutes. Suppose there are two different such mappings ψ_1 and ψ_2, then there exists a chain homotopy $h : C^*(\mathcal{U},\mathcal{F}) \to C^{*-1}(\mathcal{V},\mathcal{F})$ such that $\psi_1 - \psi_2 = h\delta - \delta h$, therefore the two different mappings agree on cohomology.

Definition 9.8. Under the refinement $\lhd$, we have a directed system $\{\mathcal{U}\}$, then the Čech cohomology in general is defined by

$$H^q(X,\mathcal{F}) = \varinjlim_{\{\mathcal{U}\}} H^q(\mathcal{U},\mathcal{F}).$$

Theorem 9.9 (Leray)**.** Given an open cover $\mathcal{U} = \{U\}$, if $\mathcal{F}$ is acyclic with respect to the cover $\mathcal{U}$, i.e., $H^q(U_{i_0\cdots i_p}, \mathcal{F}|_{U_i}) = 0$ for all p and all $q > 0$, then $H^q(\mathcal{U},\mathcal{F}) \to H^q(X,\mathcal{F})$ is an isomorphism.

Example 9.10. For $X = \mathbb{P}^1$ with standard basis $\mathcal{U} = \{U_0, U_1\}$, and let $\mathcal{F} = \mathcal{O}$. We claim that $H^q(U_i, \mathcal{O}) = H^q(U_{01}, \mathcal{O}) = 0$ for $q > 1$ and $i = 0, 1$. In fact, this is just the δ-Poincaré lemma. Hence, $H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C}$ and $H^q(\mathbb{P}^1, \mathcal{O}) = 0$ for $q > 0$.

We should also study the connecting homomorphism $\delta : H^p(X, \mathcal{G}) \to H^{p+1}(X, \mathcal{E})$ in the long exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

One can show that this is independent of the choice of liftings once we do the diagram chasing.

Theorem 9.11 (De Rham)**.** Let M be a C^∞-manifold, then we have an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{a}^0 \xrightarrow{d} \mathfrak{a}^1 \xrightarrow{d} \cdots$$

which factors via closed n-forms Z^n. That is, we have short exact sequences

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{a}^0 \longrightarrow Z^1 \longrightarrow 0$$

and

$$0 \longrightarrow Z^1 \longrightarrow \mathfrak{a}^1 \longrightarrow Z^2 \longrightarrow 0$$

and so on.

Proposition 9.12. $H^q(M, \mathfrak{a}^p) = 0$ for all $q > 0$.

Proof. Use the bump function to turn local properties into global ones. □

Remark 9.13. We have $\mathcal{F}(X) \cong H^0(\mathcal{U}, \mathcal{F})$ defined by $s \mapsto (s_\alpha = s|_{U_\alpha})$. Indeed, $H^0(X, \mathcal{F}) = \varinjlim H^0(U, X) \cong \varinjlim \mathcal{F}(X) \cong \mathcal{F}(X)$.

Remark 9.14. If M is a topological manifold, or more generally, homotopy equivalent to a CW complex, then $\check{H}^q(M, \mathbb{R}) \cong H^q_{\text{Sing}}(M, \mathbb{R})$.

Theorem 9.15. Let M be a C^∞ manifold, then $H^p_{\text{DR}}(M, \mathbb{R}) \cong \check{H}^p(M, \mathbb{R})$.

Proof. Recall that De Rham theorem tells us that the long exact sequence is factored termwise via sheaves of closed p-forms, i.e., we have triangles

$$\begin{array}{ccc} \mathfrak{a}^0 & \xrightarrow{\quad d \quad} & \mathfrak{a}^1 \\ & \searrow \quad \nearrow & \\ & \zeta^1 & \end{array}$$

Therefore,

$$H^p_{\text{DR}}(M, \mathbb{R}) \cong Z^p(M, \mathbb{R})/dA^{p-1}(M, \mathbb{R}) \cong H^0(M, \zeta^p)/dH^0(M, \mathfrak{a}^{p-1}) \cong Z^p/dA^{p-1} \cong H^1(\zeta^{p-1}).$$

For $r = p - 1$, we look at the last short exact sequence induced from the long sequence, which gives

$$H^0(\mathfrak{a}^{p-1}) \cong A^{p-1}(M, \mathbb{R}) \xrightarrow{d} H^0(\zeta^p) \cong Z^p(M, \mathbb{R}) \longrightarrow H^1(\zeta^{p-1}) \longrightarrow H^1(\mathfrak{a}^{p-1}) \cong 0$$

For $r = p - 2$, we check that the sequence

$$H^1(\mathfrak{a}^{p-2}) \cong 0 \longrightarrow H^1(\zeta^{p-1}) \xrightarrow[\cong]{\delta} H^2(\zeta^{p-2}) \longrightarrow H^2(\mathfrak{a}^{p-2}) \cong 0$$

induced by

$$0 \longrightarrow \zeta^{p-2} \longrightarrow \mathfrak{a}^{p-2} \longrightarrow \zeta^{p-1} \longrightarrow 0$$

and so on, until we reach the short exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{a}^0 \xrightarrow{d} \zeta^1 \longrightarrow 0$$

where the induced sequence on cohomology is

$$H^{p-1}(\mathfrak{a}^0) \longrightarrow H^{p-1}(\zeta^1) \xrightarrow[\cong]{\delta} H^p(M,\mathbb{R}) \longrightarrow H^p(\mathfrak{a}^0) \cong 0$$

whenever $p \geqslant 2$, since we have

$$H^p_{\text{dR}}(M) \cong H^1(\zeta^{p-1}) \cong H^2(Z^{p-2}) \cong \cdots \cong H^{p-1}(\zeta^1).$$

When $p = 0$, we have $H^0_{\text{dR}}(M) \cong Z^0(M,\mathbb{R}) \cong \mathbb{R} \cong H^0(M,\mathbb{R})$. When $p = 1$, we have

$$H^0(\mathfrak{a}^0) \longrightarrow H^0(\zeta^1) \longrightarrow H^1(M,\mathbb{R}) \longrightarrow H^1(\mathfrak{a}^0) \cong 0$$

and therefore $H^1(M,\mathbb{R}) \cong Z^1(M,\mathbb{R})/dA^0(M,\mathbb{R}) \cong H^1_{\text{dR}}(M)$. □

Theorem 9.16 (De Rham)**.** There is an isomorphism

$$H^p_{\text{DR}}(M) \to H^p_{\text{Sing}}(M,\mathbb{R})$$
$$[\omega] \mapsto \gamma \mapsto \int_\gamma \omega)$$

where $\omega \in Z^p(M,\mathbb{R})$.

Since $d\omega = 0$, then this induces $d\eta \mapsto \int_{\partial\eta} \omega = \int_\eta d\omega = 0$ where $\gamma = d\eta$ as a cocycle in $C_p(M,\mathbb{R})$.

Theorem 9.17 (Dulbeault)**.** Let M be a complex manifold and let Ω^p be its sheaf of holomorphic p-forms. Consider the long exact sequence

$$0 \longrightarrow \Omega^p \longrightarrow \mathfrak{a}^{p,0} \xrightarrow{\bar\partial} \mathfrak{a}^{p,1} \xrightarrow{\bar\partial} \cdots$$

Then $H^q(M,\Omega^p) \cong Z^{p,q}_{\bar\partial}(M)/\bar\partial A^{p,q-1}(M) \cong H^{p,q}_{\bar\partial}(M)$.

Theorem 9.18. Let M be a Riemann surface, then consider the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{O}$$

where $\mathcal{M}/\mathcal{O}$ is the sheaf of principal parts. Given points p_i's, set $f_i = \sum_{j=-n_i}^{-1} a_i(z-p_i)^j$, then there exists $f \in \mathcal{M}(X)$ such that the principal part of f over p_i is f_i. That is, we want to find some f such that the first map in

$$H^0(\mathcal{M}) \longrightarrow H^0(\mathcal{M}/\mathcal{O}) \xrightarrow{\delta} H^1(\mathcal{M},\mathcal{O})$$

sends f to $\{p_i, f_i\}$ for each index i. Such f exists if and only if the composite evaluated at f is zero in $H^1(\mathcal{M},\mathcal{O})$.

Example 9.19. Let $M = \mathbb{P}^1$, then $H^1(\mathbb{P}^1,\mathcal{M}) = 0$.

Corollary 9.20. For any $q > n = \dim(M)$, then $H^q(M,\Omega^p) = H^n(M,\Omega^p) \cong H^{p,n}_{\bar\partial}(M) = 0$. Here we should assume $p \leqslant n$, since $\Omega^p = 0$ whenever $p > n$.

Corollary 9.21. $H^q(\mathbb{C}^n,\mathcal{O}^\times) = 0$ whenever $q > 0$.

Example 9.22. Note that $H^2(\mathbb{P}^2,\Omega^2) \cong \mathbb{C}$, so the bound above is strict.

Proof. Look at the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

which induces

$$H^q(\mathcal{O}) \cong H^{0,q}_{\overline{\partial}}(\mathbb{C}^n) \cong 0 \longrightarrow H^q(\mathcal{O}^*) \longrightarrow H^{q+1}(\mathbb{Z}) \cong 0$$

since any $\overline{\partial}$-closed q-form is $\overline{\partial}$-exact, therefore $H^{0,q}_{\overline{\partial}}(\mathbb{C}^n)$ is zero, and that $H^{q+1}(\mathbb{Z}) \cong 0$ since it is contractible. This forces $H^q(\mathcal{O}^*) \cong 0$.

For an analytic hypersurface $V \subseteq \mathbb{C}^n$, we have $V = Z(f)$ for some holomorphic function $f \in \mathcal{O}(\mathbb{C}^n)$. Cover $\mathbb{C}^n$ by U_α's by the standard covering, and let $U_\alpha \cap V = Z(f_\alpha)$ for some square-free $f_\alpha \in \mathcal{O}(U_\alpha)$. Let us define $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$: the multiplicity is 1 in each case, so the zeros cancel out. Therefore, the coboundary is $(\delta g)_{\alpha\beta\gamma} = g_{\alpha\beta} g_{\beta\gamma} g_{\alpha\gamma}^{-1} = 1$, thus $[g] \in H^1(\mathbb{C}^n, \mathcal{O}^*) \cong 0$ represents a cohomology class, which forces it to be the trivial class. Thus, $g = \delta h$, so let us write $\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\alpha}{h_\beta}$ where the h_i's are nowhere-vanishing since they are from $\mathcal{O}^*$, therefore the local functions $f_\alpha h_\alpha = f_\beta h_\beta$ agree: over each U_α, we know that $U_\alpha \cap Z(f) = Z(f_\alpha) = Z(f_\alpha h_\alpha)$. We now define the global meromorphic function f using either side of this equality, so that $V = Z(f)$. ☐

Example 9.23. Let us compute $H^*(\mathbb{P}^1, \Omega^1)$ using a Leray cover for Ω. Set $\mathcal{U} = \{U_0, U_1\}$ by the standard covering, then $U_i \cong \mathbb{C}$ for $i = 0, 1$ and $U_{01} \cong \mathbb{C}^*$. By $\overline{\partial}$-Poincaré lemma, we note $H^q(U_i, \Omega^1) = H^{1,q}(\mathbb{C}) = 0$ for all $q > 0$, and therefore $H^q(U_{01}, \Omega^1) = H^{1,q}_{\overline{\partial}}(\mathbb{C}^*) = 0$ for all $q > 0$. Therefore, $H^*(\mathbb{P}^1, \Omega^1) = \check{H}(\mathcal{U}, \Omega^1)$ and we note $C^0 = \Omega^1(U_0) \oplus \Omega^1(U_1)$ and $C^1 = \Omega^1(U_{01})$. Take local coordinate z on $\Omega^1(U_0)$ and $w = \frac{1}{z}$ on $\Omega^1(U_1)$, then we write $f(z)dz = \sum\limits_{n \geqslant 0} a_n z^n dz$ and $g(w)dw = \sum\limits_{m \geqslant 0} b_m w^m dw$. Now $\delta(f, g) = (g - f)|_{U_{01}} = (\sum\limits_{n \geqslant 0} a_n w^{-n-2} + \sum\limits_{m \geqslant 0} b_m w^m) dw$. Therefore $H^1(\mathcal{U}, \Omega^1) = \mathbb{C}[\frac{1}{w}] \cong \mathbb{C}$ and $H^0(\mathcal{U}, \Omega^1) = 0$.

More generally, we can compute

$$H^q(\mathbb{P}^n, \Omega^p) = \begin{cases} \mathbb{C}, & 0 \leqslant p = q \leqslant n \\ 0, & \text{otherwise} \end{cases}$$

The interesting case is when $p = q = n$. Take the standard cover $\mathcal{U} = (U_0, \ldots, U_n)$ and the local projective coordinates $z_i = \frac{x_i}{x_0}$ for $i = 1, \ldots, n$, then $w = \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n} \in C^n(\mathcal{U}, \Omega^n)$ such that $\delta w = 0$. Thus, w represents a non-zero cohomology class in $H^n(\mathbb{P}^n, \Omega^n)$, so this generates the 1-dimensional vector space.

10 Intersection of Cycles on Manifolds

Let M be a compact oriented manifold, then we have singular homology $H_*(M,\mathbb{Z})$. If M is of C^∞ and of dimension n, then we have piecewise smooth cycles.

Definition 10.1. Suppose M is of C^∞, and suppose A is a k-cycle and B is a $(n-k)$-cycle, then we say A and B intersect transversally at $p \in M$ if $A \cap B = \{p\}$ in a neighborhood of p, and that $T_pA \oplus T_pB = T_pM$. If A and B intersect transversally (at every point), then we define the intersection number $\#(A \cdot B) \in \mathbb{Z}$ at p as

$$\#(A\cdot B)_p = \begin{cases} 1, & T_pA \oplus T_pB \text{ is oriented} \\ -1, & T_pA \oplus T_pB \text{ is oppositely oriented} \end{cases}$$

In this case, we define the transversal intersection number to be

$$\#(A\cdot B) = \sum_{p\in A\cap B} \#(A\cdot B)_p.$$

Remark 10.2. $\#(A\cdot B)$ only depends on the homology classes $[A] \in H_k(M,\mathbb{Z})$ and $[B] \in H_{n-k}(M,\mathbb{Z})$.

Remark 10.3. In fact, we do not require the intersection to be transverse to define the intersection number. To do this, we need to replace an arbitrary A and B by homologous transversally intersected algebraic cycles.

Regardless, we have an intersection pairing

$$\begin{aligned} H_k(M,\mathbb{Z}) \times H_{n-k}(M,\mathbb{Z}) &\to \mathbb{Z} \\ ([A],[B]) &\mapsto \#(A\cdot B) \end{aligned}$$

This is a purely topological notion.

Example 10.4. For a Riemann curve of genus 3, we have three horizontal cycles A_i's and three vertical cycles B_i's, hence $H_1(M,\mathbb{Z}) \cong \mathbb{Z}^6$. Then $\#(A_i,A_j)=0$, $\#(B_i,B_j)=0$, and $\#(A_i,B_j)=\delta_{ij}$, and $\#(B_i,A_j)=-\delta_{ij}$.

Theorem 10.5 (Poincaré Duality)**.** The intersection pairing above is unimodular. That is, writing down the pairing as a matrix, then it has determinant ± 1. One can also write down the pairing as

$$\begin{aligned} \alpha: H_k(M,\mathbb{Z}) &\to \operatorname{Hom}(H_{n-k}(M,\mathbb{Z}),\mathbb{Z}) \\ (\alpha(A))(B) &= \#(A\cdot B). \end{aligned}$$

Remark 10.6. The intersection cycles kills the torsion cycles. Say A is torsion, then $NA = 0$ for some $N \in \mathbb{N}$, thus $N\#(A\cdot B) = \#(NA\cdot B) = 0$, hence $\#(A\cdot B)=0$. Therefore α is a surjection, and $\ker(\alpha) = H_k(M,\mathbb{Z})_{\text{tor}}$.

In fact, we have simpler statements when we kill the torsion one way or another.

Remark 10.7. Tensoring both sides by $\mathbb{Q}$, then we have

$$H_k(M,\mathbb{Q}) \cong H_{n-k}(M,\mathbb{Q})^* \cong H^{n-k}(M,\mathbb{Q})$$

by the universal coefficient theorem. Therefore, given a cycle $A \in H_k(M,\mathbb{Q})$, we get some cohomology class $\eta \in H^{n-k}(M,\mathbb{Q})$, namely the Poincaré dual of A.

If we pass it on to $\mathbb{R}$, then the isomorphism above is

$$H_k(M,\mathbb{R}) \cong H_{n-k}(M,\mathbb{R})^* \cong H^{n-k}(M,\mathbb{R}) \cong H^{n-k}_{\text{dR}}(M).$$

In this case, we have $A \mapsto [\varphi]$ such that $d\varphi = 0$. Therefore, for any $B \in H_{n-k}(M,\mathbb{R})$, we have $\#(A\cdot B) = \int_B \varphi$.

Suppose we have $A \in H_k(M,\mathbb{R})$ and $[\eta_A] \in H^{n-k}_{\text{dR}}(M)$, along with $B \in H_{n-k}(M,\mathbb{R})$ and $[\eta_B] \in H^k_{\text{dR}}(M)$, then $\#(A\cdot B) = \int_M \eta_A \wedge \eta_B$ since this is a top form.

For $A \in H_k(M, \mathbb{R})$ and $B \in H_\ell(M, \mathbb{R})$, we can also calculate $A \cdot B \in H_{\dim(A \cap B)}(M, \mathbb{Z})$. We think of A as being given with $(n-k)$ real equations and B as $(n-\ell)$ real equations, so $A \cdot B$ should be given by $2n-k-\ell$ conditions. They are independent by transversality conditions. We should therefore expect $\dim(A \cap B) = n - (2n-k-\ell) = k+\ell-n$.

In the special case where $\ell = n-k$, then $k+\ell-n = 0$, therefore we get $H_0(M, \mathbb{Z}) \cong \mathbb{Z}$, identifying points where we count them with multiplicity.

Therefore, given $[\eta_A] \in H_{\mathrm{dR}}^{n-k}(M)$ and $[\eta_B] \in H_{\mathrm{dR}}^{n-\ell}(M)$, we would expect $\eta_A \wedge \eta_B \in H_{\mathrm{dR}}^{2n-k-\ell}(M)$. To see that, we need to check this form is closed. Indeed,

$$d(\eta_A \wedge \eta_B) = d\eta_A + (-1)^{n-k} \eta_A \wedge d\eta_B = 0.$$

In particular, the Poincaré dual of $H_{\mathrm{dR}}^{2n-k-\ell}(M)$ is $H_{k+\ell-n}(M, \mathbb{R})$.

Remark 10.8. We have $[A] \cdot [B] = [A \cap B]$.

Definition 10.9. Suppose $S, T \subseteq M$ are submanifolds where $\dim(S) = k$ and $\dim(T) = \ell$ such that $k+\ell \geqslant n$. We say S and T intersect transversally if for any $p \in S \cap T$, we have a surjection $T_pS \oplus T_pT \to T_pM$.

Remark 10.10. $S \cap T$ is a submanifold of dimension $k+\ell-n$. Therefore, we may write

$$T_pM = \operatorname{span}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{k+\ell-n}}, \cdots, \frac{\partial}{\partial x_k}, \cdots, \frac{\partial}{\partial x_n}\right),$$

where the first $k+\ell-n$ terms are from $S \cap T$, the first k terms are from S, and the rest of the terms should stay in T.

We may assign an orientation to $S \cap T$ based on the orientation on M, S, and T, then we obtain a general intersection pairing

$$H_k(M, \mathbb{Z}) \times H_\ell(M, \mathbb{Z}) \to H_{k+\ell-n}(M, \mathbb{Z})$$

by moving cycles into transversal intersection positions. By Poincaré duality, there is an associated pariing

$$H_{\mathrm{dR}}^{n-k}(M) \times H_{\mathrm{dR}}^{n-\ell}(M) \to H_{\mathrm{dR}}^{2n-k-\ell}(M)$$

that is given by the wedge product $\wedge$.

Finally, let us write down a cell decomposition for $\mathbb{P}^n$. This allows to compute things using cellular homology. We have

$$\mathbb{P}^n = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n$$

where $M_i = (z_0 = \cdots = z_{i-1} = 0) \cong \mathbb{P}^{n-i}$. Here we have $M_i \backslash M_{i-1} \cong \mathbb{C}^{n-i}$, so the compactification argument gives us

$$\mathbb{P}^n \cong \mathbb{C}^n \cup \cdots \cup \mathbb{C}^0.$$

Note the closure $[\mathbb{C}^{n-i}] = \mathbb{P}^{n-i} \cong M_i$. By checking the cells, we note $C_0 = \mathbb{Z} \cdot \mathbb{P}^0$, $C_1 = 0$ and $C_2 = \mathbb{Z} \cdot \mathbb{P}^1$ and so on, i.e., $C_{2k+1} = 0$ and $C_{2k} = \mathbb{Z}$ when $k \geqslant 0$. Therefore, all boundary maps are trivial. Moreover, $H_{2k}(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z} \cdot [\mathbb{P}^k]$ is generated by $\mathbb{P}^k \subseteq \mathbb{P}^n$. Hence,

$$H_i(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} \cdot [\mathbb{P}^{\frac{k}{2}}], & 0 \leqslant i \leqslant 2n \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

The intersection over $\mathbb{Z}$ gives

$$[\mathbb{P}^k] \cdot [\mathbb{P}^\ell] = \begin{cases} \mathbb{P}^{k+\ell-n}, & k+\ell \geqslant n \\ 0, & \text{otherwise} \end{cases}$$

To see this, set $V, W \subseteq \mathbb{C}^{n+1}$ of dimension $k+1$ and $\ell+1$, respectively, then once we move them into transversal position we have $\dim(V \cap W) = k+\ell-n+1$.

Over $\mathbb{R}$, the de Rham cohomology gives

$$H_{\mathrm{dR}}^i(\mathbb{P}^n, \mathbb{R}) = \begin{cases} \mathbb{Z} \cdot [\mathbb{P}^{\frac{k}{2}}], & 0 \leqslant i \leqslant 2n \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

by duality. Let $\omega = \frac{i}{2\pi}\partial\bar{\partial}\log||z||^2$ be a $(1,1)$ form of Fubini-Study metric, then $d\omega = \frac{i}{2\pi}d\partial\bar{\partial}\log||z||^2$, therefore

$$\begin{aligned} d\partial\bar{\partial} &= (\partial + \bar{\partial})\partial\bar{\partial} \\ &= \partial^2\bar{\partial} + \bar{\partial}\partial\bar{\partial} \\ &= -\partial\bar{\partial}^2 \\ &= 0. \end{aligned}$$

We now have an explicit de Rham cohomology class $[\omega] \in H^2_{\mathrm{dR}}(\mathbb{P}^n) \cong H_{2n-2}(\mathbb{P}^n, \mathbb{R})$, which corresponds to a class $A = \mathrm{PD}([\omega])$ in homology by Poincaré duality. It turns out that

$$\eta_{\mathbb{P}^{n-1}} = [\omega] \in H^2_{\mathrm{dR}}(\mathbb{P}^n).$$

Since $\mathbb{P}^{n-2}$ is the intersection of two hyperplanes in $\mathbb{P}^n$, we note

$$\eta_{\mathbb{P}^{n-2}} = [\omega \wedge \omega] \in H^4_{\mathrm{dR}}(\mathbb{P}^n),$$

and proceeding inductively we note

$$\eta_{\mathbb{P}^{n-k}} = [\omega^{\wedge k}] \in H^{2k}_{\mathrm{dR}}(\mathbb{P}^n).$$

Theorem 10.11 (Topological Künneth Formula)**.** For manifolds M and N, we have

$$H^*(M \times N, \mathbb{Q}) \cong H^*(M, \mathbb{Q}) \otimes H^*(N, \mathbb{Q})$$

and

$$H_*(M \times N, \mathbb{Q}) \cong H_*(M, \mathbb{Q}) \otimes H_*(M, \mathbb{Q}).$$

On homology, given $[\alpha] \in H_*(M, \mathbb{Q})$ and $[\beta] \in H_*(N, \mathbb{Q})$ via $\alpha \in Z_k(M, \mathbb{Q})$ and $\beta \in Z_\ell(N, \mathbb{Z})$, then the isomorphism sends them to a class $[\alpha \times \beta] \in Z_{k+\ell}(M \times N, \mathbb{Z})$. The formula for cohomology can be interpreted as de Rham cohomology, since

$$\begin{aligned} H^*_{\mathrm{dR}}(M) \times H^*_{\mathrm{dR}}(N) &\to H^*_{\mathrm{dR}}(M \times N) \\ (\nu, \eta) &\mapsto \pi_1^*\nu \wedge \pi_2^*\eta. \end{aligned}$$

Remark 10.12. For compact complex manifolds,

- all local transversal intersections contribute $+1$ locally since the complex basis always gives rise to an orientation.
- suppose $V \subseteq M$ is an analytic subvariety of dimension k, so for any $\omega \in A^{2k}(M)$ such that $d\omega = 0$, then we may integrate ω along V, so the assignment $\omega \mapsto \int_V \omega$ descends to de Rham cohomology $(H^{2k}_{\mathrm{dR}})^*(M) \cong H^{2n-2k}_{\mathrm{dR}}(M)$ by Poincaré duality, so we get a de Rham cohomology class representing V by Poincaré duality, which is the fundamental class.

Let us go back to the fundamental class. For $\mathbb{P}^k \subseteq \mathbb{P}^n$, let ω be the associated $(1,1)$-form of the Fubini-Study metric on $\mathbb{P}^n$.

Lemma 10.13. The fundamental class $\eta_{\mathbb{P}^{n-k}} \in H^{2k}_{\mathrm{dR}}(\mathbb{P}^n)$ of $\mathbb{P}^{n-k}$ is represented by ω^k.

