MATH 229A Notes

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1 Lecture 1

Lie groups carry the idea of symmetries. We know that given a regular n-polygon, the dihedral group of order n controls its symmetry across opposite vertices, as well as the rotation operation. This can be extended to a circle (as an infinite polygon), then the symmetry forms an infinite group given by any rotation and any reflection. This can be viewed as a continuous symmetry, which is what Lie groups are studying, e.g., $\mathrm{GL}_2(\mathbb{R})$.

Lie groups are groups together with a compatible manifold structure. The corresponding Lie algebras describe the part of a Lie group close to the identity element. For instance, $S^1 \subseteq \mathrm{O}_2(\mathbb{R})$ describes the rotations of the center O. If we say the horizontal diameter cross the circle at a point M, then the operations g (rotation of angle θ) and h (rotation of angle $\frac{\theta}{2}$) corresponds to points $g(M)$ and $h(M)$, respectively. For any rotation r_θ of angle θ, this creates a curve $\langle r_\theta(M)\rangle_{0\leq\theta\leq\frac{\pi}{2}}$, which is the tangent vector of the curve at $\theta = 0$, i.e., as M in addition of some element of $\mathbb{R}^2$. In particular, these elements altogether form a tangent space of G at $\mathrm{id} \in G$, known as the Lie algebra of G. The collection of tangent vectors for $M \in \mathbb{R}^2 \setminus \{0\}$ is a vector field.

The Lie algebra also has a bracket operation $[u, v] \in \mathfrak{g}$ for $u, v \in \mathfrak{g}$. The data of a Lie algebra is given by a vector space and the associated bracket operation.

Classical examples of Lie groups include special linear group, orthogonal group, unitary group, sympletic groups, and many more, while the ones above are considered as classical.

Usually there are two versions of Lie groups, over $\mathbb{R}$ or over $\mathbb{C}$. The analytic property of a manifold may allow us to think of an open subset U in the Lie group and contains the identity, with the inclusion into $\mathbb{C}^n$, given by $1 \mapsto 0$, with coordinates $z_1, \ldots, z_n$ for elements of G that are in U. Moreover, for small enough z_i's and z_i''s, there is the operation

$$g(z_1, \ldots, z_n) \cdot g(z_1', \ldots, z_n') = g(z_1'', \ldots, z_n'').$$

where $z''_i = f_i(z_1, \ldots, z_n, z'_1, \ldots, z'_n)$. We ask that f_i to be analytic, i.e., holomorphic. With this technique, a Lie group has to do with formal power series locally around the identity:

$$(f_i(Z_1, \ldots, Z_n, Z'_1, \ldots, Z'_n))_{1 \leq i \leq n}$$

as the formal group law.

The lowest degree term in f_i's (quadratic term) generates a bilinear map $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ that sends $[(z_1, \ldots, z_n), (z'_1, \ldots, z'_n)]$ to the quadratic part of $f_i(z_1, \ldots, z_n, z'_1, \ldots, z'_n)$, which describes the Lie algebra of g.

2 Lecture 2

Common notions to remember: topology, induced topology (taken by intersection), quotient topology (taken by preimage), discrete topology (all subsets are open).

Definition 2.1 (Connected)**.** We say a space X is connected if every continuous map $X \to \{0, 1\}$ equipped with discrete topology is constant.

Definition 2.2 (Topological Group)**.** A topological group is a group G endowed with the structure of group and a structure of topological space such that the multiplication map $m : G \times G \to G$ given by $(g_1, g_2) \mapsto g_1 g_2$ and the inverse map $i : G \to G$ given by $g \mapsto g^{-1}$ are continuous.

Example 2.3.

- $G = (\mathbb{R}, +)$,
- $G = \mathrm{GL}_n(\mathbb{C})$,
- Any group G with the discrete topology,
- $G = (\mathbb{Z}_p, +)$, the p-adic integers, given as the limit of discrete groups $\mathbb{Z}/p^n\mathbb{Z}$ over n, along with the p-adic topology of evaluations.

If G is a topological group, then for any subgroup H of G, the subspace topology makes it a topological group as well.

Let G_1 and G_2 be topological groups, then $G_1 \times G_2$ is a topological group with the product topology.

Proposition 2.4. Let G be a topological group with a subgroup H. If H is open, then H is closed.