Proof. By Poincaré duality and computation of the intersection product, we know that

$$\int_{\mathbb{P}^1} \eta_{\mathbb{P}^{n-1}} = \#(\mathbb{P}^1 \cdot \mathbb{P}^{n-1}) = 1.$$

Since $H^2_{\mathrm{dR}}(\mathbb{P}^n)$ is one-dimensional, then $[\omega] = a\eta_{\mathbb{P}^{n-1}} \in H^2_{\mathrm{dR}}(\mathbb{P}^n)$ for some real number a. Moreover, we note that $\int_{\mathbb{P}^1} \omega = 1$, so $a = 1$, therefore this proves the case for $k = 1$. In general, since $\mathbb{P}^{n-k}$ is the k-fold intersection of transversally intersecting hyperplanes, then we have that

$$\eta_{\mathbb{P}^{n-k}} = (\eta_{\mathbb{P}^{n-1}})^k = [\omega]^k = [\omega^k].$$

□

What happens when the intersection is not transverse? We don't always want to solve things using geometry. To define a global intersection multiplicity that takes the local multiplicity into account, we need to wiggle one of the varieties so that we may count using a limit argument, i.e., a tangent line is a limit of secants. Essentially, the intersection number of two analytic varieties V and W of dimension k and $n-k$, when intersecting at a finite number of points, is just the sum of local multiplicities.

Locally, let us think of the only intersection to be $p = 0 \in \Delta_n$. To formalize this, let us define $\tilde{V} = \pi_1^{-1}(V) \subseteq \Delta \times \Delta$ and $\tilde{W} = \{(z,w) : w - z \in W\} \subseteq \Delta \times \Delta$, then $\tilde{V} \cap \tilde{W}$ is an analytic subvariety of $\Delta \times \Delta$, which is isomorphic to $V \times W$ naturally (as abstract varieties). Therefore, $\dim(\tilde{V} \cap \tilde{W}) = k + (n-k) = n$. Now we have a structure theorem where we look at the d-sheeted branched cover

$$\pi : \tilde{V} \cap \tilde{W} \to \Delta$$

of Δ, then we define the multiplicity at 0 to be $\operatorname{mult}_0(V,W) = d$. If we take the fiber of $\pi_1 : \tilde{W} \to \Delta$ over some point $\varepsilon \in \Delta$, then $(\pi_1|_W)^{-1}(\varepsilon) = \{\varepsilon, W + \varepsilon\}$.

Recall that $H^2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z} \cdot \ell$ for some line ℓ in $\mathbb{P}^2$. Given a curve C of degree d, its homology class $[C]$ must be a multiple of ℓ, and we will see it is $d\ell$. To see this, suppose C is given by a polynomial of degree d $f_d(x_0,x_1,x_2) = 0$. We can take a linear combination

$$t \prod_{c=1}^{d} \ell_i(x_0,x_1,x_2) + (1-t) f_d(x_0,x_1,x_2) = 0.$$

In fact, $f_d = 0$ is homologous to $\prod \ell_i = 0$, which corresponds to $t = 0$ and $t = 1$, respectively. Therefore, $[c] = \sum_{i=1}^{d} [\ell_i] = d\ell$.

We can do the same for general curves C and D of degree c and d, respectively.

Theorem 10.14 (Bezout). If C and D has a finite number of points of intersections, then $\sum_{p \in C \cap D} \operatorname{mult}_p(C \cdot D) = cd$.

Proof. We have

$$\sum \operatorname{mult}_p(C \cdot D) = \#(C \cdot D) = \#((c\ell) \cdot (d\ell)) = cd\#(\ell \cdot \ell) = cd.$$

which only depends on homology. □

Corollary 10.15. Suppose that M is a complex projective manifold, i.e., embedded in $\mathbb{P}^N$ for some N, and let V be an analytic subvariety of dimension k (that is not $\mathbb{P}^N$), then the fundamental class $[\eta_V] \in H^{2(n-k)}_{\text{dR}}(M)$ is non-zero.

Proof. For $V \subseteq M \subseteq \mathbb{P}^N$, we choose a linear subspace $\mathbb{P}^{N-k}$ in $\mathbb{P}^n$ of codimension k, such that $\mathbb{P}^{N-k}$ intersects V at finitely many points. Now let $W = \mathbb{P}^{N-k} \cap M \subseteq M$ be an analytic subvariety, then by choosing general enough subspaces, we run the same argument and conclude that $\dim(W) = n-k$, i.e., dimensionally transverse. Note that $V \cap W \neq \varnothing$, and that $\int_W \eta_V = \#(V \cdot W) > 0$, therefore $\eta_V \neq 0$. □

Corollary 10.16. The even Betti numbers of a complex projective manifold M are positive.

Proof. We can do the same thing, i.e., choosing $\mathbb{P}^{N-k}$ inside $\mathbb{P}^N$ for $k \leqslant n$ such that $\dim(\mathbb{P}^{N-k} \cap M) = n-k$ as an analytic subvariety, therefore $0 \neq \eta_{\mathbb{P}^{n-k} \cap M} \in H^{2k}_{\text{dR}}(M)$, where $b^{2k} = \dim(H^{2k}(M))$. □

We will learn about the following result later in class, which is a lot deeper since we may not work with analytic subvarieties in the first place.

Corollary 10.17. The even Betti numbers of a Kahler manifold M are positive.

Corollary 10.18. Let $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ be a hyperplane, with homology class $H = [\mathbb{P}^{n-1}] \in H^2_{\text{dR}}(\mathbb{P}^n)$. Suppose $V \subseteq \mathbb{P}^n$ is an analytic subvariety such that $[V] = H$ as well, then V is a hyperplane.

Proof. Without loss of generality, pick distinct $p_1, p_2 \in V$, and let $L = \overline{p_1 p_2} \subseteq \mathbb{P}^n$ be the line joining them. The intersection number $\#(V \cdot L) = \#(H \cdot L) = 1$. Since $|V \cap L| > 1$, then $|V \cap L| = \infty$, therefore $L \subseteq V$, hence V is a linear subspace, thus it is equivalent to a hyperplane, i.e., as a zero set of some line. □

Corollary 10.19. Let $A \in \mathrm{GL}(n+1, \mathbb{C})$, and let

$$\begin{aligned} \varphi_A : \mathbb{P}^n &\to \mathbb{P}^n \\ [x] &\mapsto [Ax] \end{aligned}$$

be a holomorphic automorphism where x is a column vector in $\mathbb{C}^{n+1}$, then $(\varphi_A)^{-1} = \varphi_{A^{-1}}$. This is a linear automorphism of $\mathbb{P}^n$, i.e., a linear fractional transformation. Note that $\varphi_A = \varphi_{\lambda A}$ for any $\lambda \in \mathbb{C}^*$, so we may think of $A \in \mathrm{PGL}(n+1, \mathbb{C}) \cong \mathrm{GL}(n+1, \mathbb{C})/\mathbb{C}^*$. With this, we have

$$\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}(n+1, \mathbb{C}).$$

Proof. Let $\varphi \in \mathrm{Aut}(\mathbb{P}^n)$. Consider the homology class of the image of a hyperplane via φ. Note that $\varphi^* : H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z} \to H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$, then this must be a multiplication by ± 1. Therefore, $\varphi^*(H) = \pm H$. We claim that $\varphi^*(H) = H$, then since it is an analytic subvariety, we must have $\varphi^{-1}(H) \subseteq \mathbb{P}^N$. Suppose $\varphi^*(H) = -H$, then

$$\begin{aligned} 0 &< H \cdot (\varphi^{-1})^*(L) \\ &= \varphi^*(H) \cdot L \\ &= (-1) \cdot L \\ &= -1 \end{aligned}$$

since $(\varphi^{-1})^*(L)$ is an analytic subvariety, therefore we have a contradiction. Hence, φ takes hyperplanes to hyperplanes. Now set $H_i = Z(x_i) \subseteq \mathbb{P}^n$, so we think of $\varphi(H_0)$ as the hyperplane H_0 without loss of generality by replacing φ by $\varphi_A \circ \varphi$. Let $y_i = \frac{x_i}{x_0}$ for $i = 1, \ldots, n$. For $\varphi(H_i) = Z(\ell_i)$ for some line $\ell = a_0 x_0 + \cdots + a_n x_n$, then we can define $\tilde{\ell} = a_1 + a_1 y_1 + \cdots + a_n y_n$ by quotienting the first coordinate so that we end up in affine coordinates. Now $\frac{\varphi^*(y_i)}{\ell_i}$ is an entire function, which must be constant. Therefore, φ is linear. □

Remark 10.20. If subvariety has homology class $\mathbb{P}^{n-k}$, it must be a linear subspace, i.e., a hyperplane H^k.

11 Complex Vector Bundles

Definition 11.1. A C^∞ complex vector bundle of rank k on a C^∞ manifold M is a C^∞ manifold D with C^∞ map $\pi : E \to M$, such that there exists a cover $\{U_\alpha\}_{\alpha\in I}$ of M such that we have trivialization

$$\begin{array}{ccc} E \supseteq \pi^{-1}(U_\alpha) & \xrightarrow[\varphi_\alpha]{\cong} & U_\alpha \times \mathbb{C}^k \\ \downarrow & & \downarrow \\ U_\alpha & = & U_\alpha \end{array}$$

and the composition

$$\begin{aligned} \varphi_\alpha \circ (\varphi_\beta|_{\pi^{-1}(U_{\alpha\beta})})^{-1} : U_{\alpha\beta} \times \mathbb{C}^k &\to U_{\alpha\beta} \times \mathbb{C}^k \\ (z, v) &\mapsto (z, g_{\alpha\beta}(z)v) \end{aligned}$$

can be described via a C^∞ map $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{C})$.

Remark 11.2. For any $m \in U_\alpha \cap U_\beta$, let $E_m = \pi^{-1}(\{m\})$, then $E_m \cong \mathbb{C}^k$ via φ_α gives a vector space structure. Moreover, even if we give a vector structure to E_m via φ_β, then there is an isomorphism between $\mathbb{C}^k$'s via $g_{\alpha\beta}(m)$, so everything should be compatible.

Conversely, given $M = \bigcup_\alpha U_\alpha$ to be smooth, then let

$$g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{C})$$

be such that on $U_{\alpha\beta\gamma}$, $g_{\gamma\alpha} \circ g_{\alpha\beta} = g_{\gamma\beta}$, then we can define $E = \bigcup_\alpha (U_\alpha \times \mathbb{C}^k)/((z, v) \sim (y, w))$ where we identify $(z, v) \in U_\alpha \times \mathbb{C}^k$ and $(y, w) \in U_\beta \times \mathbb{C}^k$ if and only if $z = y$ and $v = g_{\alpha\beta}(z)w$. Then E is a C^∞ manifold.

Equivalently, we may use the following definition.

Definition 11.3. Let $\pi : E \to M$ be map such that for any m, E_m has the structure of a $\mathbb{C}$-vector space, then we define the C^∞ map

$$\begin{aligned} m_\lambda : E &\to E \\ e &\mapsto \lambda e \end{aligned}$$

for $\lambda \in \mathbb{C}$, as a multiplication in the fiber $e_{\pi(e)}$. Taking the fiber product, we induce an addition C^∞ map

$$\begin{aligned} + : E \times_M E &\to E \\ (e, f) &\mapsto e + f \end{aligned}$$

as an addition in $E_{\pi(e)} = E_{\pi(f)}$. This is a coordinate-free definition of a vector bundle.

Example 11.4. The trivial bundle $M \times \mathbb{C}^k$.

Example 11.5. The tangent bundle $T_{\mathbb{C}}M$ for $M = \bigcup_\alpha M_\alpha$ with coordinate charts $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$ in $(x_1, \ldots, x_n)$, gives rise to isomorphisms

$$\begin{aligned} \psi_\alpha : T_{\mathbb{C}}M|_{U_\alpha} &\cong V_\alpha \times \mathbb{C}^n \\ (x, \sum v(x)\frac{\partial}{\partial x_i}) &\mapsto (x_1, \ldots, x_n, v_1, \ldots, v_n) \end{aligned}$$

where $v_i \in C^\infty(U_\alpha)$. The transition functions $\psi_\alpha \circ \psi_\beta^{-1}$ are the Jacobians $g_{\alpha\beta} := \mathrm{Jac}(\varphi_\alpha \circ \varphi_\beta^{-1})$, which is C^∞. Then $g_{\gamma\alpha}g_{\alpha\beta} = g_{\gamma\beta}$ again by the property of the Jacobian.

Just like for vector spaces, we may create new vector bundles using operations like $E \oplus F$, $E \otimes F$, $\Lambda^r E$, $\Lambda^r E \otimes \Lambda^s F$, and E^*.

Definition 11.6. A C^∞ section of vector bundle E on open subset $U \subseteq M$ is a map $\sigma; U \to E|_U$ such that $\pi \circ \sigma = \mathrm{id}_U$ for $\pi : E|_U \to U$ restricted from $E \to M$.

Remark 11.7. Let $s_i(z) = (z, e_i)$ where e_i is the standard basis vector. Then we have

$$\begin{array}{ccc} E|_{U_\alpha} & \xrightarrow[\cong]{\varphi_\alpha} & U_\alpha \times \mathbb{C}^k \\ \sigma \uparrow & \nearrow_{s_i} & \\ U & & \end{array}$$

and can define $\sigma_i = \varphi_\alpha^{-1} \circ s_i$ to be the section of E over U. Given a set $\{\sigma_1, \dots, \sigma_k\}$ in a set of C^∞ sections of E over U_α, such that for any $p \in U_\alpha$, we have a basis $\{\sigma_1(p), \dots, \sigma_k(p)\}$ for E_p. We then say σ is a frame for E over U.

Conversely, let $\sigma = (\sigma_1, \dots, \sigma_k)$ be a frame over U_α, then we can define a C^∞ map

$$\begin{aligned} E|_{U_\alpha} &\cong U_\alpha \times \mathbb{C}^k \\ e &\mapsto (\pi(e), a_1, \dots, a_n) \end{aligned}$$

as we write $e = \sum a_i \sigma_i(\pi(e))$ in $E_{\pi(e)}$. Therefore, giving a local trivialization of a bundle is the same as giving a frame.

Let $\sigma_1, \dots, \sigma_k$ be a frame over an open subset U, and let s be any C^∞ section over U, then we can write $s = \sum s_i \sigma_i$ for $s_i \in C^\infty(U)$. That is, for any $p \in U$, we have $E_{\pi(p)} \ni s(l) = \sum s_i(p)\sigma_i(p)$. Conversely, for any $s_1, \dots, s_n \in C^\infty(U)$, we get $s = \sum s_i \sigma_i$ to be a C^∞ section over U.

If σ_α's are frames corresponding to $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^k$ and similarly for σ_β with $E|_{U_\beta} \cong U_\beta \times \mathbb{C}^k$, then we may give C^∞ sections over $U_{\alpha\beta}$: for $s = \sum s_\alpha \sigma_\alpha = \sum s_\beta \sigma_\beta$, recall we relate s_α and s_β via the standard basis vectors, therefore we have $s_\alpha = g_{\alpha\beta} s_\beta$.

Hence, given $s \in C^\infty(E)(U_{\alpha\beta})$, over U_α and U_β we obtain two different isomorphisms by the local trivialization, but they themselves are isomorphic via the identification above.

Definition 11.8. A subbundle $F \subseteq E$ is a subset with a bundle structure. Correspondingly, E/F is a quotient bundle. If we look at g_E as a matrix of E, then g_F should correspond to a minor matrix in the top left corner, while $g_{E/F}$ corresponds to a minor matrix in the bottom right corner.

Definition 11.9. Given a bundle E on M with C^∞ map $f : N \to M$, the pullback f^*E on N is given by $g_{f*(E)} = g_E \circ f$.

Definition 11.10. Given C^∞ bundles E and F on M, then we say $f : E \to F$ is a C^∞ map if it is linear on the fiber.

If we just take the kernel and image fiberwise, we may not get constant rank throughout the whole bundle.

Definition 11.11. For $f : E \to F$, $\ker(f) \subseteq E$ and $\mathrm{im}(f) \subseteq F$ are subbundles if and only if for any $p \in M$, the rank of $f_p : E_p \to F_p$ is constant.

Definition 11.12. Given two vector bundles E and F, then E and F are isomorphic if there exists an isomorphism between $E \to M$ and $F \to M$.

Definition 11.13. We say E is trivial if $E \cong M \times \mathbb{C}^k$.

Let M be a complex manifold. Then everything above has a holomorphic analogue. Let $\mathcal{O}(E)$ be the sheaf of holomorphic sections of E, then the bundle $T_\mathbb{C} M = T'(M) \oplus T''(M)$ corresponds to holomorphic differentials $\frac{\partial}{\partial z_i}$ and their conjugates $\frac{\partial}{\partial \bar{z}_i}$. In particular, $T'(M)$ gives a holomorphic subbundle of $T_\mathbb{C} M$, while $T''(M)$ only gives a C^∞ subbundle. Therefore, the sheaf of C^∞-sections $\mathfrak{a}_M^{p,q}$ gives $\Lambda^p T'(M)^* \otimes \Lambda^q T''(M)^*$. This allows us to do cohomology on the holomorphic subbundle via $H^n(M, T'M)$. If $n = 1$, this is the space of first-order deformations of M.

Once we tensor with E, i.e., $\Lambda^p T'(M)^* \otimes \Lambda^q T''(M)^* \otimes E$, we get a section of this bundle E. This gives C^∞ sections $A^{p,q}(E)$.

Definition 11.14. The C^∞ sections $A^{p,q}(E)$ has a $\bar{\partial}$ operator

$$\begin{aligned} \bar{\partial} : A^{p,q}(E) &\to A^{p,q+1}(E) \\ \sum w_i \otimes e_i &\mapsto \sum \bar{\partial} w_i \otimes e_i \end{aligned}$$

where w_i's are (p, q) forms for holomorphic sections e_i on E.

To see that this is well-defined, let us change frames $e_i = \sum g_{ij} e'_j$, then the transition matrices (g_{ij}) are given by holomorphic functions. Hence, $\sum_i w_i \otimes e_i = \sum_{i,j} (g_{ij} w_j) \otimes e'_j$. Taking $\bar{\partial}$, we get

$$\begin{aligned}\bar{\partial}(\sum_i w_i \otimes e_i) &= \sum (g_{ij} \bar{\partial} w_j) \otimes e' \\ &= \sum \bar{\partial} w_j \otimes e_j\end{aligned}$$

since (g_{ij})'s are holomorphic.

Definition 11.15. Suppose $S \subseteq M$ is a complex submanifold, then there are holomorphic tangent bundles $T'S$ and $T'M$, which can then be restricted to $T'M|_S$. We have a subbundle structure $T'S \hookrightarrow T'M|_S$ by pushing forward as inclusion. The normal bundle of S in M is defined by $N_{S/M} = T'M|_S / T'_S$. If $T'S$ has rank r and $T'M|_S$ has rank n, then the tangent bundle has rank $n - r$, and is holomorphic.

Let $E \to M$ be a C^∞ $\mathbb{C}$-vector bundle, then there is a Hermitian metric on E. Let s, s' be C^∞ sections of E, then $\langle s, s' \rangle$ is a C^∞ function sending m to $\langle s_m, s'_m \rangle$ for any $m \in M$. That is,

$$\langle -, - \rangle : E_m \times E_m \to \mathbb{C}$$

is a Hermitian metric for any $m \in M$. Equivalently, given C^∞ frames $e_1, \ldots, e_r$ for E, then $h_{ij} = \langle e_i, e_j \rangle$ is C^∞.

For C^∞ subbundle $F \subseteq E$, then the orthogonal space is

$$F^\perp = \{s \in E : \langle f, s \rangle = 0 \; \forall f \in F\}.$$

Therefore, $E \cong F \oplus F^\perp$.

12 Hermitian Vector Bundles

Definition 12.1. A Hermitian vector bundle E is a holomorphic vector bundle on a complex manifold M endowed with a Hermitian metric.

Definition 12.2. Let E be a vector bundle over $\mathbb{C}$ and M be a manifold, then a connection $D : A^0(E) \to A^1(E)$ over $E \to M$ is a C-linear map that satisfies the Leibniz rule $D(fs) = df \otimes s + fD(s)$ where f is of C^∞ and s is a section of E.

Note that we can factor $A^1(E)$ as $A^{1,0}(E) \bigoplus A^{0,1}(E)$, then we have $D = D' + D''$ where $D' = \pi^{1,0} \circ D : A^0(E) \to A^{1,0}(E)$ and $D'' = \pi^{0.1} \circ D : A^0(E) \to A^{0,1}(E)$.

Definition 12.3. We say that D is compatible with the complex structure if $D'' = \delta : A^0(E) \to A^{0,1}(E)$.

Definition 12.4. We say that D is compatible with the metric if for any sections s, s' of E, $d\langle s, s'\rangle = \langle Ds, s'\rangle + \langle s, Ds'\rangle$.

Theorem 12.5. There exists a unique connection D on a Hermitian vector bundle E such that D is compatible with both the metric and the complex structure. We call such connection to be the Chern connection.

Proof. Let $e = \{e_i\}$ be holomorphic frames for E, then $De_i = \sum \theta_{ij} \otimes e_j$ for the (connection) 1-forms θ_{ij}. Since $D''e_i = \bar{\partial} e_i = 0$, then $D''e = De$, therefore θ_{ij}'s are $(1,0)$-forms. For $h_{ij} = \langle e_i, e_j\rangle$, we have that

$$\begin{aligned} dh_{ij} &= \langle De_i, e_j\rangle + \langle e_i, De_j\rangle \\ &= \left\langle \sum_k \theta_{ik} e_k, e_j \right\rangle + \left\langle e_i, \sum_\ell \theta_{j\ell} e_\ell \right\rangle \\ &= \sum_k \theta_{ik} h_{kj} + \sum_\ell \bar{\theta}_{j\ell} h_{i\ell}. \end{aligned}$$

Therefore, the $(1,0)$-forms agree, so the holomorphic derivative

$$\partial h_{ij} = \sum \theta_{ik} h_{kj}.$$

In terms of matrix equations, we know $\partial h = \theta h$, so $\theta = \partial h \cdot h^{-1}$ is the unique solution.

Similarly, for the $(0,1)$-forms, we know that $\bar{\partial} h = h\bar{\theta}^T$, so $h^{-1}\bar{\partial} h = \bar{\theta}^T$, hence $\bar{\theta} = \partial h \cdot h^{-1}$ as well. ☐

Definition 12.6. Let $D : A^0(E) \to A^1(E)$ be a connection, then this induces $A^p(E) \to A^{p+1}(E)$ such that $D(w \otimes s) = dw \otimes s + (-1)^p \wedge D(s)$.

Proposition 12.7. $D^2 : A^0(E) \to A^2(E)$ defines a section of $\Lambda^2 T^* \otimes \operatorname{Hom}(E, E)$. That is, for any $f \in C^\infty(M)$ with section $s \in A^0(E)$, we have $D^2(fs) = f(D^2 s)$.

Proof. We have

$$\begin{aligned} D^2(fs) &= D(df \otimes s + fDs) \\ &= (d^2 f \otimes s - df \wedge Ds) + (df \wedge Ds + fD^2 s) \\ &= fD^2 s. \end{aligned}$$

This gives a section of the bundle since once we trivialize E as $E|_U \cong U \times \mathbb{C}^k$, we have a mapping that satisfies the property above, which is just a linear matrix $D^2 A^0(M \times \mathbb{C}^k) \to A^2(M \times \mathbb{C}^k)$, which is a map of vector bundles $C^k \to \Lambda^2 T^* \otimes \mathbb{C}^k$, and this is equivalent to a section of $\Lambda^2 T^* \otimes \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^k)$, which respects the transition functions, i.e., $D^2(g_{\alpha\beta} s_\alpha) = g_{\alpha\beta} D^2(s_\beta)$. ☐

If we use a trivialization $E|_U \cong U \times \mathbb{C}^k$, then $D^2 A^0(M \times \mathbb{C}^k) \to A^2(M \times \mathbb{C}^k)$ is a map of vector bundles given by $\mathbb{C}^k \to \Lambda^2 T^* \otimes \mathbb{C}^k$.

Suppose E and F are vector bundles on M with sheaves of C^∞ sections $A^0(E)$ and $A^0(F)$, let $L : A^0(E) \to A^0(F)$ be a homomorphism of sheaves of $A^0(M)$-modules, so for any $f \in A^0(U)$ and $s \in A^0(E)(U)$, we have $L(f \cdot s) = f \cdot L(s)$. Recall that $D^2 : A^0(E) \to A^2(E) = A^0(\Lambda^2 T^* \otimes E)$, and we showed that $D^2(fs) = fD^2(s)$, then D^2 is a homomorphism of sheaves of $A^0(M)$-modules.

Lemma 12.8. If $L : A^0(E) \to A^0(F)$ is a homomorphism of sheaves of $A^0(M)$-modules, then L is induced by a unique section $\tilde{L} \in A^0(\mathrm{Hom}(E, F))$, i.e., $L(s) = \tilde{L}(s) \in A^0(F)$. In other words, homomorphism of sheaves of $A^0(M)$-modules corresponds to map of vector bundles.

Proof. Let $\{e_i\}$ be a frame for E on U, with $L(e_i) \in A^0(F)$, then for $s \in A^0(E)\big|_U$, $s = \sum_j f_j e_j$ and $L(s) = \sum_j f_j L(e_j)$ for some $f_j \in A^0(U)$. Thus $e_i \mapsto L(e_i)$ defines a homomorphism $\tilde{L} : E|_U \to F|_U$. One can check that this is compatible with change of frames using linearity of C^∞ frames to see $\tilde{L}(s)\Big|_{U_i}$ is well-defined. □

From the lemma, we may identify D^2 with $\tilde{D}^2 \in A^0(\mathrm{Hom}(E, \Lambda^2 T_M^* \otimes E)) \cong A^0(\mathrm{Hom}(E, E) \otimes \Lambda^2 T_M^*)$ as the canonical isomorphism. Therefore, in a frame $\{e_1, \ldots, e_k\}$, D^2 is described by a $k \times k$ matrix of 2-forms. This is the curvature of the bundle.

Definition 12.9. More explicitly, in a frame $\{e_i\}$ we have the connection matrix $[\theta_{ij}]$'s of 1-forms with $De_i = \sum_j \theta_{ji} e_i$. Therefore, $D^2(e_i) = D(\sum_j \theta_{ji} e_i) = \sum_j (d\theta_{ji} \otimes e_i - \theta_{ji} \otimes De_i) = \sum_j (d\theta_{ji} \otimes e_i - \theta_{ji} \wedge \theta_{ik} \otimes e_k)$. We obtain a final formula $D^2 = [\Theta_{ij}]$, where $\Theta_{ij} = d\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj}$.

If we change to a frame $e' = ge$ instead, then we obtain $\theta' \otimes e' = De' = dg \otimes e + gDe = dg \otimes e + g\theta \otimes e$. Moreover, we also know that $\theta' \otimes ge = \theta' \otimes e = ((dg)g^{-1} + (g\theta g^{-1}) \otimes e)$. Equating these two expressions shows that $\theta' = (dg)g^{-1} + g\theta g^{-1}$. Similarly, we find that $\Theta' = g\Theta g^{-1}$. There are two methods:

- compute the change of frames explicitly, or
- use our formula for θ' and the definition of Θ' in terms of θ'.

Now suppose M is a complex manifold, E is a holomorphic vector bundle, and D is the Chern connection. If e is a holomorphic frame, then θ is holomorphic. If e is a unitary frame, i.e., $\langle e_i, e_j \rangle = \delta_{ij}$, then

$$\begin{aligned}
0 &= d\langle e_i, e_j \rangle \\
&= \langle De_i, e_j \rangle + \langle e_i, De_j \rangle \\
&= \left\langle \sum \theta_{ik} e_k, e_j \right\rangle + \left\langle e_i, \sum \theta_{jk} e_k \right\rangle \\
&= \theta_{ij} - \bar{\theta}_{ji}.
\end{aligned}$$

Therefore, θ is a skew Hermitian matrix.

Now $\theta \in A^1(\mathrm{Hom}(E, E)|_U) = A^{1,0}(\mathrm{Hom}(E, E)|_U) \oplus A^{0,1}(\mathrm{Hom}(E, E)|_U)$, therefore we may write $\theta = \theta^{1,0} + \theta^{0,1}$. Curvature is even better: we have $\Theta = \Theta^{2,0} + \Theta^{1,1} + \Theta^{0,2}$ globally, with $\Theta^{p,q} \in A^{p,q}(\mathrm{Hom}(E, E))$.

Let $D = D' + D''$ be the Chern connection on a Hermitian bundle E, then $D'' = \bar{\partial}$, so $(D'')^2 = (\bar{\partial})^2 = 0$, hence $D^2 = (D')^2 + (D'D'' + D''D' + 0$, so $\Theta^{0,2} = 0$. On the other hand, in a unitary frame θ is skew-Hermitian, so $\Theta = d\theta - \theta \wedge \theta = -d^T\bar{\theta} + {}^T\bar{\theta} \wedge^T \bar{\theta} = -{}^T\bar{\Theta}$, with the extra sign coming from ${}^T\bar{\theta} \wedge^T \bar{\theta}$ as we compute in indices. This implies $\Theta^{2,0} = 0$, since we know $\Theta^{0,2} = 0$. Therefore, $\Theta = \Theta^{i,j}$ is a matrix of $(1,1)$-forms in $A^{1,1}(\mathrm{Hom},(E, E))$.

Remark 12.10. If E and E' are Hermitian bundles, so are $E \oplus E'$ and $E \otimes E'$. The latter is defined via $\langle e \otimes e', f \otimes f' \rangle = \langle e, f \rangle_E \langle e', f' \rangle_{E'}$.

On $E \otimes E'$, we may induce a Hermitian metric $\langle s \otimes s', t \otimes t' \rangle = \langle s, t \rangle \langle s', t' \rangle$.

Lemma 12.11. The Chern connection satisfies $D_{E \otimes E'} = D_E \otimes 1 + 1 \otimes D_{E'}$.

Proof. To see that $D_E \otimes 1 + 1 \otimes D_{E'}$ is a connection on $E \otimes E'$, we have

$$\begin{aligned}
(D_E \otimes 1 + 1 \otimes D_{E'})((fs) \otimes s') &= D_E(fs) \otimes s' + fs \otimes D_{E'} s' \\
&= (df \otimes s + fD_E s) \otimes s' + fs \otimes D_{E'} s' \\
&= df \otimes (s \otimes s') + f(D_E s \otimes s' + s \otimes D_{E'} s').
\end{aligned}$$

Moreover, $D_E \otimes 1 + 1 \otimes D_{E'}$ is compatible with the metric. Given two sections, we have

$$\begin{aligned} d\left\langle s \otimes s', t \otimes t' \right\rangle &= d(\langle s, t\rangle \left\langle s', t'\right\rangle) \\ &= \langle s, t\rangle\, d\left\langle s', t'\right\rangle + d\langle s, t\rangle \left\langle s', t'\right\rangle \\ &= \langle s, t\rangle \left(\left\langle D_{E'} s, t'\right\rangle + \left\langle s', D_{E'} t'\right\rangle\right) + \left\langle s', t'\right\rangle (\langle D_{E'} s, t\rangle + \langle s, D_E t\rangle) \\ &= \left\langle s \otimes s', (1 \otimes D_{E'})(t \otimes t')\right\rangle + \left\langle (D_E \otimes 1)(s \otimes s'), t \otimes t'\right\rangle. \end{aligned}$$

Observe that the right-hand side is compatible with the complex structure and the metric,, then

$$\begin{aligned} (D_E \otimes 1 + 1 \otimes D_{E'})'' &= \bar{\partial}_E \otimes 1 + 1 \otimes \bar{\partial}_{E'} \\ &= \bar{\partial}_{E \otimes E'}. \end{aligned}$$

To see the second equality, note that for holomorphic frames $\{e_i\}$'s of E and $\{e'_j\}$'s of E', then $\{e_i \otimes e'_j\}$'s also give a holomorphic frame for $E \otimes E'$. Locally, for $s \in A^0(E \otimes E')$, then $s = \sum f_{ij} e_i \otimes e'_j$ where f_{ij}'s are C^∞ functions. By definition of $\bar{\partial}$, we have

$$\bar{\partial}_{E \otimes E'} s = \sum \bar{\partial} f_{ij} (e_i \otimes e'_j).$$

If we write $s = \sum (g_i e_i) \otimes (h_j e'_j) = \sum g_i h_j e_i \otimes e'_j$, then

$$\begin{aligned} \bar{\partial} s &= \sum \bar{\partial}(g_i h_j) e_i \otimes e'_j \\ &= \sum (\bar{\partial} g_i \cdot h_j + g_i \cdot \bar{\partial} h_j) e_i \otimes e'_j \\ &= \sum (\bar{\partial} s' \otimes s'' + s' \otimes \bar{\partial} s'') e_i \otimes e'_j, \end{aligned}$$

hence the two equations agree. □

Let D be any connection on a complex vector bundle, then D induces a connection on its dual E^*. Let w be a section on E^*, then we define $D_{E^*} w$ by

$$\langle s, D_{E^*} w\rangle = d\langle s, w\rangle - \langle D_E s, w\rangle$$

where s is a section of E and $\langle -, \rangle : E \otimes E^* \to \mathfrak{a}^0$ is a pairing. This is defined with respect to an operator $D_{E^*} : \mathfrak{a}^0(E^*) \to \mathfrak{a}^1(E^*)$.

Lemma 12.12. D_{E^*} is a connection indeed.

Proof. It suffices to show that

$$\langle s, D_{E^*}(fw)\rangle = \langle s, df \otimes w\rangle + \langle s, f D_{E^*} w\rangle$$

for any section s. Indeed, we have

$$\begin{aligned} \langle s, D_{E^*}(fw)\rangle &= d\langle s, fw\rangle - \langle D_E s, fw\rangle \\ &= df \otimes \langle s, w\rangle + f\, d\langle s, w\rangle - f\langle D_E s, w\rangle \\ &= \langle s, df \otimes w\rangle + \langle s, f D_{E^*} w\rangle. \end{aligned}$$

□

Lemma 12.13. Let E be a Hermitian vector bundle, and D_E is its Chern connection. Given E^* with the dual Hermitian metric, then the dual connection on E^* is the same as the Chern connection of E^*, with the dual metric.

Proof. Once we write down the definition, it suffices to check that this is compatible with the metric and the complex structure. □

Definition 12.14. Suppose $\{e_i\}$'s give a unitary frame for E, then $\{e_j^*\}$'s give a dual frame for the dual bundle E^*, characterized by the fact that $\left\langle e_i, e_j^*\right\rangle = \delta_{ij}$, where this describes a pairing. We define the Hermitian metric on E^* by $h_{ij}^* = \left\langle e_i^*, e_j^*\right\rangle = \delta_{ij}$. This is then independent of the chosen unitary frame.

Alternatively, this non-degenerate pairing $\langle -,-\rangle : E \times E \to \mathbb{C}$ is characterized by the pullback of the musical isomorphism

$$\begin{aligned} h^{\#} : E &\to \bar{E}^* \\ v &\mapsto \overline{(v,-)} \end{aligned}$$

In particular, $\langle \langle v,-\rangle, \langle w,-\rangle\rangle = \langle v,w\rangle$.

Let E be a Hermitian bundle and $F \subseteq E$ be a holomorphic subbundle. F has a restricted metric from E, therefore F is also a Hermitian bundle. To write the Chern connection D_F in terms of D_E, we have

$$D_F = \pi_F \circ D_E : \mathfrak{a}^0(E) \to \mathfrak{a}^1(E) \xrightarrow{\pi_F} \mathfrak{a}^1(F),$$

where π_F is the orthogonal projection onto F, via $E \cong F \oplus F^{\perp}$. To show this, again one just have to prove this is compatible with the complex structure and the metric.

Let $\{e_i\}$'s be a frame with $\{e_j^*\}$'s be the dual frame, with connection matrix θ_{ij} for E and θ_{ij}^* for E^*, then

$$\begin{aligned} 0 &= d\delta_{ij} \\ &= d\left\langle e_i, e_j^* \right\rangle \\ &= \left\langle De_i, e_j^* \right\rangle + \left\langle e_i, D^* e_j^* \right\rangle \\ &= \left\langle \sum \theta_{ik} e_k, e_j^* \right\rangle + \left\langle e_i, \sum \theta_{jk}^* e_k^* \right\rangle \\ &= \theta_{ij} + \theta_{ji}^*. \end{aligned}$$

Therefore, $\theta^* = -^T\theta$. (Again, $\langle -,-\rangle$ is not a metric, but it is a linear pairing, so it is holomorphic in the second entry, not anti-holomorphic.)

Let $E = T'M$ and $E^* = (T')^*M$ be with a Hermitian metric on M. We have an operator $d = \partial + \bar{\partial} : A^{1,0}(M) \to A^{2,0}(M) \oplus A^{1,1}(M) = A^{2,0}(M) \oplus (A^{1,0}(M) \otimes A^{0,1}(M))$, and $D_{E*} : A^{1,0}(M) \to A^{1,0}(M) \otimes A^1(M) = (A^{1,0}(M) \otimes A^{1,0}(M)) \otimes (A^{1,0}(M) \otimes A^{0,1}(M))$. To see the connection between the two maps, recall that in a unitary frame, $\theta + ^T\bar{\theta} = 0$, then

Lemma 12.15. Let $\{\varphi_i\}$ be a unitary coframe, i.e., unitary frame for the cotangent bundl, as $ds^2 = \sum \varphi_i \otimes \bar{\varphi}_i$. Then there exists a unique matrix ψ_{ij} of 1-forms such that

- $\psi + ^T\bar{\psi} = 0$, and
- $\tau_i = d\varphi_i - \sum \psi_{ij} \otimes \varphi_j$ is a $(2,0)$-form.

The collection of τ_i's $(\tau_1, \dots, \tau_n)$ is called the torsion of connections.

Definition 12.16. A manifold M is Kahler if the torsion of connections $\tau = 0$.

Example 12.17. Consider M with the Euclidean metric $ds^2 = \sum dz_i \otimes d\bar{z}_i$, then $\varphi_i = dz_i$, so $d\varphi_i = 0$. Now $\psi = 0$ and $\tau = 0$, therefore this is Kahler. For instance, a complex torus $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ is Kahler in the Euclidean metric.

Let M be a Hermitian manifold on a Hermitian vector bundle E, with unit frame θ, thus $\theta + ^T\bar{\theta} = 0$, and that $T'M$ is Hermitian. Let φ_i's be a unitary coframe with $ds^2 = \sum d\varphi_i \otimes d\bar{\varphi}_i$.

Lemma 12.18. There exists a unique ψ_{ij} matrix of 1-forms such that $\psi + ^T\bar{\psi} = 0$. We have $d\varphi_i = \sum_j \psi_{ij} \wedge \varphi_j + \tau_i$ for $\tau_i \in A^{2,0}(M)$ for all i, via $\mathfrak{a}^{1,1} = \mathfrak{a}^{1,0} \otimes a^{0,1}$.

Conceptually, we have $\bar{\partial}\varphi_i = \sum \psi_{ij}'' \wedge \varphi_j$, where $\psi_{ij} = \psi_{ij}' + \psi_{ij}''$ as a sum of $(1,0)$- and $(0,1)$-forms, therefore ψ_{ij}'' is uniquely determined. Again, we say $\tau = (\tau_1, \dots, \tau_n)$ is the torsion.

Proof. Define ψ'' as in the equation $\psi_{ij} = \psi_{ij}' + \psi_{ij}''$, then $\psi' = -^T\psi''$. □

Definition 12.19. Let D and D' be the Chern connection on $T'M$ and $T'M^*$, respectively, with $(D^*)'' = \bar{\partial}TM^{*\prime}$.

For connection matrix θ, we have $\theta^{*\prime\prime} = \psi''$, therefore $\theta^* = \psi$ and thus $\theta = -^T\theta^* = -^T\psi$.

Example 12.20. Consider the Euclidean space with $ds^2 = \sum dz_i \otimes d\bar{z}_i$, with $\varphi_i = dz_i$, but writing down the decomposition tells us that $(\mathbb{C}^n, ds^2)$ is Kahler.

For any manifold M of dimension 1, we know it is trivial since $\tau \in A^{2,0}(M) = 0$.

Lemma 12.21. Let M be a complex manifold and E be a $\mathbb{C}$-vector space, then the set of connections of E is an affine space $A^1(\mathrm{Hom}(E, E))$.

Proof. Let D and D' be connections and s be a section, then $D(fs) = df \otimes s + fDs$ and $D'(fs) = df \otimes s + fD's$, therefore $(D - D')(fs) = f(D - D')(s)$, therefore $D - D' \in A^1(\mathrm{Hom}(E, E))$. Let D be any connection with $w \in A^1(\mathrm{End}(E))$, then $(D_w)s = Ds + w(s)$ with the pairing $A^1(\mathrm{End}(E) \times E) \to A^1(E)$. □

One often write ∇ in place of D, then for vector field X over M, we have $D_X(s) = \langle X, D(s)\rangle$ which acts as a section of E, therefore this is the idea of the directional derivative on a global form.

Example 12.22. Consider the trivial line bundle on $\mathbb{R}^2$, and let D be a connection, then the curvature $\theta = 0$ on the frame (1). For $w = Pdx + Qdy$, we can define $D_w = D + w$ and therefore $D_w f$ is a linear differential for f.

13 Hodge Decomposition

Recall we have $H^{p,q}_{\bar{\partial}}(M) = Z^{p,q}_{\bar{\partial}}(M)/\bar{\partial}A^{p,q-1}(M)$. We want to find "good" decompositions of elements of $H^{p,q}_{\bar{\partial}}(M)$. Using Harmonic forms, we have $\mathcal{H}^{p,q}_{\bar{\partial}}(M) \cong H^{p,q}_{\bar{\partial}}(M)$. The Harmonic forms satisfy $d\mathcal{H}^{p,q} = 0$ using Kahler's theory., This allows a decomposition

$$\begin{array}{ccc} \mathcal{H}^{p,q} & \longrightarrow & H^{p,q}_{dR}(M) \\ & \overset{\cong}{\searrow} & \uparrow \\ & & H^{p,q}(M) \end{array}$$

This induces the Hodge decomposition

$$H_dR^*(M) = \bigoplus_{p+q=n} H^{p,q}(M)$$

The harmonic forms are defined on real manifolds, so $\bar{\mathcal{H}}^{p,q} = \mathcal{H}^{q,p}$. Correspondingly, there is a decomposition of Harmonic k-forms. The Hodge numbers are therefore arranged in the Hodge diamond with $h^{p,q} = \dim(H^{p,q})$, such that

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{2,0} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

Note that this is symmetric with respect to the central column by complex conjugation, and the sum of each row gives the Betti number. For $H^q(\mathbb{P}^3, \Omega^p)$, we know the Betti number b_i is 1 is $0 \leqslant i \leqslant 2n$ is even, and 0 otherwise, therefore we recover the Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Let $V \subseteq M$ be an analytic subvariety of codimension K. The fundamental class $\eta_V \in H^{2k}_{dR}(M) \cong H^{2n-2k}_{dR}(M)^*$ can be defined via Poincaré duality by the mapping $\int_V w = \int_{V*} w \leftarrow w \in A^{2n-2k}$, where V^* is a complex manifold of dimension $n-k$. Note $w|_{V*} \in A^{n-k,n-k}$, then by Poincaré duality, $\eta_V \in H^{k,k}_{dR}(M)$.

Remark 13.1 (Hodge Conjecture)**.** Given $\eta_V \in H^{p,p}(M) \cap H^{2p}(M, \mathcal{D})$, then there exists an analytic subvariety V_i with $r_i \in \mathcal{D}$ such that $\eta = \sum r_i \eta_{V_i}$. This induces an analytic way to find algebraic subvarieties.

Remark 13.2. In the case where $\dim(M) = 1$, then M is Kahler, so $\mathbb{C}^{2g} \cong H^1(M, \mathbb{C}) = H^{1,0}(M) \oplus H^{0,1}(M)$. This gives $\dim(H^{1,0}(M)) = g$. One can also identify $H^{1,0}(M)$ to be the global holomorphic 1-forms. In fact, the harmonic 1-forms without $\bar{\partial}$ are just holomorphic forms.

Example 13.3. Let M be a Hermitian manifold and g be the associated Riemannian metric. Let $M \cong \mathbb{C}^n$ be with the Euclidean metric $\sum dz_j \otimes d\bar{z}_j$, then it is Riemannian with $\sum dx_j \otimes dx_j + \sum dy_j \otimes dy_j$. Its orthogonal frames are given by the partial derivatives $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\}$. The dual orthogonal frame for T^* is then given by $\{dx_j, dy_j\}$. Note $||dy_j||^2 = ||dx_j||^2 = \langle dx_j, dy_j \rangle = 1$. This gives a Hermitian metric after complexifying. Now

$$||dz_j||^2 = \langle dx_j + idy_j, dx_j + idy_j \rangle = 1 + 1 = 2.$$

Let M be a manifold with $\{\varphi_j\}$ as a unitary coframe, then $ds^2 = \sum \varphi_j \otimes \bar{\varphi}_j$, hence $||\varphi_j||^2 = 2 = ||\varphi_j||^2$.

Note $A^{p,q}(M)$ is an infinite-dimensional vector space with an inner product structure. An element ψ is now a global section of $\Lambda^p T' \otimes \Lambda^q T''$. We start with an inner product on $\Lambda^p T'_m \otimes \Lambda^q T''_m$ for any $m \in M$, then we have a frame

$$\left(\varphi_{i_1}(m) \wedge \cdots \wedge \varphi_{i_p}(m)\right) \otimes \left(\bar{\varphi}_{j_1}(m) \wedge \cdots \wedge \bar{\varphi}_{j_q}(m)\right)$$

as a basis. They are orthogonal with squared norm as 2^{p+q}.

Now let $\psi, \eta \in A^{p,q}(M)$, then $\langle \psi(m), \eta(n) \rangle \in \mathbb{C}$. Given a volume form $\Phi = \frac{\omega^n}{n!}$ where ω is the associated $(1,1)$-form, then

$$\langle \psi(m), \eta(n) \rangle \, \Phi$$

is an (n,n)-form and is integrable. This allows us to define an inner product

$$\langle \psi, \eta \rangle = \int_M \langle \psi(m), \eta(n) \rangle \, \Phi.$$

which is Hermitian since each term is Hermitian. Note that $||\psi||^2 = \langle \psi, \psi \rangle \geqslant 0$, and is 0 if and only if $\psi = 0$. This makes $A^{p,q}(M)$ into a pre-Hilbert space.

Let $\bar{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$, then $\bar{\partial}^* : A^{p,q+1}(M) \to A^{p,q}(M)$ be a formal adjoint of $\bar{\partial}$ with respect to metrics on $A^{p,q}$ and $A^{p,q+1}$, that is,

$$\left\langle \bar{\partial}, \bar{\psi}, \eta \right\rangle = \left\langle \psi, \bar{\partial}^* \eta \right\rangle$$

for all $\psi \in A^{p,q}$ and $\eta \in A^{p,q+1}$. We now want a canonical representation of $\varphi \in H^{p,q}_{\bar{\partial}}(M)$ by some $\psi \in Z^{p,q}_{\bar{\partial}}(M)$.

Lemma 13.4. ψ has minimal norm in its $\bar{\partial}$-cohomology class if and only if $\bar{\partial}^* \psi = 0$.

Proof. Suppose $\bar{\partial}^* \psi = 0$, then

$$\begin{aligned} ||\psi + \bar{\partial}\eta||^2 &= \left\langle \psi + \bar{\partial}\eta, \psi + \bar{\partial}\eta \right\rangle \\ &= \langle \psi, \psi \rangle + \left\langle \bar{\partial}\eta, \psi \right\rangle + \left\langle \psi, \bar{\partial}\eta \right\rangle + \left\langle \bar{\partial}\eta, \bar{\partial}\eta \right\rangle \\ &\geqslant \langle \psi, \psi \rangle + \left\langle \bar{\partial}\eta, \bar{\partial}\eta \right\rangle. \end{aligned}$$

Suppose ψ has minimal norm in the cohomology class, then $\operatorname{Re} \left\langle \eta, \bar{\partial}^* \psi \right\rangle = 0$ and $\operatorname{Im} \left\langle \eta, \bar{\partial}^* \psi \right\rangle = 0$, hence $\bar{\partial}^* \psi = 0$. ☐

Now ψ satisfies $\bar{\partial}\psi = 0$ and $\bar{\partial}^* \psi = 0$, then $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^* \bar{\partial} : A^{p,q}(M) \to A^{p,q}(M)$.

Definition 13.5. ψ is a harmonic (p,q)-form if $\Delta_{\bar{\partial}} \psi = 0$.

Lemma 13.6. ψ is harmonic if and only if $\bar{\partial}\psi = \bar{\partial}^* \psi = 0$.

Proof. Note that ψ is harmonic if and only if

$$\begin{aligned} 0 &= \langle \Delta_{\bar{\partial}} \psi, \psi \rangle \\ &= \left\langle (\bar{\partial}^* + \bar{\partial}^* \bar{\partial})\psi, \psi \right\rangle \\ &= \left\langle \bar{\partial}\bar{\partial}^* \psi, \psi \right\rangle + \left\langle \bar{\partial}^* \bar{\partial}\psi, \psi \right\rangle, \end{aligned}$$

if and only if $\left\langle \bar{\partial}^* \psi, \bar{\partial}^* \psi \right\rangle = 0$ and $\left\langle \bar{\partial}\psi, \bar{\partial}\psi \right\rangle = 0$, if and only if $\bar{\partial}\psi = \bar{\partial}^* \psi$. ☐

To construct $\bar{\partial}^*$, we need a Hodge $*$ operator characterized by

$$* : A^{p,q}(M) \to A^{n-p,n-q}(M)$$
$$\langle \varphi, *\eta \rangle = \int_M \varphi \wedge *\eta$$

where η is a global (p,q)-form. Therefore, $*$ is $\mathbb{C}$-anti-linear. In terms of coframes, we have

$$*((\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p}) \otimes (\bar{\varphi}_{j_1} \wedge \cdots \wedge \bar{\varphi}_{j_q})) = 2^{p+q-n} \varepsilon_{IJ} \varphi_{I^c} \otimes \bar{\varphi}_{J^c}$$

Here ε_{IJ} denotes the sign of the permutation

$$(1 \ \cdots \ n; 1 \ \cdots \ n) \to (I, I^c; J, J^c).$$

Now $**$ as an operator on $A^{p,q}(M)$ satisfies $** = (-1)^{p+q}$.

Definition 13.7. We define $\bar{\partial}^* = - * \bar{\partial} *$ via

$$A^{p,q} \xrightarrow{*} A^{n-p,n-q} \xrightarrow{\bar{\partial}} A^{n-p,n-q+1} \xrightarrow{*} A^{p,q-1}$$

Lemma 13.8. $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$.