Proof. Consider $G = \bigcup_{gH \in G/H} gH$, then

$$\begin{aligned} l_g : G &\to G \\ g' &\mapsto gg' \end{aligned}$$

is a homeomorphism. In particular, $l_g(H) = gH$, so gH is open. Therefore, $\bigcup_{\substack{gH \in G/H \\ gH \neq H}} gH$ is open, therefore the complement H is closed. □

The map l_g gives a bijection between neighborhoods of 1 in G and the neighborhoods of g in G.

We denote G° as the largest connected topological subspace of G containing 1.

Proposition 2.5. *G° is a normal subgroup of G.*

Proof. Fix $g \in G^\circ$, then $g^{-1}G^\circ \ni 1$ is connected, then $g^{-1}G^\circ \subseteq G^\circ$. Similarly, $gG^\circ \subseteq G^\circ$, so $gG^\circ = G^\circ$. Moreover, the argument above shows that $g^{-1} \in G^\circ$, so G° is a subgroup of G.

Moreover, $gG^\circ g^{-1}$ is connected and contains 1, so so $gG^\circ g^{-1} \subseteq G^\circ$, therefore G° is a normal subgroup of G. □

Example 2.6. O(3) is not connected, and SO(3) = O(3)°.

Exercise 2.7. If 1 admits a connected open neighborhod, then G° is open.

Proposition 2.8. *If G is connected, then it is generated by any open neighborhood of 1 (as a group).*

Proof. Let V be an open neighborhood of 1 and H is a subgroup of G generated by V. Therefore,

$$H = \{v_1^{\varepsilon_1} \cdots v_n^{\varepsilon_n} \mid n \geq 0, \varepsilon_i \in \{\pm 1\}, v_i \in V\}.$$

Now

$$H = \bigcup_{\substack{n \geq 0 \\ \varepsilon_i \in \{\pm 1\} \\ v_i \in V}} v_1^{\varepsilon_1} \cdots v_n^{\varepsilon_n} V,$$

which is a union of open sets, so H is open, and therefore H is closed. Therefore, $H = G$. □

Let G be a group with subgroup H, There is an induced topological space G/H with the quotient topology, along with the map

$$\begin{aligned} G &\to G/H \\ g &\mapsto gH \end{aligned}$$

In particular, if H is normal, then G/H is a topological group.

Proposition 2.9. Let H be a closed subgroup of G. If H and G/H are connected, then G is also connected.

Proof. Consider the continuous map $f : G \to \{0, 1\}$, and we show that f is constant. Since H is connected, so there exists $a \in \{0, 1\}$ such that $f(H) = \{a\}$. More generally, for $g \in G$, gH is connected, so there exists a_g such that $f(gH) = \{ag\}$. Therefore, the map is constant on cosets. Hence, there exists a map f' such that

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr] & & \{0,1\} \\ & G/H \arrow[ur, dashed, "f'"'] & \end{tikzcd}$$

commutes. So f' is continuous, and since G/H is connected, then f' is constant, therefore f is constant, hence G is connected. ☐

3 Lecture 3

Example 3.1. Let $G = \mathrm{GL}_n(\mathbb{C})$ with multiplication map as the matrix multiplication, sending $(a_{ij})(b_{ij})$ to (c_{ij}). This is continuous since we can express c_{ij} as a polynomial with respect to a_{ij} and b_{ij}. Moreover, there is an inverse map that gives $(a_{ij})^{-1} = \frac{1}{\det((a_{ij}))}(\text{comatrix})^t$, where the coefficients in the comatrix are polynomials in a_{ij}'s. Therefore, coefficients of $(a_{ij})^{-1}$ are rational functions of a_{ij}'s, so the inverse map is also continuous. This makes $\mathrm{GL}_n(\mathbb{C})$ into a topological group.

There are involution operations (as automorphisms of topological group) $(a_{ij}) \mapsto (\overline{a_{ij}})$, and $(a_{ij}) \mapsto (a_{ij})^* = ((a_{ij})^t)^{-1}$.