Proof. Let $\varphi \in A^{p,q-1}$ and $\psi \in A^{p,q}(M)$, then

$$\int \bar{\partial}\varphi \wedge *\psi = \int \bar{\partial}(\varphi \wedge *\psi) + (-1)^{p+q} \int \varphi \wedge \bar{\partial} * \psi.$$

Since $\bar{\partial} = d$ on $\varphi \wedge *\psi$, a $(n, n-1)$-form, then the first term is zero. Therefore,

$$\begin{aligned} \int \bar{\partial}\varphi \wedge *\psi &= \int \bar{\partial}(\varphi \wedge *\psi) + (-1)^{p+q} \int \varphi \wedge \bar{\partial} * \psi \\ &= (-1)^{p+q} \int \varphi \wedge \bar{\partial} * \psi \\ &= - \int \varphi \wedge * * \bar{\partial} * \psi \\ &= \langle \varphi, - * \bar{\partial} * \psi \rangle \\ &= \langle \varphi, \bar{\partial}^* \psi \rangle \\ &= \langle \bar{\partial}\varphi, \psi \rangle . \end{aligned}$$

□

Theorem 13.9 (Hodge)**.** Let M be a compact Hermitian manifold, then

- $\dim(\mathcal{H}^{p,q}) < \infty$, and $\mathcal{H} : A^{p,q}(M) \twoheadrightarrow \mathcal{H}^{p,q}$ is an orthogonal projection;
- there exists a linear operator $G : A^{p,q}(M) \to A^{p,q}(M)$ such that $I = \mathcal{H} \oplus \Delta_\delta G$, $G|_{\mathcal{H}^{p,q}} \equiv 0$, and that $\mathcal{G}$ commutes with $\bar{\partial}$ and $\bar{\partial}^*$, i.e., with $\Delta_{\bar{\partial}}$.

Corollary 13.10. $A^{p,q}$ is the orthogonal direct sum of $\mathcal{H}^{p,q}$, $\bar{\partial}(A^{p,q-1}(M))$, and $\bar{\partial}^*(A^{p,q+1}(M))$.

Proof. Note that $\langle h, \bar{\partial}\varphi \rangle = \langle \bar{\partial}^* h, \varphi \rangle = 0$, $\langle h, \bar{\partial}^* \varphi \rangle = \langle \bar{\partial} h, \varphi \rangle = 0$, and $\langle \bar{\partial}\varphi, \bar{\partial}^* \psi \rangle = \langle \bar{\partial}^2 \varphi, \psi \rangle = 0$.
For any $\omega \in A^{p,q}$, we write $\omega = \mathcal{H}(\omega) + \Delta_{\bar{\partial}} G(w)$, then this is a direct sum

$$\mathcal{H}(\omega) + \bar{\partial}\partial^* G(\omega) + \bar{\partial}^* \bar{\partial} G(\omega)$$

of the three terms, as desired. □

Corollary 13.11. For $\eta = \Delta_{\bar{\partial}}\psi$, we may solve for $\psi \in A^{p,q}(M)$ if and only if $\mathcal{H}(\eta) = 0$. In this case, $\psi = G(\eta)$ is the unique solution with $\mathcal{H}(\psi) = 0$.

Proof. Suppose $\mathcal{H}(\eta) = 0$, then $\eta = \Delta_{\bar{\partial}} G(\eta)$, therefore $\psi = G(\eta)$ is a solution. Suppose we may solve for ψ, then $\eta = \Delta_{\bar{\partial}}\psi = \bar{\partial}\bar{\partial}^*\psi + \bar{\partial}^*\bar{\partial}\psi$ which gives a orthogonal decomposition only in two components, therefore this means $\mathcal{H}(\eta) = 0$.

To show that $\psi = G(\eta)$ is the unique solution, suppose $\mathcal{H}(\psi) = 0$, and $\Delta_{\bar{\partial}}\psi = \eta$, then $\eta = \mathcal{H}(\eta) + \Delta_{\bar{\partial}} G(\eta) = \Delta_{\bar{\partial}}\psi$, hence $G(\eta) - \psi \in \mathcal{H}^{p,q}$. Moreover, $\mathcal{H}(G(\eta) - \psi) = 0$, hence $G(\eta) = \psi = 0$. □

To prove Theorem 13.9, we need to solve $\Delta_{\bar{\partial}}\psi = \eta$ in L^2, and then prove that if $\eta \in A^{p,q}(M)$, then the solution $\psi \in A^{p,q}(M)$.

To show the second part, consider $ds^2 = \sum dz_i \otimes d\bar{z}_i$. Since $\bar{\partial}^* = - * \bar{\partial} *$ on $A^{0,0}(\mathbb{C}^n)$, then $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} = \bar{\partial}^*\bar{\partial}$. Now

$$\begin{aligned} \Delta_{\bar{\partial} f} &= \bar{\partial}^* \bar{\partial} f \\ &= \bar{\partial}^* (\sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j) \end{aligned}$$

$$\begin{aligned}
&= - * \bar{\partial} * (\sum f \bar{z}_j d\bar{z}_j) \\
&= 2^{1-n} * \bar{\partial} \sum \varepsilon \bar{f}_{\bar{z}_j} dz - 1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n \\
&= 2^{1-n} * \sum \varepsilon \frac{\partial \bar{f}_{\bar{z}_j}}{\partial \bar{z}_j} dz_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \\
&= 2 \sum \overline{\frac{\partial \bar{f}_{\bar{z}_j}}{\partial \bar{z}_j}} \\
&= 2 \sum \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j}.
\end{aligned}$$

Since $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$, then

$$\frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right).$$

Therefore $\Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d$. In general, this is true if M is Kahler.

Let us now discuss the Hodge theorem for real Laplacian Δ_d. The proof for complex Laplacian would be the same up to some coefficient. Therefore, set $M = (\mathbb{R}/2\pi\mathbb{Z})^n$ with local coordinates $(x_1, \dots, x_n)$ and orthonormal (unitary) frame $\{dx_i\}$. Consider the Fourier coefficients $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{Z}^n$, then $e^{i\xi\cdot x}$ is periodic with period 2π, and therefore the set $\{e^{i\xi\cdot x}\}$ is pairwise orthogonal since

$$\left\langle e^{i\xi\cdot x}, e^{i\xi'\cdot x}\right\rangle = \frac{1}{(2\pi)^n}\int e^{i(\xi-\xi')\cdot x} dx_1 \cdots dx_n = 0.$$

If $\varphi \in C^\infty(M)$ has a Fourier expansion (which it does for C^∞ functions)

$$\varphi(x) = \sum \varphi_\xi \cdot e^{i\xi\cdot x},$$

then $||\varphi||_{L^2}^2 = \sum |\varphi_\xi|^2$ by the orthogonality calculation, which is called Parseval's identity. This gives the Sobolev space $H_s \subseteq \{\varphi_\xi : \xi \in \mathbb{Z}^n\}$, which is a Hilbert space

$$H_s = \{\varphi_\xi : \sum_\xi (1 + |\xi|^2)^s |\varphi_\xi|^2 < \infty\},$$

with

$$\langle \varphi, \psi \rangle_s = \sum (1 + |\xi|^2)^s \varphi_\xi \bar{\psi}_\xi.$$

Therefore, $H_s \subseteq H_r$ if $s > r$, and $H_0 = L^2(T)$. In general, we have Fourier series

$$G(\varphi)_\xi = \begin{cases} \frac{1}{||\xi||^2}\varphi_\xi, & \xi \neq 0 \\ 0, & \xi = 0 \end{cases}$$

that is, $G(\varphi) = \sum_{\xi\neq 0} \frac{\varphi_\xi}{||\xi||^2} e^{i\xi x}$. If $\varphi \in C^\infty$,then $G(\varphi) \in C^\infty$, with Laplacian $\Delta_d = -\sum \frac{\partial^2}{\partial x_i^2}$. The Fourier coefficients

$$\begin{aligned}
(\partial_j \varphi)_\xi &= \frac{1}{(2\pi)^n}\int \partial_j \varphi e^{-i\xi x} dx \\
&= -\frac{1}{(2\pi)^n}\int (-i\xi_j)\varphi e^{-i\xi x} dx \\
&= i\xi_j \varphi_\xi.
\end{aligned}$$

So $(\partial_j^2 \varphi)_\xi = -\xi_j^2 \varphi_\xi$ and so $(\Delta_d \varphi)_\xi = ||\xi||^2 \varphi_s$.

Definition 13.12. Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator on a general Hilbert space $\mathcal{H}$. We say T is bounded if for any $\psi \in \mathcal{H}$, $||T(\psi)|| \leqslant C||\psi||$.

Note that the unit ball is not compact if $\dim(\mathcal{H}) = \infty$. For instance, $\{\tau^{i\xi x}\}$ has no limit points.

Definition 13.13. A bounded operator T is said to be compact if applying T on a bounded sequence has a convergence subsequence.

Example 13.14. Suppose $T : \mathcal{H} \to V \subseteq \mathcal{H}$ where $\dim(V) < \infty$.

Definition 13.15. We say T is self-adjoint if for any $\varphi, \psi \in \mathcal{H}$, we have $\langle T\varphi, \psi\rangle = \langle \varphi, T\psi\rangle$.

Definition 13.16. We say T is positive if $\langle T\varphi, \varphi\rangle \geqslant 0$.

Theorem 13.17. If T is a compact, positive, self-adjoint operator, then $\mathcal{H} = \bigoplus V_j$ where each V_j is an eigenspace for T with eigenvalue $\lambda_i \geqslant 0$.

Lemma 13.18. We have $C^s(T) \subseteq H_s(T)$ for any $s \geqslant 0$.

Lemma 13.19 (Sovolev)**.** $H_{s+\lfloor \frac{n}{2}\rfloor+1}(T) \subseteq C^s(T)$. In particular, the intersection $\bigcap\limits_{s\geqslant 1} H_s = C^\infty(T)$. In particular, $\bigcap\limits_{s\geqslant 1} \mathcal{H}_s^{p,q}(M) = A^{p,q}(M)$, as $\mathcal{H}_s^{p,q}(M)$ is the completion of $A^{p,q}(M)$ in global SObolsev s-norm.

Lemma 13.20 (Rellich)**.** The inclusion $\mathcal{H}_{s+2}^{p,q} \subseteq \mathcal{H}_s^{p,q}$ is a compact embedding.

The idea being, for $G : H_s \to H_{s+2}$ as a smoothing operator, we have a bound

$$\begin{aligned} ||G\varphi||_{s+1}^2 &= \sum_\xi (1+||\xi||^2)^{s+2}|(G\varphi)_\xi|^2 \\ &= \sum_{\xi\neq 0}(1+|\xi|^2)^s\frac{(1+||\xi||^2)^2}{||\xi||^4}|\varphi_\xi|^2 \\ &\leqslant 4||\varphi||_s^2. \end{aligned}$$

If $\varphi \in C^\infty = \bigcap\limits_{s\geqslant 0} H_s$, then $G(\varphi) = \bigcap\limits_{s\geqslant 0} H_{s+2} = C^\infty$.

For a compact Hermitian manifold M, the Hodge theorem for Hermitian vector bundle E, e.g., $E = TM$, gives $A^{p,q}(M)$ as the C^∞ sections of $\Lambda^p T' \otimes \Lambda^q T' \otimes E$. To give this a norm, we have to introduce connections using the Sobolev s-norm

$$||\psi||_s^2 = ||\psi||^2 + ||\nabla\psi||^2 + \cdots + ||\nabla\psi||^2.$$

Theorem 13.21 (Riesz Representation Theorem)**.** Let $\mathcal{H}$ be a Hilbert space, and $T : \mathcal{H} \to \mathbb{C}$ be a bounded linear operator, then there exists a unique element $\psi \in \mathcal{H}$ such that $T(\varphi) = \langle \varphi, \psi\rangle$.

Definition 13.22. The Dirichlet norm is defined by $\mathcal{D}(\psi) = \langle (I+\Delta)\psi, \psi\rangle = \langle \psi, \psi\rangle + \langle \bar\partial\psi, \bar\partial\psi\rangle + \langle \bar\partial^*\psi, \bar\partial^*\psi\rangle$. This also gives the function space a Hilbert space structure. Similarly, there is a Dirichlet inner product $\mathcal{D}\langle \psi, \psi'\rangle$.

Theorem 13.23 (Gårding's inequality)**.** $||\varphi||_1^2 \leqslant \mathcal{D}(\varphi)$. In fact, the two norms $||\cdot||$ and $\mathcal{D}$ are equivalent, meaning they define the same topology.

We want to invert $I+\Delta$ to solve $(I+\Delta)\psi = \varphi$ for ψ, so we should solve it weakly first. That is, for $\eta \in A^{p,q}(M)$, we want $\langle \eta, \varphi\rangle = \langle (I+\Delta)\eta, \psi\rangle$. To estimate this, we have

$$|\langle \eta, \varphi\rangle| \leqslant ||\eta||_0||\varphi||_0 \leqslant C||\varphi||_0^2\mathcal{D}(\eta).$$

Therefore, $\varphi \mapsto \langle \eta, \varphi\rangle$ is a bounded linear function on $\mathcal{H}_1^{p,q}$ using the equivalent $\mathcal{D}$-norm. There exists a unique $\psi \in \mathcal{H}_1^{p,q}$ such that $\langle \eta, \varphi\rangle = \mathcal{D}\langle \eta, \psi\rangle = \langle (1+\Delta_{\bar\gamma}\eta, \psi\rangle$. Therefore, we solved the equation $(I+\Delta)\psi = \varphi$ weakly.

Now linear operator $T : \mathcal{H}_0^{p,q}(M) \to \mathcal{H}_1^{p,q}(M)$ with $T(\varphi) = \psi$ means that $(I+\Delta)\psi = 0$ weakly. In particular, T is bounded. Just like in the local case, we have

$$\mathcal{H}_0^{p,q}(M) \xrightarrow{T} \mathcal{H}_1^{p,q}(M) \hookrightarrow \mathcal{H}_0^{p,q}(M)$$

as a self-adjoint positive operator. By the spectral theorem, $\mathcal{H}_0^{p,q}(M) = \bigoplus V_{\rho_i}$ for $\rho_i > 0$. Therefore, there is no kernel. In particular, $\dim(V_{\rho_i}) < \infty$ with $1 \geqslant \rho_1 > \rho_2 > \cdots$. By compactness, $\lim\limits_{i\to\infty}\rho_i = 0$.

For $\varphi \in V_{\rho_i}$, let $T(\varphi) = \rho_i \varphi$. Then $\langle (I+\Delta)\eta, \rho_i\varphi\rangle = \langle \eta, \varphi\rangle$, so $\langle \Delta\eta, \rho_i\varphi\rangle = \langle \eta, (1-\rho_i)\varphi\rangle$. Therefore, $\langle \Delta\eta, \varphi\rangle = \left\langle \eta, \frac{1-\rho_i}{\rho_i}\varphi\right\rangle$, so $\Delta\varphi = \frac{1-\rho_i}{\rho_i}\varphi$ is a weak solution. Locally,

$$G|_{W_{\rho_i}} = \begin{cases} \frac{\rho_i}{1-\rho_i}\,\mathrm{Id}, & \rho_i \neq 1 \\ 0, & \rho_i = 1 \end{cases}.$$

In case where $\rho_i = 1$, we have $\lambda_i = 0$.

Lemma 13.24 (Regularity Lemma)**.** Suppose $\Delta\psi = \varphi$ is a weak solution for $\varphi \in \mathcal{H}_s^{p,q}(M)$, then $\psi \in \mathcal{H}_{s+2}^{p,q}(M)$.

First suppose $\Delta\psi = 0$ is given by a weak harmonic form $\psi \in \mathcal{H}_s^{p,q}$, then since $0 \in \mathcal{H}_s^{p,q}$, then $\psi \in \mathcal{H}_{s+2}^{p,q}$, so $\psi \in \bigcap_{s\geqslant 1} \mathcal{H}_s$, which means $\psi \in C^\infty$. If $\varphi \in A^{p,q}(M) \subseteq \mathcal{H}_s^{p,q}(M)$, then $\psi = G(\varphi)$ satisfies $\Delta\psi = \varphi$ weakly, hence $\psi \in \mathcal{H}_{s+2}^{p,q}$ for all s, that is, $\psi \in A^{p,q}(M)$. This proves the Hodge theorem.

Corollary 13.25. The Hodge star operator commutes with the Laplacian: $*\Delta = \Delta *$. In particular, $* : \mathcal{H}^{p,q} \to \mathcal{H}^{n-p,n-q}$ is an anti-linear isomorphism.

Corollary 13.26. Note $\mathcal{H}^{p,q} \cong \mathcal{H}_{\bar\partial}^{p,q}(M)$ sends $h \mapsto [h]$, and its inverse is defined by $\omega \mapsto \mathcal{H}(\omega)$.

Corollary 13.27. The Hodge star operator $* : \mathcal{H}^{p,q}(M) \to \mathcal{H}^{n-p,n-q}$ is $\mathbb{C}$-antilinear, and restricts to an isomorphism

$$\begin{aligned} * : \mathcal{H}^{0,0} &\to \mathcal{H}^{n,n} \cong \mathbb{C}\cdot\Phi \\ \mathbb{C}(1) &\mapsto \mathbb{C}(\Phi) \end{aligned}$$

where Φ is the volume form.

Corollary 13.28 (Serre Duality)**.** We have an isomorphism $H^q(M,\Omega^p) \to H^{n-q}(M,\Omega^{n-p})^*$. Here $\mathcal{H}^{p,q} \cong H^q(M,\Omega^p)$.

Note that we have a commutative diagram and a pairing

$$\begin{array}{ccc} H^q(M,\Omega^p) \times H^{n-q}(M,\Omega^{n-p}) & \longrightarrow & H^n(M,\Omega^n) \xrightarrow{\cong} \mathbb{C} \\ & \searrow^{\smile} \quad \nearrow_{\wedge} & \\ & H^n(M,\Omega^p \otimes \Omega^{n-p}) & \end{array}$$

To check the isomorphism, we first need to show that it is well-defined. That is, for $\omega \in A^{n,n}(M)$, we have $[\omega] \mapsto \int_M \omega$ and for $\eta \in A^{n,n-1}(M)$l, we have $[\bar\partial\eta] \mapsto \int -M\bar\partial\eta = \int_M d\eta = 0$. To show this is an isomorphism, we note that $\Phi \in \mathcal{H}^{n,n} \cong H_{\bar\partial}^{n,n}(M)$, so $\int_M \Phi = \mathrm{vol}(M) > 0$.

Corollary 13.29. Serre duality gives a perfect pairing.

Indeed, we have $(\psi, *\psi) \mapsto \int_M \psi \wedge *\psi = ||\psi||^2 = 0$, therefore $\psi = 0$.

Corollary 13.30 (Künneth Formula)**.** Let M and N be Hermitian manifolds, then we have a canonical isomorphism $H^q(M\times N, \Omega^p) \cong \bigoplus_{\substack{p'+p''=p \\ q'+q''=q}} (H^{q'}(M,\Omega^{p'}) \otimes H^{q''}(M,\Omega^{p''}))$

To prove this, we look at the decomposable forms on $M \times N$, given by $\pi_M^*\psi \wedge \pi_N^*\eta$. This depends on the metric. We have

$$\Delta_{M\times N}(\pi_M^*\psi \wedge \pi_N^*\eta) = \pi_M^*\Delta_M\psi \wedge \pi_N^*\eta + \pi_M^*\psi \wedge \pi_N^*\Delta_N\eta,$$

and the set of decomposable forms is dense in L^2-metric of the space of all forms. Given these information, the decomposable harmonic forms $\Delta_M\psi = \lambda\psi$ and $\Delta_N\eta = \alpha\eta$ induces $\Delta_{M\times N}(\pi_M^*\psi \wedge \pi_N^*\eta) = (\alpha+\lambda)\pi_M^*\psi \wedge \pi_N^*\eta$, so $\alpha + \eta = 0$, and $\alpha = \eta = 0$. By density argument, $\pi_M^*\psi \wedge \pi_N^*\eta$ are all possible eigenfunctions of $\Delta_{M\times N}$, so $\mathcal{H}^{p,q}(M\times N)$ is spanned by decomposable forms.

14 Kahler Maniflolds

Proposition 14.1. Let M be a Hermitian manifold, then the following are equivalent.

1. Unitary coframe $\tau = 0$.
2. $d\omega = 0$.
3. locally near any $z_0 \in M$, there exists coordinates $(z_1, \ldots, z_n)$ centered at z_0 such that
$$ds^2 = \sum (\delta_{ij} + g_{ij}(z)) dz_i \wedge d\bar{z}_j$$
where $g(z_0) = dg(z_0) = 0$, and the metric is Euclidean up to order 2.

Definition 14.2. If any of the conditions above is satisfied, we say M is a Kahler manifold.

Example 14.3.

- $\mathbb{C}^n$ with Euclidean metric;
- $\mathbb{C}^n/\Lambda$ for lattice Λ;
- $\mathbb{P}^n$ with Fubini-Study metric $\omega = \frac{i}{2\pi}\partial\bar{\partial}\log||z||^2$; alternatively, we may choose $\omega = dd^c \log||z||^2$ where $d^c := \frac{i}{4\pi}(\bar{\partial} - \partial)$, then $\frac{i}{4\pi}(\partial + \bar{\partial})(\bar{\partial} - \partial)\log||z||^2 = \frac{i}{4\pi}(\partial\bar{\partial} - \bar{\partial}\partial)\log||z||^2 = \frac{i}{2\pi}\partial\bar{\partial}\log||z||^2$.
- For any submanifold S of a Kahler manifold M. We choose $\omega_S = \omega_M|_S$, then $d\omega_S = d\omega_M|_S = 0$, so by the proposition, we note that S is Kahler as well.
- If M and N are Kahler, then so is $M \times N$.

Proof.

- 1. $\Leftrightarrow$ 2. Choose a unitary coframe $\{\varphi_j\}$, then $d\varphi_j = \sum \psi_{ij} \wedge \wedge \varphi_j + \tau_j$ with $\psi + {}^T\bar{\psi} = 0$. Now
$$d\left(\frac{2\pi}{i}\omega\right) = \sum (\psi_{ij} \wedge \varphi_j + \tau_j) \wedge \bar{\varphi}_j - \varphi_j \wedge (\bar{\psi}_j \wedge \bar{\varphi}_i + \bar{\tau}_j) = \sum \tau_j \wedge \bar{\varphi}_j + \varphi_j \wedge \bar{\tau}_j.$$
This is a sum of a $(2,1)$-form with a $(1,2)$-form. Therefore, this is 0 if and only if $d\omega = 0$.
- 3. $\Rightarrow$ 2.: set $\omega = \frac{i}{2}\sum(\delta_{ij} + g_{ij}) dz_i \wedge d\bar{z}_j$, then $d\omega(z_0) = 0$, so $d\omega = 0$.
- 2. $\Rightarrow$ 3.: without loss of generality, consider the coordinates $ds^2 = \sum\limits_{i,j,k} (\delta_{ij} + a_{ijk}z_k + a_{ij\bar{k}}\bar{z}_k + \ell_{ij}(z)) dz_i \otimes d\bar{z}_j$ as a first-order Taylor approximation, where $\ell_{ij}(0) = d\ell_{ij}(0) = 0$. Since the metric is Hermitian, then $a_{ij\bar{k}} = \bar{a}_{jik}$. Since $d\omega = 0$, then $a_{ijk} = a_{kji}$. To prove this, one should look at the expansion of coefficients of $dz_i \wedge dz_k \wedge d\bar{z}_j$ in dw. By a change of coordinates $z_i = w_i + \sum\limits_{j,k} b_{ijk} w_j w_k$, then $b_{ijk} = b_{ikj}$ and $b_{jki} = -a_{ijk}$, then ds^2 has the desired form in the coordinates with respect to w.

□

Theorem 14.4. Suppose M is a compact Kahler manifold.

1. The even Betti numbers $b_{2i}(M)$ are positive for $0 \leqslant i \leqslant n$.
2. The global holomorphic forms $H^0(M, \Omega^q) \hookrightarrow H^q_{\mathrm{dR}}(M0)$ injects into the de Rham cohomology.
3. Let $V \subseteq M$ be a non-empty analytic subvariety, then the fundamental class $\eta_V \neq 0 \in H^{2n-2k}_{\mathrm{dR}}(M)$ where $\dim(V) = k$.

Proof. For $0 \leqslant i \leqslant n$, then $d(\omega^i) = 0$ as M is Kahler, hence

$$0 \neq n! \operatorname{vol}(M) = \int_M \omega^n.$$

Therefore, $[\omega^i] \in H^{2i}_{\text{dR}}(M)$. Suppose, towards contradiction, that $\omega^i = d\psi$, then

$$\int_M \omega^n = \int d\psi \wedge \omega^{n-i} = \int d(\psi \wedge \omega^{n-i}) = 0,$$

contradiction. Therefore, $[\omega^i] \neq 0$.

Now let $\eta \in H^0(M, \Omega^q)$ be a holomorphic q-form, and choose a unitary coframe $\{\varphi_i\}$, then we can write $\eta = \sum_{|I|=q} \eta_I \varphi_I$. We look at the integral $\int_M \eta \wedge \bar{\eta} \wedge \omega^{n-q}$ of the top form, then by expanding the wedge, we get $C \int (\sum \eta_I \bar{\eta}_J \varphi_I \wedge \bar{\varphi}_J) \wedge (\sum (\varphi_i \wedge \bar{\varphi}_i))^{n-q}$, but after reordering, we have

$$C' \int \sum |\eta_I|^2 \varphi_1 \wedge \bar{\varphi}_1 \wedge \cdots \wedge \varphi_n \wedge \bar{\varphi}_n.$$

Suppose $\eta = d\psi$ is exact, then again this is the same as $\int d\psi \wedge d\bar{\psi} \wedge \omega^{n-q} = \int d(\psi \wedge d\bar{\psi} \wedge \omega^{n-q}) = 0$. Hence, $\eta_I = 0$ since the wedge sum above is a multiple of the volume form, therefore $\eta = 0$. Therefore, any non-zero form is not exact.

Now using the same argument again, suppose $d\eta = \partial\eta \in H^0(\Omega^{q+1})$ since $\bar{\partial}\eta = 0$, but any exact form must be zero, so $d\eta = \partial\eta = 0$. We conclude that η is closed. Therefore,

$$H^0(M, \Omega^q) \to H^q_{\text{dR}}(M)$$

since every holomorphic form is closed and represent a cohomology class, and the injectivity follows from the exactness.

We have $k \operatorname{vol}(V) = \int_V \omega^k = \langle \eta_V, \omega^k \rangle$, with $\eta_V \in H^{2n-2k}$ and $\omega^k \in H^{2k}$, therefore $\eta_V \neq 0$. □

Let M be a compact Kahler manifold, then we will show that $\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$. Therefore, holomorphic forms are compatible with decompositions into types. We define an operator

$$\begin{aligned} L : A^{p,q}(M) &\to A^{p+1,q+1}(M) \\ \eta &\mapsto \eta \wedge \omega. \end{aligned}$$

Lemma 14.5. Define

$$\Lambda : A^{p,q}(M) \to A^{p-1,q-1}(M)$$

to be $(-1)^{p+q} * L*$, then this is the adjoint of L.

Proof. Consider $\psi \in A^{p,q}(M)$ and $\eta \in A^{p-1,q-1}(M)$, therefore

$$\begin{aligned} \langle L\eta, \psi \rangle &= \int_M (\eta \wedge \omega) \wedge *\psi \\ &= \int_M \eta \wedge (-1)^{p+q} * *(\omega \wedge *\psi) \\ &= \left\langle \eta, ((-1)^{p+q} * L*)\psi \right\rangle. \end{aligned}$$

□

Proposition 14.6 (Kahler Hodge Identities). Let $d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$, then $[\Lambda, d] = -4\pi(d^c)^* = -4\pi(-\frac{i}{4\pi}(\bar{\partial}^* - \partial^*)) = -i(\bar{\partial}^* - \partial^*)$. The first equality is known for compact Kahler manifolds.

We see that

$$A^{p,q}(M) \xrightarrow{d} (A^{p+1,q} \oplus A^{p,q+1})(M) \xrightarrow{\Lambda} (A^{p,q+1} \oplus A^{p-1,q})(M).$$

Equivalently, we can check the maps separately, where $[\Lambda, \bar{\partial}] = -i\partial^*$ and $[\Lambda, \partial] = i\bar{\partial}^*$. Moreover,

$$[\Lambda, d]^* = [d^*, \Lambda^*] = -[\Lambda^*, d^*] = -[L, d^*],$$

and

$$(-4\pi(d^c)^*) = -4\pi d^c,$$

therefore the Kahler identity $[\Lambda, d] = -4\pi(d^c)^*$ is equivalent to $[L, d^*] = 4\pi d^c$.

Lemma 14.7. $[L, \Delta_d] = 0$.

Proof. Equivalently, we just need to show that the adjoint $[\Lambda, \Delta_d] = 0$ since the Laplacian is self-adjoint. First note that L commutes with d, i.e., $[L, d] = 0$. This is because $Ld\eta = d\eta \wedge \omega = d(\eta \wedge \omega) = dL\eta$ since the form is closed. We have

$$\begin{aligned} \Lambda\Delta_d &= \Lambda(dd^* + d^*d) \\ &= (\Lambda - 4\pi(d^c)^*)d^* + d^*\Lambda d \\ &= d\Lambda d^* + d^*\Lambda d - 4\pi(d^c)^*d^* \\ &= \Delta_d\Lambda. \end{aligned}$$

□

Remark 14.8. $(d^c)^*d^* \neq 0$ since its adjoint dd^c is non-zero.

Suppose we know that $[\Lambda, \bar{\partial}] = -i\partial^*$ and $[\Lambda, \partial] = i\bar{\partial}^*$, then $[\Lambda, \Delta_d] = 0$. Note that $\bar{\partial}\partial^* + \partial^*\bar{\partial} = 0$, then $-i(\bar{\partial}\partial^* + \partial^*\bar{\partial}) = 0$ as well, which means $\bar{\partial}(\Lambda\bar{\partial} - \bar{\partial}\Lambda) - (\Lambda\bar{\partial} - \bar{\partial}\Lambda)\bar{\partial} = 0$, so by expansion we have

$$\begin{aligned} \Delta_d &= (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) \\ &= \Delta_\partial + \Delta_{\bar{\partial}} + 0 \\ &= 0. \end{aligned}$$

Therefore, $\Delta_d = \Delta_\partial + \Delta_{\bar{\partial}}$, and it suffices to show that $\Delta_\partial = \Delta_{\bar{\partial}}$. This is true because

$$\begin{aligned} i\Delta_{\bar{\partial}} &= i(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) \\ &= \bar{\partial}(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)bar\partial \\ &= i(\partial\partial^* + \partial^*\partial) \\ &= i\Delta_\partial \end{aligned}$$

as $\partial\bar{\partial} = -\bar{\partial}\partial$. It remains to show the two fact we supposed at the start. To see $[\Lambda, \bar{\partial}] = -i\partial^*$ on the Euclidean space, we may write down the metric with form $\eta = \sum f_{IJ} dz_I \wedge d\bar{z}_J$, and let us take compactly-supported forms

$$\begin{aligned} e_i : A_c^{p,q}(\mathbb{C}^n) &\to A_c^{p+1,q}(\mathbb{C}^n) \\ \eta &\mapsto dz_i \wedge \eta \end{aligned}$$

and

$$\begin{aligned} \bar{e}_i : A_c^{p,q}(\mathbb{C}^n) &\to A_c^{p+1,q}(\mathbb{C}^n) \\ \eta &\mapsto d\bar{z}_i \wedge \eta \end{aligned}$$

Similarly, in the general case, we may write down the unit coframe instead using the Kahler conditions, so we have $d\varphi_i(z_0) = 0$ locally. For any $(z_1, \dots, z_n)$ and $\eta = \sum f_{IJ}\varphi_I \wedge \bar{\varphi}_J$, then we may define

$$\begin{aligned} e_i : A^{p,q}(U) &\to A^{p+1,q}(U) \\ \eta &\mapsto d\varphi_i \wedge \eta \end{aligned}$$

and

$$\begin{aligned}\bar{e}_i : A^{p,q}(U) &\to A^{p+1,q}(U)\\ \eta &\mapsto d\bar{\varphi}_i \wedge \eta\end{aligned}$$

instead on some local subset U of M. Note that even if we have a global Kahler metric, we are choosing the coordinates locally, and therefore a locally-defined coframe. Now $L = \frac{i}{2}\sum e_j\bar{e}_j$ on both cases by the same calculations, and the adjoints $\iota_j = e_j^*$ and $\bar{\iota}_j = \bar{e}_j^*$ in both cases. We then calculate that

$$i_k(dz_I \wedge d\bar{z}_J) = 0$$

if $k \notin I$; in the case where $k \in I$, then this is $i_k(dz_K \wedge dz_I \wedge d\bar{z}_J) = 2dz_I \wedge d\bar{z}_J$. Therefore, $\Lambda = i^* = -\frac{i}{2}\sum \bar{i}_k i_k$. We then calculate $\partial_k(\eta) = \sum \frac{\partial f_{IJ}}{\partial z_K} dz_I \wedge d\bar{z}_J$. Similarly,

$$i_k(\varphi_I \wedge \bar{\varphi}_J) = 0$$

if $k \notin I$, and $i_k(\varphi_k \wedge \varphi_I \wedge \bar{\varphi}_J) = 2\varphi_I \wedge \bar{\varphi}_J$, and that $\Lambda = -\frac{i}{2}\sum \bar{i}_k i_k$. We then calculate $\partial_k(\eta) = \sum \frac{\partial f_{IJ}}{\partial z_K} d\varphi_I \wedge d\bar{\varphi}_J$.

However, the difference being, in the Euclidean case we have $\partial = \sum \partial_k e_k$ and $\bar{\partial} = \sum \bar{\partial}_k \bar{e}_k$, but in the general case, we have

$$\begin{aligned}\bar{\partial}(f\varphi_k) &= \bar{\partial} f \varphi_k + f\bar{\partial}\varphi_k\\ &= \sum \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k \wedge d\varphi_k + f\bar{\partial}\varphi_k\\ &= \sum \bar{\partial}_l \bar{e}_k\end{aligned}$$

where the second term vanishes at z_0. Finally, we check that they anti-commute.

Let us now reformulate Hodge's theorem. For the d-closed forms $Z_d^{(p,q)}(M)$, we have a map $Z_d^{(p,q)}(M) \to H_{\mathrm{dR}}^{p+q}(M)$. Since the kernel is just the exact (p,q)-forms, we may mod out the kernel $Z^{p,q}(M) \cap dA^{p+q-1}(M)$. Therefore, we identify the quotient with the image $H^{p,q}(M)$. This is now defined without the metric present.

Theorem 14.9. Let M be a compact Kahler manifold, then the natural map

$$\bigoplus_{p+q=k} H^{p,q}(M) \cong H_{dR}^k(M)$$

is an isomorphism. Furthermore, $\mathcal{H}^{p,q}(M) \cong H^{p,q}(M)$. Moreover, for any $\eta \in \mathcal{H}^k(M)$, we write $\eta = \sum\limits_{p+q=k} \eta^{p,q}$ for $\eta^{p,q} \in A^{p,q}(M)$, then η is harmonic if and only if $\eta^{p,q}$'s are all harmonic. Finally, the Hodge structure gives $H^{p,q}(M) = \bar{H}^{q,p}(M)$ on the vector spaces of cohomology.

Example 14.10. Consider the Hodge diamond for $n = 2$.

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2}\\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

We have $h^{0,0} = 1$, and by Serre duality we know that $H^q(\Omega^p) \cong H^{n-q}(\Omega^{n-p})^*$, and by complex conjugacy we know it mirrors across vertical axis, so it really looks like

$$\begin{array}{ccccc} & & 1 & & \\ & h^{1,0} & & h^{1,0} & \\ h^{2,0} & & h^{1,1} & & h^{2,0}\\ & h^{1,0} & & h^{1,0} & \\ & & h^{0,0} & & \end{array}$$

and we know it adds up to the corresponding Betti number on each row.

We have commutator $[L,\Lambda]: A^k(M) \to A^k(N)$ as $h = (n-k)\cdot\mathrm{id}$. The commutator $[h,L] = hL - Lh = (n-k-2)L - L(n-k) = 2L$. Similarly, $[h,\Lambda] = 2\Lambda$. This is the representation of $sl(2,\mathbb{C})$, which is the set of traceless 2×2-matrices with basis elements

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

such that $[E,F] = H$, $[H,E] = 2F$, and that $[H,F] = -2F$. Therefore, we have an assignment

$$\begin{aligned} \mathfrak{sl}_2(\mathbb{C}) &\to \mathrm{End}(H^*_{\mathrm{dR}}(M)) \\ H &\mapsto h \\ E &\mapsto \Lambda \\ F &\mapsto L \end{aligned}$$

Now finite-dimensional representations of $\mathfrak{sl}_2$ are classified, so this will give us the hard Lefschetz theorem for compact Kahler manifolds.

15 Representations and Lefschetz Theorem

Definition 15.1. A representation of $\mathfrak{sl}_2$ is irreducible if it has no proper subrepresentations.

Example 15.2. We have

$$\begin{aligned}\mathfrak{sl}_2 &\to \mathrm{End}(\mathbb{C}^2)\\ A\cdot v &\mapsto av\end{aligned}$$

An important fact being, if $W \subseteq V$ is an irreducible subrepresentation, then $V \sim_{\mathfrak{sl}_2} W \oplus W^{\perp}$, therefore the finite-dimensional $\mathfrak{sl}_2$ representations decompose as direct sums of irreducible representations. The key idea in the proof is to use weight spaces, i.e., H-eigenspaces $V_\lambda = \{v \in V : Hv = \lambda v\}$. Since $[H, E] = 2E$ and $[H, F] = -2F$, we see $E : V_\lambda \to V_{\lambda+2}$ and $F : V_\lambda \to V_{\lambda-2}$. Thus, there exists eigenvectors $v \in V$ such that $Ev = 0$. For instance, take $\omega \in V_\mu$ such that $E^{k+1}_\omega = 0$, so let $v = E^k_\omega$.

Definition 15.3. We say v is primitive or of highest weight if v is an H-eigenvector and $Ev = 0$.

Lemma 15.4. Let V be irreducible and $v \in V$ be primitive, then $V = \mathrm{span}(v, Fv, F^2v, \ldots)$.

Proof. Let W be the span on the right-hand side, so it suffices to show that W is invariant under $\mathfrak{sl}_2$. Clearly it is invariant under $F + H$. For v, we show $EF^kv \in W$ by induction on k. We have $Ev = 0$ and $EEF^kv = (FE + H)F^{k-1}v = FEF^{k-1}v + HF^{k-1}v \in W$ by induction. □

Since V is of finite-dimensional, then there exists n such that $F^nv \neq 0$ but $F^{n+1}v = 0$, then $EFv = (FE+H)v = \lambda_v$ and $EF^kv = (k\lambda - k^2 + k)F^{k-1}v$ by induction, therefore $EF^{n+1}v = 0 = ((n+1)\lambda - (n+1)^2 + (n+1))F^nv$, therefore $\lambda = n$. We conclude that $V = V_n \oplus V_{n-2} \oplus V_{n-4} \oplus \cdots \oplus V_{-n}$ for some $n \in \mathbb{Z}$, where E and F move between the weight spaces. Moreover, E^k and F^k are isomorphisms for any finite-dimensional representations of $\mathfrak{sl}_2$. We summarize our results as follows:

Theorem 15.5 (Lefschetz Decomposition)**.** If V is a finite-dimensional representation of $\mathfrak{sl}_2$ and $PV := \ker(E) \subseteq V$, then $V = PV \oplus FPV \oplus F^2PV \oplus \cdots$, and $\ker E \cap V_n = \ker(F^{n+1}|_{V_n}) : V_n \to V_{n-2}$.

Theorem 15.6 (Hard Lefschetz)**.** Let M be a compact Kahler manifold, then $L^k : H^{n-k}(M) \to H^{n+k(M)}$ is an isomorphism: define the primitive cohomology $P^{n-k} \subseteq H^{n-k}(M)$ to be the kernel of $L^{k+1} : H^{n-k}(M) \to H^{n+k+2}(M)$, then $H^r(M) = \bigoplus_{0\leqslant r\leqslant \frac{r}{2}} L^kP^{r-2k}(M)$. Even better, since $L : H^{p,q} \to H^{p+1,q+1}$ and $\Lambda : H^{p,q} \to H^{p-1,q-1}$, then we have primitive (p, q)-classes $P^{p,q}(M) \subseteq P^{p+q}(M)$ and $P^k(M) = \bigoplus_{p+q=k} P^{p,q}(M)$.

Proof. This is immediate from the representation theory of $\mathfrak{sl}_2$ above and the fact that $H^*(M)$ is a finite-dimensional $\mathfrak{sl}_2$-representation. □

In the Hodge diamond, L and Λ represent vertical moves, so for example $H^{1,1} = P^{1,1} + LP^{0,0}$.

Definition 15.7 (Hodge-Riemann Bilinear Relations)**.** Let

$$\begin{aligned}Q : H^{n-k}(M) \times H^{n-k}(M) &\to \mathbb{C}\\ (\eta, \xi) &\mapsto \int_M \eta \wedge \xi \wedge \omega^k,\end{aligned}$$

then $Q(H^{p,q}(M), H^{p',q'}(M)) = 0$ unless $p = q'$ and $q = p'$. Moreover, Q is skew-symmetric.

Theorem 15.8. If $\xi \in P^{p,q}(M)$ is non-zero, then $i^{p,q}(-1)^{(n-p-q)(n-p-q-1)/2Q(\xi,\bar{\xi})>0}$. We call the constant factor c.

Note that $Q(L^r\xi, L^r\eta) = Q(\xi, \eta)$ if $p + q + 2r \leqslant n$, thus if $p + q$ is even and $W := \{\xi + \bar{\xi} \in P^{p,q} + P^{q,p}\}$, then cQ is positive definite on W. People call the decomposition $P^k = \bigoplus_{p+q=k} P^{p,q}$ with these properties a polarized Hodge structure of weight k.

If M is a compact oriented manifold of dimension $4k$, then $Q : H^{2k}(M)\times H^{2k}(M) \to \mathbb{R}$ is a bilinear form, and it signature, i.e., the difference in the number of positive and negative eigenvalues of Q, is called the index of M. If M is a compact Kahler manifold of dimension $2k$, then the results above show that the index is $I(M) = \sum_{2n\geqslant p+q\equiv 0 \pmod 2} (-1)^p \dim(P^{p,q})$.

Now by the decomposition we have $h^{p,p+j} = \sum_{i=0}^{p} \dim(P^{i,i+j})$, so along a vertical line in the Hodge diamond, we have

$$\sum_{i=0}^{p-1}(-1)^i \dim(P^{i,i+j}) = (-1)^p h^{p,p+j} + 2\sum(-1)^i h^{i,i+j},$$

so finally

$$I(M) = \sum_{p+q=2n}(-1)^p h^{p,q} + 2\sum_{2n\geqslant p+q\equiv 0 \pmod 2}(-1)^p \dim(P^{p,q}) = \sum_{p+q\equiv 0 \pmod 2}(-1)^p h^{p,q}.$$

16 Divisors

Let M be a complex manifold (not necessarily complex) of dimension n. Let $V \subseteq M$ be an analytic subvariety such that $\dim(V) = n-1$, then V is a hypersurface: for every point $p \in V$, there exists an open neighborhood $U \subseteq M$ such that there exists an analytic function f on U such that $U \cap V = Z(f)$. In fact, V is a union of irreducible subvarieties. Locally $f \in \mathcal{O}_{M,p}$

Definition 16.1. A divisor on M is a locally finite $\mathbb{Z}$-linear combination of irreducible subvarieties of dimension $n-1$, then we can write $D = \sum_i a_i V_i$. The set of all divisors on M gives an abelian group. We say a divisor D is effective if $a_i \geqslant 0$ for all i.

If $g \in \mathcal{O}_{M,p'}$ and f is a local equation of irreducible subvariety V, then we can write $g = f^a \cdot u$ such that a is maximal and $u \in \mathcal{O}_{M,p}$. Such a is called the order of g at V, denoted $\operatorname{ord}_{V,p}(g) \in \mathbb{Z}_{\geqslant 0}$. This is independent of the choices. Moreover, one has $\operatorname{ord}(g_1 g_2) = \operatorname{ord}(g_1) + \operatorname{ord}(g_2)$.

Locally, for any meromorphic function can be written as $\frac{g'}{g}$ with $g', g \in \mathcal{O}_{M,p}$, then we define the order to be $\operatorname{ord}_V\left(\frac{g'}{g}\right) = \operatorname{ord}(g') - \operatorname{ord}(g)$.

Remark 16.2. If $\operatorname{ord}_V(g) = a > 0$, then g has a zero of order a along V; if $\operatorname{ord}_V(g) = a < 0$, then g has a pole of order $-a$ along V.

This creates a functor $\mathfrak{M}(M) \to \operatorname{Div}(M)$, from the meromorphic functions to the divisors. We denote $(f) = \sum_V \operatorname{ord}_V(f) \cdot V \in \operatorname{Div}(M)$. In particular, $(f)_0 = \sum_{\operatorname{ord}(f)>0} \operatorname{ord}_V(f) \cdot V$, and $(f)_\infty = \sum_{\operatorname{ord}(f)<0} -\operatorname{ord}_V(f) \cdot V$. Therefore, the order of f is just $(f) = (f)_0 - (f)_\infty$.

Let $\mathfrak{M}_M^*$ be the sheaf of meromorphic function on M, not identically zero on any non-empty open subset. There is also a subset $\mathcal{O}_M^* \subseteq \mathfrak{M}_M^*$, given by the nowhere zero meromorphic functions.

Lemma 16.3. $\operatorname{Div}(M) \cong \check{H}^0(M, \mathfrak{M}_M^*/\mathcal{O}_M^*) = \Gamma(M, \mathfrak{M}_M^*/\mathcal{O}_M^*)$, where the right-hand side gives the Cartier divisors.

Proof. Let $s \in H^0(M, \mathfrak{M}^*/\mathcal{O}^*)$. Choose $M = \bigcup_\alpha U_\alpha$, such that $s|_{U_\alpha}$ is represented by $f_a^* \in \mathfrak{M}^*(U_\alpha)$. Therefore, $g_{\alpha\beta} = \left.\frac{f_\alpha}{f_\beta}\right|_{U_{\alpha\cap\beta}} \in \mathcal{O}_{U_{\alpha\beta}^*}$. This satisfies the cocycle conditions. Define a divisor D as $D|_{U_\alpha} = (f_\alpha)$. This is well-defined: $f_\alpha|_{U_{\alpha\beta}} = (f_\beta) + (g_{\alpha\beta}) = f_\beta|_{U_{\alpha\beta}}$. Conversely, we have a correspondence by looking at the local structure. ☐

Given a cover $U = \{U_i\}$ with $f_\alpha \in \mathfrak{M}^*(U_\alpha)$, we have $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ which satisfies the cocycle conditions. The set of these elements is an element in $\check{H}^1(U, \mathcal{O}^*)$.

Given a holomorphic line bundle $L \to M$, the $g_{\alpha\beta}$'s give rise to a line bundle $[D]$. Given a short exact sequence

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathfrak{M}^* \longrightarrow \mathfrak{M}^*/\mathcal{O}^* \longrightarrow 0$$

we has a coboundary map $\delta : H^0(\mathfrak{M}^*/\mathcal{O}^*) = \operatorname{Div}(M) \to H^1(M, \mathcal{O}^*)$. In particular, $\delta(D) = (g_{\alpha\beta}^{-1}) = [D]^* \in H^1(M, \mathcal{O}^*) = \operatorname{Pic}(M)$, where $\operatorname{Pic}(M)$ is the group of line bundles over tensor products. In particular, $[D] = (\delta(D))^*$. Since L and L' correspond to $g_{\alpha\beta}$'s and $g'_{\alpha\beta}$'s, $L \otimes L'$ corresponds to $g_{\alpha\beta} g'_{\alpha\beta}$'s, and that L^* corresponds to $g_{\alpha\beta}^{-1}$. In particular, $M \times \mathbb{C}$ is given by 1.

Given an element $D \in \operatorname{Div}(M)$, we have $D \sim D'$ as equivalent if $D' = D + (f)$ for some principal divisor $f \in \mathfrak{M}^*$.

Lemma 16.4. $[D] = [D']$ if and only if $D' \sim D$, i.e., $[D' - D] = [0]$.

If $\frac{f_\alpha}{f_\beta} = 1$ over $U_{\alpha\beta}$ for $U = \{U_\alpha, U_\beta\}$, then we have $[D] = M \times \mathbb{C}$, then $g_{\alpha\beta} \sim 1 \in H^1(M, \mathcal{O}^*)$, so $g_{\alpha\beta} = \delta(g_\alpha)$ for $g_\alpha \in C^0(U, \mathcal{O}^*)$. Given $\frac{f_\alpha}{f_\beta} = \frac{g_\beta}{g_\alpha}$, then $f_\alpha g_\alpha = f_\beta g_\beta$, which is just $h|_{U_{\alpha\beta}}$ for $h \in \mathfrak{M}^*(M)$. To see that $D = (h)$, we note $D|_{U_\alpha} = (f_\alpha)$, but

$$(h)|_{U_\alpha} = (h_\alpha) = (h)|_{U_\alpha} = (f_\alpha g_\alpha) = (f_\alpha) + (g_\alpha) = (f_\alpha)$$

since $g_\alpha \in C^0(U, \mathcal{O}^*)$, therefore $D|_{U_\alpha} = (h)|_{U_\alpha}$, so $D = (h)$.

Example 16.5. Let $M = \mathbb{P}^n$. We have a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

which gives

$$H^1(\mathcal{O}) \cong H^{0,1} \cong 0 \longrightarrow H^1(\mathcal{O}^*) \cong \operatorname{Pic}(\mathbb{P}^n) \longrightarrow H^2(\mathbb{Z}) \cong \mathbb{Z} \longrightarrow H^2(\mathcal{O}) \cong H^{0,2} = 0$$

Therefore $\operatorname{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$. We have a mapping

$$\begin{aligned} \operatorname{Div}(\mathbb{P}^n) &\to \operatorname{Pic}(\mathbb{P}^n) \\ H_i = Z(X_i) &\mapsto [H_i]. \end{aligned}$$

Therefore, if $H_i \sim H_j$, then $H_i = H_j + \left(\frac{x_i}{x_j}\right)$. It is important that the principal divisor has numerator's and denominator's degrees agree.

Let $D \subseteq \mathbb{P}^n$, we have $D = \sum a_i V_i$ for some hypersurfaces V_i's, then $\deg(D) = \sum a_i \deg(V_i)$. Therefore, $D \sim D'$ if and only if $\deg(D) = \deg(D')$. This gives rise to the line bundle-divisor correspondence. One can then define meromorphic sections analogous to the holomorphic sections, except that we allow poles.

Now let us assume that M is compact. Let $L(D) = \{0\} \cup \{f \in \mathfrak{M}^* : (f) + D \geqslant 0\} \subseteq \mathfrak{M}(M)$, then this is a vector space by the Archimedean property of the valuation. Note that $L(D) = H^0(M, \mathcal{O}(D)$. Also, $\mathcal{O}(D) \subseteq \mathfrak{M}$ is a subsheaf whose sections on any open set U is given by $\mathcal{O}(D)(U) = \{0\} \cup \{f \in M^*(U) : (f) + D|_U \geqslant 0\}$. There is now a mapping

$$\begin{aligned} H^0(M, \mathcal{O}(D)) \backslash \{0\} &\to \operatorname{Div}(M) \\ f &\mapsto (f) + D \end{aligned}$$

The kernel of the map is given by global non-vanishing holomorphic functions, so by compactness that is $\mathbb{C}$. Therefore, the projective space $\mathbb{P}^(H^0(M, \mathcal{O}(D)))$ gives a surjective mapping onto effective divisors $\operatorname{EffDiv}(M)$, which gives a set of divisors $|D|$ called the complete linear systems. That is, $|D| = \{D' \in \operatorname{Div}(M) : D' \sim D, D' \geqslant 0\}$.