- The subgroup $\mathrm{O}(n, \mathbb{C})$ is invariant under $*$. This is the group of elements $g \in \mathrm{GL}_n(\mathbb{C})$ preserving the form $\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = x_1y_1 + \cdots + x_ny_n$.
- The subgroup $\mathrm{U}(n)$ is invariant under the map $(a_{ij}) \mapsto (\overline{a_{ij}})^*$. This is the group of elements preserving $x_1\bar{y}_1 + \cdots + x_n\bar{y}_n$.
- The subgroup $\mathrm{GL}_n(\mathbb{R})$ is invariant under the map $(a_{ij}) \mapsto (\overline{a_{ij}})$. The symplectic group $\mathrm{Sp}_n(\mathbb{C}) \subseteq \mathrm{GL}_{2n}(\mathbb{C})$ preserves $x_1y_{n+1} + \cdots + x_ny_{2n} - x_{n+1}y_1 - \cdots - x_{2n}y_n$. So $\mathrm{Sp}(n) = \mathrm{Sp}_n(\mathbb{C}) \cap \mathrm{GL}_{2n}(\mathbb{R})$.
- $\mathrm{O}(n) = \mathrm{GL}_n(\mathbb{R}) \cap \mathrm{O}(n, \mathbb{C})$.

Proposition 3.2. $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{Sp}(n)$ are compact.

Proof. $\mathrm{U}(n) = \{a \in \mathrm{M}_n(\mathbb{C}) : a \cdot \bar{a}^t = 1\}$ is a closed subspace of $\mathrm{M}_n(\mathbb{C})$. To see $\mathrm{U}(n)$ is bounded, for $a \in \mathrm{U}(n)$, $\sum\limits_j a_{ij}\overline{a_{ji}} = 1$, so $|a_{ij}| \leq 1$ for all i, j. Therefore, $\mathrm{U}(n)$ is compact.

Since $\mathrm{O}(n) \subseteq \mathrm{U}(n)$, then $\mathrm{O}(n)$ is bounded as well. Moreover, $\mathrm{O}(n) = \mathrm{GL}_n(\mathbb{R}) \cap \mathrm{U}(n)$, so is closed in $\mathrm{M}_n(\mathbb{C})$. Therefore, $\mathrm{O}(n)$ is compact as well.

The fact that $\mathrm{Sp}(n)$ is compact is left as an exercise. ☐

Remark 3.3.
- $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ is not compact.
- $\mathrm{SL}_n(\mathbb{C})$ is a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$.
- $\mathrm{SL}_n(\mathbb{R})$ is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$.
- $\mathrm{SO}(n) = \mathrm{SL}_n(\mathbb{C}) \cap \mathrm{O}(n)$.
- $\mathrm{SO}(n, \mathbb{C}) = \mathrm{SL}_n(\mathbb{C}) \cap \mathrm{O}(n, \mathbb{C})$.

Proposition 3.4. $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SL}_(n)$, $\mathrm{Sp}(n)$ are connected.

Proof. O(n) acts on $\mathbb{R}^n$, stabilizes at $S^{n-1} = \{x \in \mathbb{R}^n, |x| = 1\}$. This induces a map

$$\mathrm{O}(n-1) \xrightarrow{\sim} \mathrm{Stab}_{\mathrm{O}(n)}((0, \dots, 0, 1))$$

so $\mathrm{O}(n)$ acts transitively on S^{n-1}. Therefore, there is an embedding $\mathrm{GL}_{n-1} \hookrightarrow \mathrm{GL}_n$ where matrices in the image are $n \times n$ matrices where entries in the last row and last column are all 0 except the bottom right one. The stabilizing map is given by

$$\begin{aligned} \mathrm{O}(n) &\to S^{n-1} \\ g &\mapsto g \cdot (0, \dots, 0, 1) \end{aligned}$$

and is continuous (because this is the projection on the last column) and open, therefore induces $\mathrm{O}(n)/\mathrm{O}(n-1) \to S^{n-1}$, which is a bijective continuous (open) map. Therefore, this induced map is a homeomorphism.

For $n \geq 2$, similarly we have a homeomorphism

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) \to S^{n-1}$$

By induction, we note that $\mathrm{SO}(1) = \{1\}$ is connected, and suppose $\mathrm{SO}(n)$ is connected, then since S^n is connected, so $\mathrm{SO}(n+1)$ is also connected.