Suppose M is compact. If $D + (f) = D + (f')$, then $\frac{f}{f'} \in \mathbb{C}$. Therefore, the induced map

$$\mathbb{P}(L(D)) = \mathbb{P}(H^0(M, \mathcal{O}(D))) \hookrightarrow |D| \subseteq \operatorname{Div}(M)$$

has no kernel, therefore it is an isomorphism. Hence, $|D|$ has the structure of a projective space.

Definition 16.6. A linear system is a linear subspace of some complete linear system $|D|$, i.e., the image of $\mathbb{P}V \to |D|$ for some subspace $V \subseteq L(D)$.

In particular, $\dim(|D|) = \dim(L(D)) - 1$, and so $\dim(\mathbb{P}(V)) = \dim(V) - 1$. In particular, in projective dimension 1 we have a pencil; in projective dimension 2 we have a net; in projective dimension 3 we have a web.

We can find an isomorphism $\mathcal{O}(D)(U) \cong \mathcal{O}([D])(U)$ for each open subset U by $h \mapsto \{(hf_\alpha)\}$ for $hf_\alpha \in \mathcal{O}(U_\alpha \cap U)$.

Let V be a linear system, then the base locus B of projective P is $\bigcap_{\lambda \in P} D_\lambda \subseteq M$. For instance, for $P = \mathbb{P}^0$, we have $B = D = D_0$ for $0 \in \mathbb{P}^0$.

A fixed component of P is divisor $F \subseteq B$. For $V \subseteq H^0(\mathcal{O}(p))$, we have $\dim(V) = 1$ and $\dim(\mathbb{P}V) = 0$ where we call p a basepoint.

Theorem 16.7 (Bertini)**.** The generic member of a linear system is smooth away from the base locus.

Suppose a collection of generic configuration is parametrized by an analytic variety. To say that the generic configuration has a property means, there exist a countable collection of proper subvarieties $H_i \subsetneq G$ such that every configuration converges to a part of $G \backslash \bigcup_i H_i$ has the property.

Proof. We reduce to the case of a pencil: that is,

$$M \backslash B \to \mathbb{P}^1$$

$$p \mapsto \lambda$$

for p outside of the base locus, where λ corresponds to D_λ, the unique divisor with $p \in D_\lambda$. It now suffices to show that generic D_λ is smooth at p. For $V \subseteq (M \backslash B) \times \mathbb{P}^1$, elements are of the form (p, λ) where p is contained in the singular locus of

We trivialize L, the line bundle corresponding to the linear system near $p \in U$, which is given by $\{(s) : s \in W \subseteq H^0(U)\}$ for $s \neq 0$ and $\dim(W) = 2$. Therefore,

$$D_\lambda|_U = \{(\mu f + \lambda g) : f, g \in \mathcal{O}(U), (\mu, \lambda) \in \mathbb{P}^1\}.$$

In particular, this locus contains the common zeros of f and g. Studying the equation of V, we have $f(p) + \lambda g(p) = 0$ for $p \in D_\lambda$. Taking the derivative, we have

$$\frac{\partial f}{\partial z^j}(p) + \lambda \frac{\partial g}{\partial z^j}(p) = 0$$

for p in the singular locus of D_λ. One should then show that π is constant on the connected component of V. Therefore, $\pi(V) \subseteq \mathbb{P}^1$ is countable subset, so now we take $\lambda \in \mathbb{P}^1 \backslash \pi(V)$ with D_λ smooth on $p \in M \backslash B$.

Remark 16.8. This is one reason why we ask manifolds to be second countable.

Now consider $\pi(p, \lambda) = \lambda : -\frac{f}{g} \in \mathbb{P}^1$. Therefore,

$$\frac{\partial}{\partial z_j}\left(\frac{f}{g}\right) = \frac{\frac{\partial f}{\partial z_j} - \frac{f}{g}\frac{\partial g}{\partial z_j}}{g} = 0,$$

for $p, \lambda \in V$. □

Let L be a line bundle, then

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

There is an associated long exact sequence with connecting homomorphism

$$\begin{aligned} \delta : H^1(M, \mathcal{O}^*) &\to H^2(M, \mathbb{Z}) \\ L \in \operatorname{Pic}(M) &\mapsto \delta(L) =: c_1(L) \end{aligned}$$

where the image is defined to be the first Chern class of L. In fact, the Chern class is a purely topological construction. We have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}^* & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathfrak{a}^0_M & \xrightarrow[e^{2\pi i}]{} & \mathfrak{a}^{0,\times}_M & \longrightarrow & 0 \end{array}$$

Taking the long exact sequence, the connecting homomorphism gives a square which shows that every holomorphic bundle is a C^∞-bundle.

Theorem 16.9. Choose a holomorphic line bundle L on a complex manifold, and choose a Hermitian connection and curvature Θ, then $c_1(L) = \frac{i}{2\pi}[\Theta]$. Furthermore, if $L = [D]$, then $c_1(L) = \eta_D$.

Proof. Given an open cover $M = \bigcup U_\alpha$, let θ_α be the connection form, and we know $\theta_\alpha = g_{\alpha\beta}\theta_\beta g_{\alpha\beta}^{-1} + dg_{\alpha\beta}g_{\alpha\beta}^{-1} = \theta_\beta + dg_{\alpha\beta}g_{\alpha\beta}^{-1}$. Take the long exact sequence, we get $A^2_d/dA^1 \hookrightarrow H^1(Z^1) \xrightarrow{\delta} H^2(R)$, and this gives a commutative square

$$\begin{array}{ccc} \theta_\alpha & \xrightarrow{\quad d \quad} & \Theta_\alpha \\ {\scriptstyle\delta}\downarrow & & \\ \theta_\beta - \theta_\alpha = dg_{\alpha\beta}g_{\alpha\beta}^{-1} = h_{\alpha\beta} & & \end{array}$$

Computing the coboundary again, we have

$$\begin{array}{ccc} \log\theta_\alpha & \xrightarrow{\quad d \quad} & h_{\alpha\beta} = dg_{\alpha\beta}g_{\alpha\beta}^{-1} \\ \downarrow{\scriptstyle\delta} & & \\ 2\pi i c_1(L) = \log_{\alpha\beta} + \log_{\beta\gamma} - \log_{\alpha\gamma} & & \end{array}$$

□

Corollary 16.10. Given a principal divisor $f \in \mathcal{M}^*(M)$, the fundamental class $\eta_{(f)} = 0$. Moreover, $[(f)]$ is the trivial bundle, so $c_1(f) = 0$.

For $D = \sum n_i p_i$ for $p_i \in M$ and $n_i \in \mathbb{Z}$, we had $\deg(D) = \sum n_i$, then $\langle c_1([D]) \frown [M]\rangle = \langle \eta_D \frown [M]\rangle = \deg(D)$.

Example 16.11. For $\mathbb{P}^n$, we get

$$H^1(\mathbb{P}^,\mathcal{O}) = H^{0,1} = 0 \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}^*) \xrightarrow{\cong} H^2(\mathbb{P}^2, \mathbb{Z}) \longrightarrow H^2(\mathbb{P}^2, 0) = H^{0,2} \cong 0$$

Therefore, the two middle terms agree. Moreover, $H^2(\mathbb{P}^2, \mathbb{Z})$ is generated by the class of hyperplane H, therefore so is $H^1(\mathbb{P}^n, \mathcal{O}^*)$, so we define $\mathcal{O}(1) = \mathcal{O}(H)$. Similarly, $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ for $n > 0$ and $(\mathcal{O}(1)^*)^n$ for $n < 0$.

Recall we also had $\mathcal{O}(-1) = \mathcal{O}(J)$ as defined in a homework question, where $J \subseteq \mathbb{P}^n \times \mathbb{C}^m$ is a subspace of the form $\{(z, \mathbb{C} \cdot z) : [z] \in \mathbb{P}^n\}$. In particular $J \cong [-H]$, so $\mathcal{O}(J) \cong \mathcal{O}_{\mathbb{P}^n}(-1)$. Then $s([z]) = \left(1, \frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right)$ and so $s \in J(U_0)$.

Suppose U_1 has local coordinates $y_1, \ldots, y_n$, then this is given by $(y_1, 1, y_2, \ldots, y_n)$, therefore $s = \left(1, y_1^{-1}, \frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1}\right)$ for $s \in \mathcal{O}(U_{01})$. There is a correspondence $\varphi_1 : U_1 \times \mathbb{C} \cong J|_{U_1}$ given by $1 \mapsto e = \left(\frac{z_0}{z_1}, 1, \cdots, \frac{z_n}{z_1}\right)$. Under this trivialization, we have $s = y_1^{-1}e$ corresponding to y_1^{-1} with a first-order pole on $H = Z(z_0)$, so $(s) = -H$.

17 Adjunction Formula

Suppose $V \subseteq M$ is a smooth hypersurface, i.e., submanifold locally defined by a holomorphic zero with some partial derivatives as 0. Suppose V has dimension $n-1$ and M has dimension n.

Definition 17.1. We define the canonical bundle of M to be $K_M := \bigwedge^n (T'_M)^*$, and the sheaf associated to the canonical bundle as $\mathcal{O}(K_M) \cong \Omega^n_M$.

Theorem 17.2 (Adjunction Formula)**.** There is an isomorphism $K_V \cong (K_M \otimes [V])|_V$.

Example 17.3. Suppose we have a smooth hypersurface $V \subseteq \mathbb{P}^n$ of degree d, then $K_V \cong (K_{\mathbb{P}^n} \otimes [V])|_V \cong (K_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d))|_V$. By calculation, we get that

$$K(\mathbb{P}^n) = [-(n+1)H]$$

and

$$\Omega^n_{\mathbb{P}^n} = \mathcal{O}(K_{\mathbb{P}^n}) \cong \mathcal{O}(-n-1).$$

Pick $y_i = \frac{z_i}{z_0}$ on U_0, with $s = dy_1 \wedge \cdots \wedge dy_n \in \Omega^n_{\mathbb{P}^n}(U_0)$, then $\Omega^n_{\mathbb{P}^n}|_U$ is trivialized by $d_1 \wedge \cdots \wedge dx_n$ with local coordinates y_i's on U_1, and one note that $dy_1 \wedge \cdots \wedge dy_n = -\frac{1}{x_1^{n+1}} dx_1 \wedge \cdots \wedge dx_n$, so $(s) = -(n+1)H$.

When does V have nowhere vanishing holomorphic forms of top degree $n-1$? We have

$$\mathcal{O}_V \cong K_V \cong \mathcal{O}_V(d-n-1),$$

so we must have $d = n+1$.

- If $d = 3$, then $n = 2$, we have elliptic curves.
- If $d = 4$, then $n = 3$, we have K3 surfaces.
- If $d = 5$, then $n = 4$, which is a quintic 3-fold, which is the simplest example of Calabi-Yau 3-folds.

Proof of Adjunction Formula. We have a short exact sequence

$$0 \longrightarrow T'_V \longrightarrow T'_M|_V \longrightarrow N_{V/M} \longrightarrow 0$$

where $N_{V/M}$ is the normal bundle of V over M, then

$$\bigwedge^n T'_M|_V \cong \bigwedge^{n-1} T'_V \otimes N_{V/M}.$$

Indeed, a linear algebra argument shows that for subspace $Y \subseteq W$ of dimension $n-1$ and n respectively, there is a canonical isomorphism $\bigwedge^n W \cong \bigwedge^{n-1} V \otimes (W/V)$ defined by $v_1 \wedge \cdots \wedge v_{n-1} \wedge w \leftarrow (v_1 \wedge \cdots \wedge v_{n-1}) \otimes \bar{w}$. By dualizing everything, we have

$$\bigwedge^n T'^*_M|_V \cong \bigwedge^{n-1} T'^*_V \otimes N^*_{V/M}$$

where $N^*_{V/M}$ is the conormal bundle, so $K_M|_V \cong K_V \otimes N^*_{V/M}$. The formula now follows from $N^*_{V/M} \cong [-V]|_V$. (Note that it has a dual form $N_{V/M} \cong [V]|_V$.) □

In terms of the Poincaré residue map, there is a sheaf version of the adjunction formula. The residue map is given by

$$\operatorname{Re} \frac{f dz_1 \wedge \cdots \wedge dz_n}{g} = (-1)^{n-1} \left. \frac{f dz_1 \wedge \cdots \wedge dz_{n-1}}{\frac{\partial g}{\partial z_n}} \right|_V .$$

Taking the short exact sequence

$$0 \longrightarrow \Omega^n_M \longrightarrow \Omega^n_M(V) \cong \Omega^n \otimes \mathcal{O}(V) \longrightarrow \Omega^{n-1}_V \longrightarrow 0$$

we get $\Omega_V^{n-1} \cong \Omega_M^n(V)|_V$.

For a smooth hypersurface $V \subseteq M$ and any open subset $U \subseteq M$, we define $I_V(U) = \mathcal{O}_M(-V)(U) \subseteq \mathcal{M}_M$. In particular, this is the set described by

$$\{0\} \cup \{f \in \mathcal{M}^*(U) : (f) - V \geqslant 0\},$$

then $f \in \mathcal{O}(U)$, so $(f) \geqslant V \geqslant 0$, and in particular $f|_V = 0$. This fits into a short exact sequence

$$0 \longrightarrow \mathcal{O}_M(-V) \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_V \longrightarrow 0$$

Therefore the line bundles on M give information on restrictions to V.

18 Positive Bundles

Definition 18.1. Given a bundle L, we say it is positive if there exists a connection whose curvature Θ is such that $\frac{i}{2\pi}\Theta$ is a positive $(1,1)$-form. We say L is negative if L^* is positive. We say D is positive (respectively, negative) if the associated line bundle $[D]$ is positive (respectively, negative).

Remark 18.2. The existence of a positive bundle L on M implies M is Kahler: we have a metric from the closed $(1,1)$-form, and the curvature is always closed.

Remark 18.3. Every bundle L has the first Chern class $c_1(L)$ represented by a $(1,1)$-form. Just pick a metric on L and choose the Chern connection.

Example 18.4. Let $[H] \in \operatorname{Pic}(\mathbb{P}^n)$ be the hyperplane bundle, correspondingly $\mathcal{O}([H]) = \mathcal{O}(1)$. Let J be the universal bundle, then $J \simeq [-H]$, so $[H] = J^*$, and we pick a metric via $J \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}$, where we restrict the natural Euclidean metric. After restriction, we take the Chern connection, with $\theta = \partial \log h$, then the curvature is $\Theta = d\theta - \theta \wedge \theta = d\theta$. Therefore, $\Theta^* = d\partial \log ||z||^2$. In the holomorphic frame given by the section $U \to J$ over $\mathbb{P}^n \times (\mathbb{C}^{n+1} \backslash \{0\})$, so $\Theta^* = -\partial\bar{\partial} \log ||z||$, as the connection form of J. Our formula says that Θ on $[H]$ is given by $\underset{\Theta}{*} = \partial\bar{\partial}\log ||z||^2$, so $\frac{i}{2\pi}\Theta = \frac{i}{2\pi}\partial\bar{\partial}\log||z||^2$, which is just the Fubini-Study metric. This proves that the hyperplane bundle is positive.

Proposition 18.5. Let M be a compact Kahler manifold. Suppose $L \in \operatorname{Pic}(M)$, and that $c_1(L) \in H^2_{\mathrm{dR}}(M)$ is represented by $\omega \in H^{1,1}(M)$, then there exists a metric on L whose Chern connection satisfies $\frac{i}{2\pi}\Theta = \omega$.

Proof. The proof makes use of the following $\partial\bar{\partial}$ lemma.

Lemma 18.6. Let M be a compact Kahler manifold. Suppose $\omega \in A^{p,q}(M)$ that is d-, ∂-, or $\bar{\partial}$-exact, then there exists $\eta \in A^{p-1,q-1}(M)$ such that $\omega = \partial\bar{\partial}\eta$.

Subproof. We note $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$, then $G_d = \frac{1}{2}G_\partial = \frac{1}{2}G_{\bar{\partial}}$. We get to an explicit formula $\eta = \pm\partial\bar{\partial}(\partial^*\bar{\partial}^*G^2_{\bar{\partial}}\eta)$. ■

To prove the proposition, choose any metric h, then $\Theta = d\partial \log h = -\partial\bar{\partial}\log h$, and we know $[\frac{i}{2\pi}\Theta] = [\omega] \in H^2_{\mathrm{dR}}(M)$ are the same in de Rham cohomology, since they both represent the Chern class. Now take a general metric $h' = e^\rho h$ for any ρ, so $\Theta' = -\partial\bar{\partial}\rho + \Theta$, therefore $\frac{i}{2\pi}\Theta - \omega$ is d-exact. Therefore, we may write ρ as $\frac{2\pi}{i}\partial\bar{\partial}\sigma$ for some specific σ according to Lemma 18.6, so $\frac{i}{2\pi}\Theta = \omega$. This gives $\frac{i}{2\pi}\Theta' = \frac{i}{2\pi}(\partial\bar{\partial}\frac{2\pi}{i}\sigma + \Theta) = \frac{i}{2\pi}((\omega - \Theta) + \Theta)$. □

Suppose E is a holomorphic vector bundle on a compact Kahler manifold with metric and connection chosen, then we have

$$\bar{\partial}_E : A^{p,q}(E) \to A^{p,q+1}(E).$$

There is an adjoint $\bar{\partial}_E^* = \pm * \partial_E^*$, where $* : A^{p,q}(E) \to A^{n-p,n-q}(E^*)$, such that for any $\omega, \eta \in A^{p,q}(E)$, then we have a global inner product

$$\langle \omega, \eta \rangle = \int (\omega, \eta) \operatorname{vol} = \int_M \omega \wedge *\eta,$$

where $\wedge : E \times E^* \to \mathbb{C}$. In this context, we have Laplacian $\Delta_E = \bar{\partial}_E\bar{\partial}_E^* + \bar{\partial}_E^*\bar{\partial}_E$, harmonic forms $\mathcal{H}^{p,q}(E)$. The Hodge theorem now says $I = \mathcal{H}^{p,q} + G\Delta_E$ with $\dim(\mathcal{H}^{p,q}(E)) < \infty$. Moreover, we study the cohomology via exact sequences

$$0 \longrightarrow \mathcal{O}(E) \longrightarrow A^{0,0}(E) \xrightarrow{\bar{\partial}_E} A^{0,1}(E) \xrightarrow{\bar{\partial}_E} \cdots$$

and

$$0 \longrightarrow \Omega(E) \longrightarrow A^{p,0}(E) \xrightarrow{\bar{\partial}_E} A^{p,1}(E) \xrightarrow{\bar{\partial}_E} \cdots$$

which defines $H^p(M, \mathcal{O}(E))$ and $H^q(M, \Omega^p(E)) \cong H^{p,q}_{\bar{\partial}}(E) \cong \mathcal{H}^{p,q}_E$ via $\Omega^p(E)$, the holomorphic sections of $\Lambda^p T'^*_M \otimes E$. Note that $** = \pm 1$, then $H^q(\Omega^p(E)) \cong H^{n-q}(\Omega^{n-p}(E^*))^*$ where the perfect pairing is given by

$$H^q(\Omega^p(E)) \times H^{n-q}(\Omega^{n-p}(E^*)) \to H^n(\Omega^P(E) \otimes \Omega^{n-p}(E^*)) \to H^n(\Omega^n) \cong \mathbb{C}$$

as in Serre duality.

Theorem 18.7 (Kodaira Vanishing)**.** Suppose M is a compact Kahler manifold and L is a positive bundle, then $H^q(M, \Omega^p(L)) = 0$ for $p+q > n = \dim(M)$.

Proof. See text. □

Corollary 18.8. There is a version of Kodaira vanishing theorem for negative bundles. That is, $H^{n-q}(M, \Omega^{n-p}(L^*))^* = H^q(M, \Omega^p(L)) = 0$ for $p+q < n$ as $(n-q)+(n-p) > n$.

Theorem 18.9 (Lefschetz Hyperplane Theorem)**.** Let M be compact of dimension n and $V \subseteq M$ be a smooth hypersurface. (For instance, a projective space M with a hypersurface V.) Suppose $[V]$ is positive, then

$$r_i : H^i(M, \mathbb{Q}) \to H^i(V, \mathbb{Q})$$

is an isomorphism for $i \leqslant n-2$ and an injection for $i = n-1$.

Proof. Since $[V]$ is positive, we may define a Kahler metric on M so that it becomes a Kahler manifold, and V is also Kahler by restriction. By Hodge theorem,

$$H^i(M, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(M) \cong \bigoplus_{p+q=i} H^q(M, \Omega^p)$$

and similarly for $H^i(V, \mathbb{C})$. Therefore, we look at the map

$$H^q(M, \Omega^p) \cong H^{p,q}(M) \xrightarrow{A} H^q(M, \Omega^p|_V) \cong H^q(V, \Omega^p|_V) \xrightarrow{B} H^q(V, \Omega^p_V) \cong H^{p,q}(V)$$

where $\Omega^p|_V$ is given by extension by zero. Therefore, we represent r_i as a direct sum of maps $h^{p,q} : H^{p,q}(M) \to H^{p,q}(V)$ over $i = p+q$, and it suffices to prove the theorem upon maps $h^{p,q}$, which can be done by proving this for maps A and B. From the sequence

$$0 \longrightarrow \Omega^p(-V) \longrightarrow \Omega^p_M \longrightarrow \Omega^p_M|_V \longrightarrow 0$$

we get

$$\cdots \longrightarrow H^q(M, \Omega^p(-V)) \longrightarrow H^q(M, \Omega^p) \longrightarrow H^q(M, \Omega^o|_V) \longrightarrow H^{q+1}(M, \Omega^p(-V)) \longrightarrow \cdots$$

From Theorem 18.7, we know $H^q(M, \Omega^p(-V)) = 0$ for $p+q < n$. Therefore, in this range, we know $H^q(M, \Omega^p) \to H^q(M, \Omega^o|_V)$ is an injection. Moreover, we know that $H^{q+1}(M, \Omega^p(-V)) = 0$ in the case $p+q+1 < n$, therefore the map $A : H^q(M, \Omega^p) \to H^q(M, \Omega^p|_V)$ is an injection when $p+q = n-1$, and an isomorphism if $p+q \leqslant n-2$.

To do this for map B, we look at the the exact sequence

$$0 \longrightarrow N^* \cong [-V] \longrightarrow T'_M|_V^* \longrightarrow T_V^* \longrightarrow 0$$

On the level of exterior powers, recall that given a sequence

$$0 \longrightarrow U \longrightarrow W \longrightarrow V \longrightarrow 0$$

of dimension $1, n, n-1$, respectively, we take the exterior power and get

$$0 \longrightarrow U \otimes \bigwedge^{p-1} V \longrightarrow \bigwedge^p W \longrightarrow \bigwedge^p V \longrightarrow 0$$

where the first map is defined by $u \otimes (\bar{v}_1 \wedge \cdots \wedge \bar{v}_{p-1}) \mapsto u \wedge v1 \wedge \cdots \wedge v_{p-1}$ by representing each element in a class in W. In our case, we have

$$0 \longrightarrow \Omega_V^{p-1} \longrightarrow \Omega^p_M|_V \longrightarrow \Omega^p_V \longrightarrow 0$$

and get a sequence

$$\cdots \longrightarrow H^q(V, \Omega_V^{p-1}(-V)) \longrightarrow H^q(V, \Omega^p_M|_V) \longrightarrow H^q(V, \Omega^p_V) \longrightarrow H^{q+1}(V, \Omega_V^{p-1}(-V)) \longrightarrow \cdots$$

Since V is negative, then $H^q(V, \Omega_V^{p-1}(-V))$ vanishes when $(p-1)+q < n-1$. Similarly, $H^{q+1}(V, \Omega_V^{p-1}(-V))$ vanishes when $p+q = (p-1)+(q+1) < n-1$, and using the same idea we are done. □

Example 18.10. We know the Hodge diamond for $\mathbb{P}^3$.

$$\begin{matrix} &&&1&&& \\ &&0&&0&& \\ &0&&1&&0& \\ 0&&0&&0&&0 \\ &0&&1&&0& \\ &&0&&0&& \\ &&&1&&& \end{matrix}$$

How do we find this for hypersurface $V \subseteq \mathbb{P}^3$? By Theorem 18.9, we know it looks like

$$\begin{matrix} &&?&& \\ &?&&?& \\ ?&&?&&? \\ &0&&0& \\ &&1&& \end{matrix}$$

by duality, we have

$$\begin{matrix} &&1&& \\ &0&&0& \\ h^{2,0}&&h^{1,1}&&h^{0,2} \\ &0&&0& \\ &&1&& \end{matrix}$$

By Kahler condition, we know $h^{1,1} \geqslant 1$. Finally, by adjunction formula, we find the rest and get

$$\begin{matrix} &&1&& \\ &0&&0& \\ 1&&20&&1 \\ &0&&0& \\ &&1&& \end{matrix}$$

Theorem 18.11 (Serre's theorem B)**.** For compact manifold M and positive line bundle L, and holomorphic vector bundle E over M, then there exists some n_0 such that for all $n \geqslant n_0$, $H^q(M, E \otimes L^{\otimes n}) = 0$ for any $q > 0$.

Given a divisor, we already know how to find a line bundle. Conversely, given a line bundle, we may find a divisor. Suppose $M \subseteq \mathbb{P}^N$ has a positive line bundle, then

$$\begin{aligned} \operatorname{Div}(M) &\twoheadrightarrow \operatorname{Pic}(M) \\ D &\mapsto [D] \end{aligned}$$

Equivalently, any $E \in \operatorname{Pic}(M)$ has a non-zero meromorphic section s such that $[(s)] \simeq E$. It suffices to show that $H^0(M, E \otimes \mathcal{O}(n)) \neq 0$ for $n \gg 0$. This was done by induction on n.

Theorem 18.12 (Lefschetz $(1,1)$ Theorem)**.** Suppose M is a compact manifold that may be embedded in $\mathbb{P}^N$, so it is Kahler in particular. Suppose $\gamma \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})$, then $\gamma = \eta_D$ for some divisor D of M.

Proof. Consider

$$H^1(M,\mathcal{O}) \longrightarrow H^1(M,\mathcal{O}^*) \longrightarrow H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathcal{O})$$

and we will show that the diagram

$$\begin{array}{ccc} H^2(M,\mathbb{Z}) & \longrightarrow & H^2(M,\mathcal{O}) \\ \downarrow & & \\ H^2(M,\mathbb{C}) & & \downarrow \cong \\ \cong \downarrow & & \\ H^2_{\mathrm{dR}}(M,\mathbb{C}) & \xrightarrow[\pi^{0,2}]{} & H^{0,2}_{\bar\partial}(M) \end{array}$$

commutes, where the bottom map is defined by $A^2 \mapsto A^{0,2}$. Indeed, we perform a diagram chase from $Z_{\alpha\beta\gamma} \in Z^2(U,\mathbb{Z})$, which maps down to $z \in Z^2(U,\mathbb{C})$, then by identification it runs to $(\bar{w}) \in H^2_{\mathrm{dR}}(M,\mathbb{C})$ via $H^2(M,\mathbb{C}) \cong H^1(M,Z^1_d) \cong H^2_{\mathrm{dR}}(M,\mathbb{C})$. On other other hand, using Dolbeault cohomology, we map it to $z \in Z^2(U,\mathcal{O})$ and therefore to $[w^{02}] \in H^{0,2}_{\bar\partial}(M)$ via the identification $H^2(M,\mathcal{O}) \cong H^1(Z^1_{\bar\partial}) \cong H^{0,2}_{\bar\partial}(M)$. □

Finally, we prove the Hodge conjecture for $(n-1,n-1)$-form, which follows from the Lefschetz $(1,1)$ Theorem. Set $L = \omega \wedge - = c_1(H) \wedge -$. We have an operator $L^{n-2} : H^2(M,\mathbb{C}) \cong H^{2n-2}(M,\mathbb{C})$, which restricts to an isomorphism $H^{1,1}(M) \cong H^{n-1,n-1}(M)$. Similar identification happens under cohomology $L^{n-1} : H^2(M,\mathbb{Q}) \cong H^{2n-2}(M)$ with rational coefficient. Therefore, we define an isomorphism

$$L^{n-2} : H^{1,1}(M) \cap H^2(M,\mathbb{Q}) \to H^{n-1,n-1}(M) \cap H^{2n-2}(M,\mathbb{Q}).$$

In particular, $\gamma = L^{n-2}\eta \in H^{n-1,n-1}(M) \cap H^{2n-2}(M,\mathbb{Q})$, where $\eta = \sum a_i \eta_{D_i}$ for some rational numbers a_i. We may now choose $n-2$ generic hyperplanes $H_1,\dots,H_{n-2}$, such that $\dim(H_1 \cap \cdots \cap H_{n-2} \cap D_i) = 1$ for all i, since $\dim(D_i) = n-1$: we choose generic hyperplanes to cut down the dimension by 1 each time by avoiding containing the entire existing set. Therefore,

$$\begin{aligned} \gamma &= L^{n-2}\eta \\ &= L^{n-2}\left(\sum a_i \eta_{D_i}\right) \\ &= \left(\prod_j \eta_{H_j}\right)\left(\sum a_i \eta_{D_i}\right) \\ &= \sum a_i \eta_{C_i} \end{aligned}$$

for $C_i = H_1 \cap \cdots \cap H_{n-2} \cap D_i$.

19 Algebraic Variety

Definition 19.1. An algebraic variety $M \subseteq \mathbb{P}^n$ is the zero locus of a set of homogeneous polynomials in homogeneous coordinates $(x_0 : \cdots : x_n)$.

Remark 19.2. An algebraic variety is an analytic subvariety of $\mathbb{P}^n$.

Proposition 19.3. $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is the vector space of degree d homogeneous polynomials on $\mathbb{P}^n$.

Proof. Following the textbook's definition, we take the bundle $J \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}$, then the hyperplane bundle is $H = J^*$, hence $\mathcal{O}(1) = \mathcal{O}(J^*)$. For $p = (p_0 : \cdots : p_n) \in \mathbb{P}^n$, we have $J_p = \mathbb{C} \cdot (p_0 : \cdots : p_n)$, so we give a global section $s_\ell \in H^0(\mathbb{P}^n, \mathcal{O}(J^*))$ defined by $s_\ell(p) = \ell|_{J_p}$. Let V be the collection of linear homogeneous polynomials, then it injects into $H^0(\mathbb{P}^n, \mathcal{O}(1))$. In particular, $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n} = \mathcal{O}((J^*)^{\otimes n}) = \mathcal{O}(\mathrm{Sym}^n(J^*))$, therefore the set of degree d homogeneous polynomials $\mathrm{Sym}^d(V)$ injects into $H^0(\mathbb{P}^n, \mathcal{O}(1))$. We claim that this is also a surjection. For any choice of $F, s \in H^0(\mathbb{P}^n, \mathcal{O}(d))$ where we assume $d > 0$ without loss of generality, then this gives a meromorphic function $\frac{s}{F}$ over $\mathbb{P}^n$.

$$\begin{array}{ccc} \mathbb{C}^{n+1}\backslash\{0\} & \longrightarrow & \mathbb{C} \\ {\scriptstyle\pi}\downarrow & \nearrow_{f} & \\ \mathbb{P}^n & & \end{array}$$

In particular, fF is meromorphic section of $\mathcal{O}(d)$, with $(f \circ \pi)F(x_0 : \cdots : x_n) \in \mathcal{O}(\mathbb{C}^{n+1}\backslash\{0\})$. By Hartog's theorem, we extend this to $\mathcal{O}(\mathbb{C}^{n+1})$, therefore we have a holomorphic section of $\mathcal{O}(d)$. Take $G = (f \circ \pi)F$, then we examine G as a homogeneous polynomial such that $G(\lambda\vec{x}) = \lambda^d G(\vec{x})$, therefore $G = G_d$, and in particular $s = G$. $\square$

Corollary 19.4. $\dim(H^0(\mathbb{P}^n, \mathcal{O}(d))) = \binom{n+d}{d}$.

Theorem 19.5 (Chow)**.** Consider an analytic variety $V \subseteq \mathbb{P}^n$, then V is an algebraic variety.

Proof. Suppose $\dim(V) = n - 1$, i.e., we have a hypersurface, then $[V] = [H]^{\otimes d}$, then $\mathcal{O}([V]) = \mathcal{O}(d)$ for some d. In particular, $V \geqslant 0$, so $s_V \in H^0(\mathcal{O}_d)$, hence $(F) = (s_V) = V$, so this is given by homogeneous polynomials: $V = Z(F)$. In general, suppose $\dim(V) = k < n - 1$, then pick a $\mathbb{P}^{n-k-1}$, i.e., taking $k + 1$ hyperplane sections, therefore in general position we have $\mathbb{P}^{n-k-1} \cap V = \varnothing$. That is, it suffices to showt aht there exists some homogeneous polynomial F such that $F|_V \equiv 0$ but $F(p) \neq 0$. Once we have that, we may project from hyperplane $\mathbb{P}^{n-k-2}$ to $\mathbb{P}^{k+1}$ via

$$\mathbb{P}^n\backslash\mathbb{P}^{n-k-2} \to \mathbb{P}^{k+1}$$
$$(x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_{k+1})$$

where the domain contains both V and p. Since V is compact, then by the proper mapping theorem, we note $\pi(V) \subseteq \mathbb{P}^{k+1}$ compact, hence we have an analytic variety of dimension k. $\square$

20 GAGA Principle

Slogan: "analytic objects in $\mathbb{P}^n$ are algebraic."

Example 20.1. If $V \subseteq \mathbb{P}^n$ if an analytic variety, then it is an algebraic variety.

Definition 20.2. A rational function on $\mathbb{P}^n$ is a function $\frac{F}{G}$ where F, G are homogeneous polynomials of the same degree $G \not\equiv 0$.

Remark 20.3. These functions are algebraic in the sense that they are holomorphic on $\mathbb{P}^n \backslash Z(G)$.

Theorem 20.4. Meromorphic functions on $\mathbb{P}^n$ are rational.

Proof. Suppose f is meromorphic, then $(f) = (f)_0 - (f)_\infty$. Note that $[(f)_0] = [dH]$ for some $d \geqslant 0$, i.e., it is a multiple of the hyperplane divisor, and also recall that the bundle of a principal divisor is trivial, hence $[(f)_\infty] = [(f)_0] = [dH]$. Therefore, there exists a homogeneous function[6] F of degree d with $(F) = (f)_0$, and a homogeneous function G of degree d with $(G) = (f)_\infty$. Therefore, $\left(\frac{F}{G}\right) = (f)$, so $\left(\frac{F}{Gf}\right) = 0$, therefore $f = c\frac{F}{G}$. □

Definition 20.5. Suppose $V \subseteq \mathbb{P}^n$ is an irreducible variety, then a rational function on V is the restriction to V of a rational function $\frac{F}{G}$ on $\mathbb{P}^n$ such that $G|_V \not\equiv 0$.

Theorem 20.6. Meromorphic functions on irreducible variety $V \subseteq \mathbb{P}^n$ are rational.

Example 20.7. Suppose $f : M \to N$ is a holomorphic mapping of varieties, where $M \subseteq \mathbb{P}^m$ is irreducible, and $N \subseteq \mathbb{P}^n$, then f is rational: for any standard open subset $U_i \subseteq \mathbb{P}^n$, we have

$$\begin{array}{ccc} f^{-1}(U_i) & \xrightarrow{f|_{f^{-1}(U_i)}} & N \cap U_i \\ & \searrow & \downarrow \pi_j \\ & & \mathbb{C} \end{array}$$

such that the compositions are all rational for all j's.

Example 20.8. Suppose $M \subseteq \mathbb{P}^n$ is an irreducible variety, and $E \to M$ is a holomorphic vector bundle, then E is algebraic: for some trivialization, the transition matrices $g_{\alpha\beta}$ consist of rational functions. This gives rise to $i_{|E|} : M \to \mathbb{P}^n$.

Let M be a compact complex manifold, and let $|E| = \mathbb{P}V$ be a basepoint-free linear system of $V \subseteq \mathbb{P}(H^0(M, L))$, such that $\dim(|E|) = n + 1$. The dual projective space $(\mathbb{P}^n)^*$ of $\mathbb{P}^n$ can be understood as the parametrized space for hyperplanes in $\mathbb{P}^n$. For $Z \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^*$, there is a projection into $(\mathbb{P}^n)^*$, then the preimage of any point a in the dual space is $(H, \{a\})$ where H is the hyperplane equation $H = \sum a_i z_i = 0$ that defines Z. We now have a function

$$\begin{aligned} i_{|E|} : M &\to \mathbb{P}(V)^* \\ p &\mapsto \{\bar{s} \in \mathbb{P}(V) : s \in V \subseteq H^0(M, L), s(p) = 0\} \end{aligned}$$

to hyperplanes. Given a trivialization L near p, we pick a basis $s_0, \ldots, s_n$ of V in $H^0(M, L)$. This allows us to identify them as holomorphic functions in neighborhoods of p. Locally, they gives $\sum a_i s_i(p) = 0$. To see that they are hyperplanes in $\mathbb{P}(V)$, note that p is not a basepoint, therefore some $s_i(p) \neq 0$.

Less intrinsically, we may compute from the holomorphic mapping

$$\begin{aligned} M &\to \mathbb{P}^n \\ p &\mapsto (s_0(p), \ldots, s_n(p)) \end{aligned}$$

which depends on the choice of basis. We have $s_i(p) \in L_p \cong \mathbb{C}$. For $s' = As$ for some $A \in \mathrm{GL}(n+1)$, we view A as an automorphism on $\mathbb{P}^n$ as matrix multiplication, then we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{i_{|E|,s}} & \mathbb{P}^n \\ & \searrow^{i_{|E|,s'}} & \downarrow A \\ & & \mathbb{P}^n \end{array}$$

[6] Given any $D \geqslant 0$, there exists a unique (up to multiplication of scalar) section $s \in H^0(\mathcal{O}(D))$ such that $(s) = D$. Moreover, note that $H^0(\mathcal{O}(D))$ is exactly the collection of homogeneous polynomials of degree d.

Definition 20.9. We say $f : M \to \mathbb{P}^n$ is non-degenerate if $f(M)$ is not contained in any hyperplane.

Theorem 20.10. There is a one-to-one correspondence between the non-degenerate holomorphic functions $f : M \to \mathbb{P}^n$ modulo the projective automorphisms $\mathrm{PGL}(n+1, \mathbb{C})$, as well as the basepoint-free linear systems $|E|$ for $E \subseteq H^0(M, L)$ of dimension $n + 1$ over $L \in \mathrm{Pic}(M)$.

Proof. Given a basepoint free linear system, we have found a holomorphic mapping unique up to projective automorphisms. To see why $i_{|E|,s}$ is non-degenerate, suppose otherwise, then $i_{|E|,s} \subseteq Z(\sum a_i z_i)$ for $z = (s_0, \ldots, s_n)$, then $i^*_{|E|,s}[H] = L$. Once we interpret z_i as the transform by $[H]$, we note $i^*(z_i) = s_i$. Viewing $L = i^*[H]$, we have $i^*(\sum a_i z_i) \in H^0(M, L)$ such that it is contained in $H^0(\mathbb{P}^n, \mathcal{O}(H))$, therefore $\sum a_i s_i = 0$. However, s_i forms a basis, so this is not possible.

Given a non-degenerate holomorphic map, define $L = f^*[H]$, then coordinates $z_i \in H^0(\mathbb{P}^n, \mathcal{O}(H))$, so we have pullbacks $s_i := f^* z_i \in H^0(M, L)$. Define the linear system $|E|$ to be the set of divisors $(f^* \sum a_i z_i)$, then $\dim(|E|) = n + 1$. The assignment $E \mapsto (f(p) \mapsto (s_0(p), \ldots, s_n(p))$ defines the inverse. $\square$

Example 20.11. Consider the twisted cubic

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ (s, t) &\mapsto (s^3, s^2 t, st^2, t^2) \end{aligned}$$

then up to change of coordinates we may identify it as $i_{|3H|}$. Note that $H^0(\mathbb{P}^1, \mathcal{O}(3H))$ is the collection of degree-3 homogeneous polynomials, then the image defined above gives a basis already, namely with dimension 4. Moreover, the linear system is basepoint-free: given any point, we can find a holomorphic polynomial that is non-vanishing.

Example 20.12. Consider $M = \mathbb{P}^n$, then the only line bundles we have are $L = [dH]$ for $d \geqslant 0$. For any mapping $\mathbb{P}^n \to \mathbb{P}^N$, we have $E \subseteq H^0(\mathbb{P}^n, \mathcal{O}(d))$ where the space of sections have dimension $\binom{n+d}{d} \geqslant N + 1$ whenever $d \geqslant 1$, and $\dim(E) = n + 1$. The complete linear system $|dH|$ is basepoint-free: given any point $p \in \mathbb{P}^n$, without loss of generality we may take $p = (1, 0, \ldots, 0)$, then a non-zero section can be written down, namely $z_0^d \in H^0(\mathcal{O}(d))$. This gives a non-degenerate mapping $i_{dH} : \mathbb{P}^n \to \mathbb{P}^N$ of dimension $N\binom{n+d}{d} - 1$.

- If $n = 1$ and $d = 3$, we cover the twisted cubic in Example 20.11. In fact, i_{dH} is an embedding, usually called the (d-uple) Veronese embedding.
- If $n = 2$ and $N = 5$, then the image of i_{2H} is called Veronese surface.

Example 20.13. We know by Theorem 19.5 that the twisted cubic is an algebraic variety. Denoting the mapping by $(s, t) \mapsto (z_0, z_1, z_2, z_3)$, then we have two identifications as surfaces, given by Q_1 defined by $z_0 z_2 = z_1^2$ and Q_2 defined by $z_1 z_3 = z_2^2$. We know $H^*(\mathbb{P}^3) = \mathbb{C}[H]/H^4$, where the image of the embedding C has class $[C] = 3H^2$. Looking into the perfect pairing $H^2 \times H^4 \to \mathbb{C}$, we note $[Q_1], [Q_2] = 2H$, so $[Q_1 \cap Q_2] = 4H^2 \neq 3H^2$, so $Q_1 \cap Q_2$ is the union of C with some line, namely the line $Z(z_1, z_2)$.

21 Blow-ups

We now ask: when is $i_E : M \to \mathbb{P}^N$ an embedding? We need that

- for any distinct points $p, q \in M$, $i_E(p) \neq i_E(q)$, i.e., i_E separates points;
- $i_{E*} : T'_p M \hookrightarrow T'_{i_E(p)}(\mathbb{P}^N)$, i.e., i_E separates tangents.

We define

$$\begin{aligned} f : M &\to \mathbb{P}^N \\ p &\mapsto (s_0(p), \dots, s_N(p)), \end{aligned}$$

then $I_p(L) = \{s \in \mathcal{O}(L) : s(p) = 0\} \subseteq \mathcal{O}(L)$. Therefore,

$$H^0(M, I_p(L)) = \{s \in H^0(L) : s(p) = 0\}.$$

Locally we get $d_p : H^0(M, I_p(L)) \to T'^*_p \otimes L_p$. Looking at the local trivialization, we note the map is well-defined.

Lemma 21.1. The complete linear system $|L|$ is

1. basepoint-free if and only if for any $p \in M$, $H^0(M, L) \twoheadrightarrow L_p$;
2. such that i_L separates points if and only if for any distinct $p, q \in M$, we have $H^0(M, L) \twoheadrightarrow L_p \oplus L_q$;
3. such that i_L separates tangents if and only if for any $p \in M$, $d_p : H^0(M, I_p(L)) \twoheadrightarrow T'^*_p \otimes L_p$.

Moreover, the results hold for non-complete linear systems.

Proof.

1. Pick $0 \neq \alpha \in L_p$, then there exists $s \in H^0(M, L)$ such that $s(p) = \alpha \neq 0$, so p is not a basepoint.
2. Pick $0 \neq \alpha \in L_p$, we find $s \in H^0(M, L)$ such that $s(p) \neq 0$ but $a(q) = 0$, then $i_L(p) \neq i_L(q)$, where we think of $i_L(x) = (s(x), \dots, s(x))$, $i_L(p) = (1, 0, \dots, 0)$, and $i_L(q) = (0, 0, \dots, 0)$.
3. It suffices to show that $(i_L)^* : T'^*_{\mathbb{P}^N, i_L(p)} \to T'^*_{M,p}$ is a surjection. We have

$$\begin{array}{ccc} T'^*_{\mathbb{P}^N, i_L(p)} \otimes L_p & \xrightarrow{d_p^* = (i_L)^* \otimes \mathbb{1}_{L_p}} & T'^*_{M,p} \otimes L_p \\ {\scriptstyle d_{i_L(p)}} \uparrow & & \uparrow {\scriptstyle d_p} \\ H^0(\mathbb{P}^N, I_{i_L(p)}(1)) & \xrightarrow[i_L^*]{} & H^0(M, I_p(L)) \end{array}$$

   where we identify $L_p = (H)_{i_L(p)}$ since $i_L^*[H] = L$. It suffices to show that d_p^* is surjective, since we are only tensoring by a one-dimensional vector space. Note that this diagram commutes, and d_p^* is surjective if and only if d_p is surjective, if and only if i_L^* is surjective, as n-dimensional vector spaces. Here

   - $i_L^* z_i = s_i$ and $d_{i_L(p)}$ sends z_i to $dz_i \otimes 1$;
   - d_p sends s_i to $ds_i \otimes 1$, where $H^0(M, I_p(L))$ can be thought of as the span of $s_0, \dots, s_N$, since $H^0(I_p(L))$ is the span of $z_0, \dots, z_N$.

□

Example 21.2. Consider

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^2 \\ (s, t) &\mapsto (s^3, st^2, t^3) \end{aligned}$$

This map separates points but does not separate tangents.

Example 21.3. Consider $\mathbb{P}^n \times \mathbb{P}^m$ with projections π_n, π_m to the corresponding components. We have a bundle

$$\mathcal{O}(d, d') := \mathcal{O}(\pi_n^*[dH_n] \otimes \pi_m^*[d'H_m])$$

with coordinates $(z, y) \in \mathbb{P}^n \times \mathbb{P}^m$. Now one can show that $H^0(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{O}(d, d'))$ to be the set of bihomogeneous polynomials of degree n in z's and degree m in y's.

Example 21.4. The Segre embedding is the case where $d = d' = 1$: we have $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$, where $N = (n+1)(m+1)+1$, which is the dimension of $H^0(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{O}(1, 1))$, since that would be the dimension of $H^0(\mathbb{P}^n, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^m, \mathcal{O}(1))$, given by $(n + 1) \times (m + 1)$ but subtracted by 1 by projectifying.

In the case

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^3 \\ (z, y) &\mapsto (w_0, w_1, w_2, w_3) = (z_0, y_0, z_0y_1, z_1y_0, z_1y_1) \end{aligned}$$

this is an algebraic variety by Theorem 19.5, and this should be a hyperplane, thus defined by one equation. Namely, this is just given by $w_0w_3 = w_1w_2$. This explains the picture where the surface is given by two rulings.

Theorem 21.5 (Kodaira Embedding Theorem)**.** Let M be a compact complex manifold and let $L \in \mathrm{Pic}(M)$ be a positive line bundle, then for $k \gg 0$, $i_{L^k} : M \to \mathbb{P}^N$ is an embedding.

We need to show that for $k \gg 0$ and any x, y, we have

$$H^0(M, L^k) \longrightarrow L_x \oplus L_y \longrightarrow 0$$

and for any x, we have

$$H^0(M, L^k) \xrightarrow{\ d\ } T_x'^* \oplus L_x^k \longrightarrow 0$$

Incorrect Proof. Consider the exact sequence

$$0 \longrightarrow I_{x,y}(L^k) \longrightarrow \mathcal{O}(L^k) \xrightarrow{\ r\ } L_x^k \oplus L_y^k \longrightarrow 0$$

where $I_{x,y}(L^k)$ is the sheaf of sections of L^k vanishing at x, y. Since the sheaf is supported in x and y only, then we have

$$\cdots \longrightarrow H^0(M, \mathcal{O}(L^k)) \longrightarrow H^0(L_x^k \oplus L_y^k) \cong L_x^k \oplus L_y^k \longrightarrow H^1(M, I_{x,y}(L^k)) \longrightarrow \cdots$$

so we just need to show that $H^1(M, I_{x,y}(L^k)) = 0$. However, this is not a sheaf of vector bundles, so we cannot apply our vanishing theorem. However, if $\dim(M) = 1$, then $I_{x,y}(L^k) \cong \mathcal{O}(L^k \otimes [-x - y]$, so we do get a line bundle, then $H^1(\mathcal{O}|L^k \otimes [-x - y]| = 0$ for $k \gg 0$ by the vanishing theorem. This proves it separates points. One can also show that it separates tangents. We look at

$$0 \longrightarrow I_x^2(L^k) \qquad I_x(L^k) \xrightarrow{\ d_x\ } T_x'^* \otimes L_x \longrightarrow 0$$

where

$$\begin{aligned} d_x : I_x(L^k) &\to T_x'^* \otimes L_x \\ s &\mapsto ds|_x \end{aligned}$$

takes the derivative. Therefore, the kernel is given by the elements with zero derivative, which are sections $I_x^2(L^k)$ that are vanishing to second order. By the same argument as before, we have

$$\cdots \longrightarrow H^0(M, I_x(L^k)) \cong \longrightarrow H^0(T_x'^k \oplus L_x^k) \cong T_x'^* \otimes L \longrightarrow H^1(M, I_x^2(L^k)) \longrightarrow \cdots$$

Again, we ask $H^1(M, I_x^2(L^k)) = 0$, but this is again not a vector bundle, but since we are in dimension 1, then $I_x^2(L^k) \cong \mathcal{O}(L \otimes [-2x])$, so this vanishes when $k \gg 0$.

However, points are not divisors in general, so we cannot easily separates points/tangents by running the $\dim(M) = 1$ argument, which requires this assumption. □

Proof. Instead, we blow-up at $0 \in \mathbb{C}^n$, then we get $\tilde{\mathbb{C}}^n \subseteq \mathbb{C}^n \times \mathbb{P}^{n-1}$ with coordinates $((z_1, \dots, z_n), [\ell_1 : \cdots : \ell_n])$, then the equation is given by $z_i\ell_j = z_j\ell_i$ for all i, j. We study this space by projection $\pi : \tilde{\mathbb{C}}^n \to \mathbb{C}^n$, then we will see that $\pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}$, and $\pi|_{\tilde{\mathbb{C}}^n\setminus\pi^{-1}(0)} : \tilde{\mathbb{C}}^n\setminus\pi^{-1}(0) \xrightarrow{\cong} \mathbb{C}\setminus\{0\}$ is an isomorphism.

We do need to define a complex manifold structure this space before saying it is an isomorphism. Therefore, set $V_i = \{(z, \ell) : \ell_i \neq 0\}$, and we want to define a chart $V_i \xrightarrow{\cong} \mathbb{C}^n$, where we have coordinates $(y_1, \dots, y_n) \in \mathbb{C}^n$ that go to $((0, \dots, 0, y_i, 0, \dots, 0), [y_1 : \cdots : y_{i-1} : 1 : y_{i+1} \cdots : y_n])$ where the specified coordinates are at position i in each tuple.