The rest are left as exercises. ☐

Claim 3.5. $\mathrm{O}(n)^\circ = \mathrm{SO}(n)$. In particular, $\mathrm{O}(n)$ is not connected.

Proof. The relation $\mathrm{SO}(n) \leq \mathrm{O}(n)^\circ \lhd \mathrm{O}(n)$ has index 2 in total. $\square$

We now consider the Hausdorff topological spaces, that is, a space X such that for all distinct $x, x' \in X$, there exists open neighborhoods $U \ni x$ and $U' \ni x'$ such that $U \cap U' = \varnothing$.

Definition 3.6 (Base)**.** A base for X is a subset B of the set of open sets such that for all $x \in X$ and every open neighborhood $U \ni x$, there exists $V \in B$ with $x \in V \subseteq U$.

Remark 3.7. A base for X determines a topology on X.

We will work over Hausdorff spaces with countable base.

Example 3.8. $\mathbb{R}$ has a countable base (since $\mathbb{Q}$ is dense in $\mathbb{R}$).

Definition 3.9 (Topological Manifold)**.** A C^k-manifold is a topological space that is homeomorphic to an open subset of $\mathbb{R}^n$ for some n and such that the transition maps are C^k.

If there exists φ such that $x \in U \subseteq X$ is homeomorphic to open subset $V \subseteq \mathbb{R}^n$ and φ' such that $x' \in U' \subseteq X$ is homeomorphic to open subset $V' \subseteq \mathbb{R}^n$, then the intersection $U \cap U'$ is homeomorphic to manifolds M_1 and M_2 with respect to the two maps. In particular, $\varphi' \varphi^{-1} : M_1 \to M_2$ on $\mathbb{R}^n$ defines a transition map.

4 Lecture 4

Let X be a topological space (Hausdorff with countable bases). An atlas for X is the data of a family of open subsets $U \in \mathcal{U}$ of X with $\{f_U : U \to \mathbb{R}^n \text{ for some } n\}_{U \in \mathcal{U}}$ such that 1) $\bigcup\limits_{V \in \mathcal{U}} V = X$ and 2) f_U gives a homeomorphism $U \xrightarrow{\sim} f(U)$ and $f(U)$ is open in $\mathbb{R}^n$.

Fix an atlas with $U, V \in \mathcal{U}$. Then $f_U(U \cap V) \subseteq \mathbb{R}^n$ and $f_V(U \cap V) \subseteq \mathbb{R}^m$ are open subsets. Therefore, the intersection $U \cap V$ is homeomorphic to $f_U(U \cap V)$ through $(f_U)|_{U \cap V}$, and is homeomorphic to $f_V(U \cap V)$ through $(f_V)|_{U \cap V}$. Therefore, we have a map

$$f_U(U \cap V) \xrightarrow{(f_v)|_{U \cap V} \circ (f_U)^{-1}|_{U \cap V}} f_V(U \cap V)$$

between the two open subsets. These are called the transition maps.

Definition 4.1 (Analytic Atlas)**.** An analytic atlas for X is an atlas such that the transition maps are analytic.

Let X be a topological space with an analytic atlas $\mathcal{U}$. We can define a sheaf $\mathcal{O}$ on X (sheaf of analytic functions on X. A sheaf on X is the data of a vector space $\mathcal{F}(U)$ for U open in X, such that given open subset $U \subseteq V$, a linear map $\mathrm{Res}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ such that given $U \subseteq V \subseteq W$, the composition restriction maps is just the restriction map from W to U. Moreover, given $U = U_1 \cup U_2$, the diagram

$$\begin{tikzcd}
\mathcal{F}(U_1 \cup U_2) \arrow[r] \arrow[d] & \mathcal{F}(U_1) \arrow[d] \\
\mathcal{F}(U_2) \arrow[r] & \mathcal{F}(U_1 \cap U_2)
\end{tikzcd}$$

is a pullback.

Example 4.2. Let $\mathcal{O}^{\text{disc}}$ be the sheaf such that for each open subset U, $\mathcal{O}^{\text{disc}}(U)$ is the set of maps from U to $\mathbb{R}$. For $U \in \mathcal{U}$, we can define $\mathcal{O}(U) = \{\varphi : U \to \mathbb{R} | f_U(U) \xrightarrow{f_U^{-1}} U \xrightarrow{\varphi} \mathbb{R}$ is analytic$\}$, so $\mathcal{O}$ is the unique subsheaf of $\mathcal{O}^{\text{disc}}$ extending the values on $U \in \mathcal{U}$ defined above.