For instance, in the case $n = 2$, we have $(x_1, x_2) \to ((x_1, x_1x_2), [1, x_2])$ in V_1 and $(y_1, y_2) \to ((y_1y_2, y_2), [y_1, 1])$ in V_2, so now $y_1 = \frac{1}{x_2}$ and $y_2 = x_1x_2$ by comparison, therefore both are holomorphic. In general, this shows that we have a complex manifold structure of dimension n, and we have an isomorphism $\pi|_{\tilde{\mathbb{C}}^n\setminus\pi^{-1}(0)}$ of complex manifolds. The inverse is given by $(z_1, \dots, z_n) \mapsto ((z_1, \dots, z_n), (z_1, \dots, z_n))$, therefore we have a biholomorphism.

We compute that $\pi^{-1}(0)\big|_{V_i}$, where V_i is given by $(y_iy_1, y_iy_2, \dots, y_iy_{i-1}, y_i, y_iy_{i+1}, \dots, y_iy_n)$, then that is just the set $\{y : y_i = 0\}$. We then say $E := \pi^{-1}(0)$ is the exceptional divisor. This turns points into divisors. $\square$

Given a point $p \in M$ in a complex manifold of dimension n, we may blow up at p, which is to choose $p \in U \subseteq M$ where $U \cong \Delta \subseteq \mathbb{C}^n$ locally, then the blow-up $\tilde{M} = (M\setminus\{p\}) \cup \tilde{\Delta}$, where we make the identification that $U\setminus\{p\} \cong \tilde{\Delta}\setminus E$. Therefore, for $\pi : \tilde{M}\setminus E \to M\setminus\{p\}$, we have

$$\begin{array}{ccc} E \cong \mathbb{P}^{n-1} & \hookrightarrow & \tilde{M} \\ \downarrow & & \downarrow \pi \\ \{p\} & \hookrightarrow & M \end{array}$$

Example 21.6. For $n = 2$, suppose C is the curve defined by $y^2 = x^2 + 2x$, then we send $(u_1, u_2) \mapsto ((u_1u_2, u_2), [u : 1])$. We say $\pi^{-1}(C)$ is the total transform. Note that this is given by $u_2 = (u_1u_2)^2 + 2(u_1u_2)$, then by gluing we get $1 = u_1^2u_2 + 2u_1$ and $v_2 = v_1^2 + 2$ which is given by coordinate transformation on the exceptional divisor via $y_1 = \frac{1}{x_2}$ and $y_2 = x_1x_2$ for $(x_1, x_2) \in V_1$ and $(y_1, y_2) \in V_2$, which gives the proper transform. In particular, C and E intersects at one point, therefore under two different coordinate systems we get $(u_1, u_2) = (\frac{1}{2}, 0)$ and $(v_1, v_2) = (0, 2)$. This is given by trivializations of sections of E, and as we will see, this respects local trivialization g_{ij}'s.

Theorem 21.7. $E \cong \mathbb{P}(T'_{M,p})$ is the projectification of the holomorphic tangent space at p.

Proof. Without loss of generality, say $M = \mathbb{C}^n$, then we may compute locally, where we have $(0, [\ell_1 : \cdots : \ell_n]) \mapsto \sum \ell_i \frac{\partial}{\partial z_i}\Big|_p$, but this assignment is not well-defined (up to scalar multiplication), so we take the projectification. $\square$

Again, we have $[E]_{\mathbb{C}^2}$ given by $g_{12} = \frac{x_1}{y_2} = \frac{1}{x_2}$ by the change of variables, over E defined by $x_1 = 0$ in U_1 and E defined by $y_2 = 0$ in U_2. Note g_{12} can be identified with $\frac{\ell_1}{\ell_2}$, therefore this is the transformation for the universal bundle on $\mathbb{P}^1$, i.e., pullback of universal bundle of $\mathbb{P}^{n-1}$ to the manifold locally defined as $\mathbb{C}^n$. That is, projecting the other way, we get $(E) = \pi_2^* J$ for $\pi_2 : \tilde{\mathbb{C}}^n \to \mathbb{P}^{n-1}$.

Now $\mathcal{O}_{\tilde{M}}(-E)|_E \cong \mathcal{O}_E(-E) \cong \mathcal{O}_E(1)$ by identifying $E \cong \mathbb{P}^{n-1}$ and restricting twice via $\mathcal{O}_{\tilde{M}}(-E) \to \mathcal{O}_{\tilde{U}}(-E) \xrightarrow{\tilde{U}\cong\tilde{\Delta}} \mathcal{O}_E(-E)$. Taking global sections $H^0(M, \mathcal{O}_E(-E))$, we get global sections of $\mathcal{O}_E(1)$ on the projective space. Since the fibers of J are coordinates, then $H^0(M, \mathcal{O}_E(-E)) \cong (T'_p)^*$, i.e., canonical isomorphism to the cotangent space. More explicitly, for local function $f \in \mathcal{O}(U)$ vanishing at p, i.e., inside $H^0(U, I_p)$, pulling back to the blow-up $\pi^* f$, since it vanishes at p, it vanishes along the holomorphic section, so we get $\pi^* f|_E \in H^0(E, \mathcal{O}_E(-E))$, and in particular, this defines a mapping $f \mapsto df|_p \in T_p'^*$. In particular, the diagram

$$\begin{array}{ccc} H^0(E, \mathcal{O}_E(-E)) & \xrightarrow{\cong} & T_p'^* \\ \uparrow & \nearrow_{f \mapsto df|_p} & \\ H^0(U, I_p) & & \end{array}$$

commutes.

Basically, we have

$$\pi^*(V) = \tilde{V} + (\mathrm{mult}_p(V)) \cdot E \in \mathrm{Div}(\tilde{M}) \cong \pi^* \mathrm{Div}(M) + \mathbb{Z} \cdot E.$$

Lemma 21.8. For $L \in \mathrm{Pic}(M)$, we have

- $\pi^* : H^0(M, L) \xrightarrow{\cong} H^0(\tilde{M}, \pi^* L)$, and as a subset, we have
- $H^0(\tilde{M}, \pi^* L \backslash E) \cong H^0(M, I_p(L))$.

Proof. Consider the pullback map π^* as mentioned, we want to construct its inverse. Take $s \in H^0(\tilde{M}, \pi^* L)$, we have $s|_{\tilde{M}\backslash E} \in H^0(\tilde{M}\backslash E, \pi^* L) \cong H^0(M\backslash\{p\}, L)$. In the case $n = 1$, this is proven by Theorem 21.5, then we assume $n \geqslant 2$. By Hartog's theorem, we identify the latter set by $H^0(M, L)$, then this defines the inverse. □

Lemma 21.9. We have $K_{\tilde{M}} = \pi^* K_M + (n-1)E$.

Proof. Let us assume that M has a global meromorphic n-form. For the general case, see text. We choose coordinates $z_1, \ldots, z_n$ centered at p where we blow up, so write $W = \frac{f}{g} dz_1 \wedge \cdots \wedge dz_n$ locally, and write the coordinates on $\tilde{M}$ as $\tilde{M}$, so $z_1 = y_1$ and $z_i = y_1 y_i$ for $i \geqslant 2$. Now let E be defined by $y_1 = 0$, then

$$\pi^* \omega = \frac{\pi^* f}{\pi^* g} (dy_1, \ldots, dy_n)(y_1^{n-1}).$$

By identifying this as a section over $K_{\tilde{M}}$ and patching the local coordinates, we get the formula. □

Lemma 21.10. For positive line bundle L on M, then for $k \gg 0$, $\pi^* L^k \otimes [-E]$ is positive on $\tilde{M}$.

Proof. This is a sketch of the proof in the text. There are two main ideas:

- $\pi^* L$ is positive on $\tilde{M}\backslash E$, and
- $[-E]$ is positive on E.

By partition of unity, we need large enough k to give a metric that ensures positivity globally, i.e., taking positive eigenvalues by tensoring. We then have a flat zero metric outside of E in $\tilde{M}$, and a (pullback of) Fubini-Study metric in a subset $\tilde{U}_\varepsilon$ of E, then we smoothen the metric for the tangent space $T_x'(E) \subseteq T_x'(\tilde{M})$ built upon the region in the middle. The construction ensures that the smoothen portion is still positive for E. Similar idea works for $\pi^* L$ we have positivity on $\tilde{M}\backslash E$, but on $T_x(E)$ it should be identified as zero, then we have positivity on $T_x'(\tilde{M})/T_x'(E)$. For $k \gg 0$, the positivity on $\tilde{M}\backslash E$ overpowers negative eigenvalues. □

Proof of Theorem 21.5. For $x \neq y \in M$, we look at the blow-up over x and y in M. This gives

$$\begin{array}{ccc} H^0(M, L^k) & \longrightarrow & L_X^k \oplus L_Y^k \\ \cong\downarrow & & \| \\ H^0(\tilde{M}, \pi^* L^k) & \xrightarrow[-|_{E_x, E_y}]{} & H^0(E_x, \pi^* L^k|_{E_x}) \oplus H^0(E_y, \pi^* L^k|_{E_x}) \cong L_x^k \oplus L_y^k \end{array}$$

Repeating the proof of the case where $n = 1$, set $E = E_x + E_y$, then we look at

$$0 \longrightarrow \mathcal{O}(\pi^* L^k \backslash E) \longrightarrow \mathcal{O}(\pi^* L^k) \longrightarrow \pi^* L^k|_E \longrightarrow 0$$

By choosing $k \gg 0$, we have surjectivity once we see $H^1(\pi^* L^k \backslash E) = 0$. We follow the steps below.

- Find k_1 such that $L^{k_1} \backslash K_M$ is positive.
- Find k_2 such that $\pi^* L^k - nE$ is positive for $k \geqslant k_2$. Note that the pullback does not spoil positivity.

- For $k \geqslant k_1 + k_2$, we apply Kodaira vanishing theorem and get that
$$\begin{aligned}\mathcal{O}_{\tilde{M}}(\pi^* L^k \backslash E) &\cong \Omega^n_{\tilde{M}}((\pi^* L^k \backslash E) \backslash K_{\tilde{M}}) \\ &\cong \Omega^n_{\tilde{M}}(\pi^*(L^{k_1} \backslash K_M)) \otimes (\pi^* L^{k-k_1} - nE).\end{aligned}$$
By choice, $k - k_1 \geqslant k_2$, $L^{k_1} \backslash K_M$ is positive, therefore $\pi^*(L^{k_1} \backslash K_M)$ is semi-positive, and since $\pi^* L^{k-k_1} - nE$ is positive, then $(\pi^*(L^{k_1} \backslash K_M)) \otimes (\pi^* L^{k-k_1} - nE)$ is positive.
- Taking the long exact sequence, we are done.

We now move on to showing the separation of tangents. We make identifications and get the map
$$d_x : H^0(\mathcal{O}_{\tilde{M}}(\pi^* L^k \backslash E)) \cong H^0(M, I_x(L^k)) \to T'^*_x \otimes L^k_x \cong H^0(E, \mathcal{O}_E(\pi^* L^k \backslash E)),$$
and we want to show that this is a surjection. Again, by the property of the blow-up, we make identification of the sections and get a short exact sequence
$$0 \longrightarrow \mathcal{O}_{\tilde{M}}(\pi^* L^k \backslash 2E) \longrightarrow \mathcal{O}_{\tilde{M}}(\pi^* L^k \backslash E) \longrightarrow \mathcal{O}_E(\pi^* L^k \backslash E) \longrightarrow 0$$
In particular, note that it suffices to show that the second map is a surjection. Again, this comes down to showing that the first term is zero, which is done using a similar argument as the situation before. ☐

Definition 21.11. Suppose M is algebraic as a complex projective manifold. Consider a line bundle L and a map $i_L : M \hookrightarrow \mathbb{P}^N$.

- We say L is ample if $i_{L^k} \mathcal{O}_{\mathbb{P}^N}(1) \cong L^*$ for some k.
- We say that a line bundle L is very ample if $i_L : M \hookrightarrow \mathbb{P}^N$ is an embedding.

Therefore, L is ample if and only if i_{L^k} is very ample for some positive integer k.

Corollary 21.12. Suppose M_1 and M_2 are algebraic as complex projective manifolds, then $M_1 \times M_2$ is algebraic as well.

Proof. Consider positive bundles L_1 and L_2 on M_1 and M_2, respectively, then the bundle $L = \pi_1^* L_1 \otimes \pi_2^* L_2$, along with pulling back positive metrics and taking tensor products, is positive. ☐

Example 21.13. For $M_i \hookrightarrow \mathbb{P}^{N_i}$, we look at the embedding
$$M_1 \times M_2 \hookrightarrow \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \xrightarrow{|\mathcal{O}(1,1)|} \mathbb{P}^{(N_1+1)(N_2+1)-1}$$

Proposition 21.14. Suppose M is algebraic and $p \in M$, and let $\tilde{M}$ be the blow-up of M, then $\tilde{M}$ is algebraic.

Proof. Take positive L on M, then for $k \gg 0$, $\pi^* L^k \backslash E$ is positive in $\tilde{M}$. ☐

Example 21.15. Consider $M = \mathbb{P}^2$ with $p = (1, 0, 0)$, then let H be the hyperplane bundle, then $k\pi^* H \backslash E$ is very ample for $k \gg 0$.

In the case $k = 1$, we have $\pi^* H \backslash E$, then $\tilde{\mathbb{P}^2} \subseteq \mathbb{P}^2 \times \mathbb{P}^1$, with coordinates $((z_0, z_1, z_2), (\ell_1, \ell_2))$. Taking affine coordinates in chart U_0 of $\mathbb{P}^2$ as $(1, x_1, x_2)$ where $x_i = \frac{z_i}{z_0}$, then this is defined by $x_1 \ell_2 = x_2 \ell_1$. In particular,
$$H^0(\tilde{\mathbb{P}^2}, \pi^* H \backslash E) \cong H^0(\mathbb{P}^2, I_p(H)) = \operatorname{span}(z_1, z_2).$$
In this case, we work out that we just need $k = 2$.

22 Riemann-Roch Theorem

Let M be compact and complex of dimension n. First suppose $n = 1$, then we know the Hodge diamond looks like

$$\begin{matrix} & 1 & \\ g & & g \\ & 1 & \end{matrix}$$

Suppose $n = 2$, let us suppose M is Kahler in addition. In this case we have Hodge diamond

$$\begin{matrix} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{matrix}$$

where $p_g = \dim(H^{2,0})$ is the geometric genus, and q is called the irregularity of M. The goal of the sections is to consider $n = 2$, and work out the Hodge numbers. We will also work out a Riemann-Roch Theorem.

Let $D = \sum n_i p_i$ be a divisor of M of degree $\sum n_i$. Recall we have $h^i(D) = \dim(H^i(M, \mathcal{O}(D))) = \dim(H^i(M, D))$, and Euler characteristic $\chi(D) = \sum(-1)^i h^i(D)$. In particular, $h^i(D) = 0$ for $i > n$.

We know the following result for $n = 1$, and we hope to generalize this to $n = 2$.

Theorem 22.1 (Riemann-Roch)**.** Let M be of dimension $n = 1$, then $\chi(D) = \deg(D) + 1 - g = \deg(D) + \chi(\mathcal{O}_M)$ since $\chi(\mathcal{O}_M) = h^0(\mathcal{O}_M) - h^1(\mathcal{O}_M) = 1 - g$.

Theorem 22.2 (Serre Duality)**.** Let M be of dimension $n = 1$, then $\chi(D) = -\chi(K_M - D)$. In particular, $\deg(D) < 0$ implies $h^0(D) = 0$.

Lemma 22.3. $\deg(K_M) = 2g - 2$.

Proof. We have $-(1 - g) = -\chi(\mathcal{O}_M) = \chi(K_M) = \deg(K_M) + 1 - g$. □

Corollary 22.4. Suppose $\deg(D) > 2g - 2$, then $h^1(D) = 0$.

Proof. Note $\deg(K_M - D) < 0$, so by Serre duality we have $h^1(D) = h^0(K_M - D)$, which has negative degree, therefore they are zero. □

Example 22.5. For $\mathbb{P}^1$, we consider $\omega = f(z)dz$ where $f \in \mathcal{M}(\mathbb{P}^1)$, then $(\omega) = (f) + (dz)$, where $(dz) = -2(\infty)$, so by argument of local coordinates we get $\deg(\omega) = 0 - 2 = -2 = 2g - 2$.

Example 22.6. For complex torus $\mathbb{C}/\Lambda$ with $g = 1$, we take $\omega = dz$, then it has no zeros or poles, so $(\omega) = 0$, hence $\deg(K) = 0$.

Proof of Theorem 22.1. This is obviously true for $D = 0$. It now suffices to show Riemann-Roch Theorem for general divisor D implies the Theorem for $D \pm p$ for a point p, as we write $D = \sum n_i p_i - \sum m_j q_j$ for $n_i, m_j > 0$. We take a short exact sequence

$$0 \longrightarrow \mathcal{O}(D - p) \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}_p(D) \longrightarrow 0$$

then $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D - p)) + \chi(\mathcal{O}_P(D))$, but

$$\chi(\mathcal{O}_P(D)) = h^0(\mathcal{O}_p(D)) - h^1(\mathcal{O}_p(D)) = h^0(\mathcal{O}_p) - h^1(\mathcal{O}_p) = 1 - 0$$

thus this says $\chi(D - p) = \chi(D) - 1 = \deg(D) + 1 - g - 1 = \deg(D) - g = \deg(D - p) + 1 - g$ by substituting D for $D - p$. Similarly, we have a proof for D implying $D + p$. □

Consider a divisor D, then a point $p \in D$ is a basepoint if and only if for any $s \in H^0(\mathcal{O}(D))$, $s(p) = 0$. That is, the collection

$$\{s \in H^0(D) : s(p) = 0\} = H^0(D - p) \subseteq H^0(D)$$

as a subspace. Therefore, p is a basepoint if and only if $h^0(D - p) \leqslant h^0(D)$.

We can say even more about this. Take a basis $s_1, \ldots, s_n$ for $[D]$ near p. We then ask when would a general section (as a linear combination) be 0 at p, which is just asking about the linear equations on a_i's. Therefore, this says $h^0(D) - 1 \leqslant h^0(D - p)$. We thereby conclude that $|D|$ is basepoint-free if and only if $h^0(D - p) = h^0(D)$.

Exercise 22.7. A linear system $|D|$ separates points if and only if for any $p \neq q$ as points, we have $h^0(D - p - q) = h^0(D) - 2$.

Exercise 22.8. A linear system $|D|$ separates tangents if and only if for any point p, we have $h^0(D - 2p) = h^0(D) - 2$.

Theorem 22.9. If $\deg(D) \geqslant 2g + 1$, then D is very ample.

Proof. Show that it separates points and tangents. Taking off two points gives $\deg(D - p - q) \geqslant 2g - 1 > 2g - 2$, so the h^1-term is still zero. □

Example 22.10. Every compact Riemann surface of genus 1 embeds in $\mathbb{P}^2$ as a plane cubic curve. For instance, we get an embedding $i_{|3p|} : \mathbb{C}/\Lambda \hookrightarrow \mathbb{P}^2$ since $h^0(3p) = 3+1-1 = 3$. Another way of seeing this is that, consider $H^0(np)$ as vector spaces for $n \geqslant 0$, then it has dimension n. Let us now list a basis for the vector space $H^0(np)$. For $n = 0$, this is given by $\{1\}$; for $n = 1$, this is given by $\{1, x\}$; for $n = 2$, this is given by $\{1, x, y\}$ where $(x) = -2p + \cdots$ and $(y) = -3p + \cdots$. Now the basis for $H^0(4p)$ can be obtained for free, which is $\{1, x, y, x^2\}$. Similarly, we have $\{1, x, y, xy\}$ for $H^0(5p)$. Note that in each case the set is linearly independent. What happens if we consider $H^0(6p)$? That means $x^3 = y^2$, therefore we have an identity of meromorphic functions, so $ax^3 + by^2 + cxy + dx^2 + ex + fy + g \equiv 0$. Now we have a mapping

$$(x, y) : M - (x)_\infty - (y)_\infty \to \mathbb{C}^2,$$

which extends uniquely to a mapping $M \to \mathbb{P}^2$. In particular, the equation we want is a plane curve.

23 Compact Complex Surfaces

Let S be a compact complex surface of dimension 2. If S is Kahler, then we have a Hodge diamond as described in the previous section. We then have intersection pairings on divisors (which is only true in dimension 2): we have

$$\mathrm{Div}(S)\times\mathrm{Div}(S)\xrightarrow{\eta\times\eta}H^2(S,\mathbb{Z})\times H^2(S,\mathbb{Z})\longrightarrow H^4(S,\mathbb{Z})\cong\mathbb{Z}.$$

There is a notion of transversal intersection for effective curves: $D_1\cdot D_2=D_1\cap D_2$.

Lemma 23.1. Suppose D_1 is an (effective) divisor and D_2 is a smooth curve of S, then $D_1\cdot D_2=\deg[D_1]|_{D_2}$.

Proof. Since D_1 is effective, then the sections below are always holomorphic. Assuming D_1 and D_2 intersect transversally, then we choose a section s of $\mathcal{O}(D_1)$ with $(s)=D_1$. In particular, $S|_{D_2}$ is a section of $\mathcal{O}([D_1]|_{D_2})$. To find the degree of this bundle, we take the section and count its divisors: the divisor of $(s|_{D_2})$ on D_2 is given by $(s)\cdot D_2$. □

Example 23.2. Consider $S=\mathbb{P}^2$, and let $H\in H^2(\mathbb{P}^2,\mathbb{Z})$ be generated by the hyperplane class, so $H^2=H\cdot H$. We now have $\deg(\mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{P}^1})=\deg(\mathcal{O}_{\mathbb{P}^1}(1))=1$.

Example 23.3. Let $\tilde{S}$ be the blow-up of S at a point p. We compute the self-intersection E^2 of the exceptional divisor. Note that we cannot move the divisor since it is not transverse, regardless $E^2=\deg[E]|_E=\deg\mathcal{J}_{\mathbb{P}^1}=-1$.

In general, we may want to compute the cohomology of the blow-up. This is given by a topological statement

$$H^2(\tilde{S},\mathbb{Z})\cong\pi^*H^2(S,\mathbb{Z})\oplus(\mathbb{Z}\cdot E).$$

which is computable by Mayer-Vietoris sequence.

Example 23.4. Computing $H^2(\tilde{\mathbb{P}}^2,\mathbb{Z})$ blowing up at a point. We have

$$H^2(\tilde{\mathbb{P}}^2,\mathbb{Z})\cong\mathbb{Z}^2\cong(\mathbb{Z}\cdot\pi^*H)\oplus(\mathbb{Z}\cdot E).$$

Example 23.5. We have $H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathbb{Z})\cong(\mathbb{Z}\cdot F_1)\oplus(\mathbb{Z}\cdot F_2)$, then $F_1^2=0$ and $F_1\cdot F_2=1$.

For $n=1$, recall we have $\chi(\mathcal{O}_M)=1-g$.

Theorem 23.6 (Noether's Formula)**.** For $n=2$, we have

$$\chi(\mathcal{O}_S)=\frac{K^2+e(S)}{12}$$

where $e(S)$ is the Euler class of S.

Example 23.7. If $S=\mathbb{P}^2$, then the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

where $\chi(\mathcal{O}_S)$ is given by the bottom-left diagonal. We have

$$\chi(\mathcal{O}_{\mathbb{P}^2})=h^0(\mathcal{O}_{\mathbb{P}^2})-h^1(\mathcal{O}_{\mathbb{P}^2})+h^2(\mathcal{O}_{\mathbb{P}^2})=1$$

But $K=-3H$, so $K^2=9H^2=9$, so we must have $e(\mathbb{P}^3)=3$.

Example 23.8. Consider $S = \mathbb{P}^1 \times \mathbb{P}^1$, then the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

where $e(S) = 4$ and $\chi(\mathcal{O}) = 1$. Indeed, $K = -2F_1 - 2F_2$, so $K^2 = (-2F_1 - 2F_2)^2 = 4F_1^2 + 4F_2^2 = 8$ since the cross-terms do not matter, thus $1 = \frac{8+4}{12}$.

Example 23.9. Consider $S = \tilde{\mathbb{P}}^2$, then the Hodge diamond is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The canonical bundle of the blow-up is then given by $K_{\tilde{\mathbb{P}}^2} = \pi^* K_{\mathbb{P}^2} + E$, then $\left(K_{\tilde{\mathbb{P}}^2}\right)^2 = 9 - 1 = 8$. Again, $e(S) = 4$.

Let us now compute the Hodge diamond of any smooth projective space.

Example 23.10. Consider $S \subseteq \mathbb{P}^3$ to be a general subspace. The irregularity $q(S) = 0$ by Lefschetz's theorem: since $H^1(S) \cong H^1(\mathbb{P}^3) = 0$, then $2q = 0$. Moreover, we know

$$K_S = \left.(K_{\mathbb{P}^3} + [S])\right|_S = \left.(-4H + 3H)\right|_S = -\left.H\right|_S,$$

and

$$p_g = h^{2,0} = \dim(H^0(S, K_S)) = 0$$

where $K_S = \Omega^2$. We now have $K_S^2 = (-H_S)^2 = H_S^2$, and the intersection number of $S \cap H_1 \cap H_2$ is 3. Therefore, the Hodge diamond looks like

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 3 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Therefore, by Theorem 23.6, we have $\chi(\mathcal{O}_S) = \frac{3+e(S)}{12}$, thus $e(S) = 9$. By Hodge Index Theorem, we note that this is the projective surface blown up at 6 points.