Definition 4.3. Two analytic atlases for X are equivalent if they have the same sheaf $\mathcal{O}$.

Definition 4.4. An analytic manifold is a topological space together with an equivalence class of analytic atlases.

Let X be an analytic manifold. Let $x \in X$. define $\mathcal{O}_x = \operatorname{colim}_{x \in U} \mathcal{O}(U) = \{(\varphi_U)_{U \ni x}, \varphi_U \in \mathcal{O}_U\}/\sim$, where $\sim$ is the relation such that $(\varphi_U) \sim (\varphi'_U)$ if and only if there exists an open neighborhood $V \ni x$ such that $\varphi_V = \varphi'_V$.

Note that if we define N_x to be the category of open neighborhoods $U \ni x$ with maps $U \to V$ if $U \subseteq V$, then $\mathcal{O}$ is now a functor $\mathcal{O} : N_x^{\text{op}} \to$ vector space, then $\mathcal{O}_x$ is the colimit of this functor. Concretely, $\mathcal{O}_x = \{(U, \varphi) \mid N \ni x, \varphi \in \mathcal{O}(U)\}/\sim$ where $(U, \varphi) \sim (U', \varphi')$ if there exists some $x \in V \subseteq U \cap U'$ such that $\operatorname{Res}_V^U(\varphi) = \operatorname{Res}_V^{U'}(\varphi')$.

Now let $X = \mathbb{R}^n$ be an analytic manifold with $\mathcal{U} = \{\mathbb{R}^n\}$, then for $x \in \mathbb{R}^n$, $\mathcal{O}_x$ is the set of power series at x that have a non-zero radius of convergence. In particular, $\mathcal{O}_0 \subseteq \mathbb{R}[[x_1, \ldots, x_n]]$.

For a general X, let $\mathcal{U}$ be an analytic atlas. For $x \in X$, there exists $x \in U \in \mathcal{U}$ with $f_U : U \to \mathbb{R}^n$ such that $\mathcal{O}_x$ is equivalent to the convergent power series at $f_U(x)$, where we corresponds $\psi \circ f_U$ with ψ.

This gives rise to a evaluation map $\text{ev}_x : \mathcal{O}_x \to \mathbb{R}$ that sends $\varphi \mapsto \varphi(x)$, then $\mathcal{O}_x$ is a commutative $\mathbb{R}$-algebra. Let $\mathfrak{m}_x$ be the kernel of ev_x, then $\mathcal{O}_x$ is a local ring with maximal ideal $\mathfrak{m}_x$.

The tangent space of X at x is $T_{X,x} = (\mathfrak{m}_x/(\mathfrak{m}_x)^2)^*$, where $\mathfrak{m}_x = \{\varphi \mid \varphi(x) = 0\}$ and $\mathfrak{m}_x^2 = \{\varphi \mid \varphi(x) = \varphi'(x) = 0\}$.

Example 4.5. Let $X = \mathbb{R}^n$ with $x = 0$, then $\mathcal{O}_x = \mathbb{R}\{x_1, \ldots, x_n\}$ is the convergent power series, with the maximal ideal $\mathfrak{m}_x = \sum_{i=1}^{n} X_i \mathbb{R}\{x_1, \ldots, x_n\}$, then this corresponds to $\mathbb{R}^n \xrightarrow{\sim} \mathfrak{m}_x/\mathfrak{m}_x^2$.

Example 4.6. $T_{S^1,x}$ for a point x on S^1 is just the set of tangent lines that passes $x \in S^1$.

5 Lecture 5

Definition 5.1. A Lie group is a group G with a structure of analytic manifold such that the multiplication and inverse maps are analytic.

Example 5.2.

- $G = (\mathbb{R}, +)$,
- $G = \mathrm{GL}_n(\mathbb{R})$,
- $G = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Instead of analytic manifold, we can ask for C^k-manifold or C^k Lie group.

Theorem 5.3 (Hilbert's 5th Problem)**.** Every C^0-Lie group can be given uniquely the structure of an analytic Lie group.

The examples above are real Lie groups. Similarly, there are complex Lie groups using complex analytic manifolds (locally are open in $\mathbb{C}^n$).

Definition 5.4. Let X and Y be manifolds. Then the morphism of manifolds $\varphi : X \to Y$ is analytic if φ is continuous and given any $x \in X$, the following holds: let $y = \varphi(x)$, then there exists open neighborhood V of y such that $f : V \hookrightarrow \mathbb{R}^n$ in an analytic atlas, and there exists open neighborhood U of x such that $f' : U \hookrightarrow \mathbb{R}^{n'}$ in an analytic atlas. Moreover, let $U' = U \cap \varphi^{-1}(V)$, then the diagram

$$\begin{tikzcd}
U' \arrow[r, "\sim"] \arrow[d, "\varphi"'] & f'(U') \subseteq \mathbb{R}^{n'} \arrow[d, "g"] \\
\varphi(U') \arrow[r, "\sim"'] & f(\varphi(U')) \subseteq \mathbb{R}^n
\end{tikzcd}$$

where g is required to be analytic.

Let G and H be Lie groups, then a morphism $G \to H$ is a morphism of groups that is analytic.

Theorem 5.5. Let G and H be Lie groups, and $\varphi : G \to H$ be a continuous morphism of groups. Then φ is analytic.

Example 5.6. Let $G = \mathbb{R}$ and $H = \mathrm{GL}_n(\mathbb{C})$. Then $\varphi : \mathbb{R} \to \mathrm{GL}_n(\mathbb{C})$ be a continuous morphism. One can show that there exists $A \in M_n(\mathbb{C})$ such that $\varphi(t) = e^{tA}$ for all $t \in \mathbb{R}$.

Example 5.7. Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $\mathbb{R} \to S^1 \times S^1$ be an injective morphism of Lie groups defined by $t \mapsto (e^{it}, e^{it\alpha})$. The image will not be called a Lie subgroup.

Theorem 5.8 (Inverse Function Theorem)**.** Let X, Y be manifolds and $\varphi : X \to Y$ be analytic. Let $x \in X$ and $y = \varphi(x)$. Then φ is locally an isomorphism at x (i.e., there exists open neighborhood $U \ni x$ such that $\varphi|_U : U \to \varphi(U)$ is an isomorphism) if and only if $T_x\varphi : T_{X,x} \to T_{Y,y}$ is an isomorphism. Therefore, for some enough U, we have open maps $U \hookrightarrow \mathbb{R}^n$ and $\varphi(U) \to \mathbb{R}^{n'}$ given by $x \mapsto 0$ and $y \mapsto 0$, respectively. Therefore, $T_x\varphi : \mathbb{R}^n \to \mathbb{R}^{n'}$ is just the Jacobian at 0.

Theorem 5.9. Let $\varphi : X \to Y$ be analytic over connected manifolds X and Y. Let $x \in X$ and $y = \varphi(x)$. Consider the map $T_x\varphi : T_x \to T_y$, and denote r as the rank of the map, $m = \dim(X) = \dim(T_x)$ and $n = \dim(Y) = \dim(T_y)$, then $T_x(\varphi)$ is a diagonal $n \times m$ matrix of the form $\mathrm{diag}(1, \ldots, 1, 0 \ldots, 0)$ under suitable coordinates, where the first r entries are 1's. Then φ is given locally around x by

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$$

where there are $n - r$ 0's.

Example 5.10. Consider $X = \mathbb{R}$ and $Y = S^1 \times S^1$. Then $\varphi(t) = (e^{it}, e^{i\alpha t})$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $x = 0$ and $y = (1, 1)$, then $V = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \hookrightarrow \mathbb{R}^2$ gives a map to Y defined by $(v_1, v_2) \mapsto e^{(iv_1, iv_2)}$, where the image is open neighborhood of $(1, 1)$. Therefore, $T_y \simeq \mathbb{R}^2$. This induces

$$\begin{tikzcd}
T_0\mathbb{R} \arrow[r] \arrow[d, equal] & T_1(S^1 \times S^1) \\
\mathbb{R} \arrow[r] & \mathbb{R}^2 \arrow[u, "\sim"]
\end{tikzcd}$$

where the bottom map is given by $t \mapsto (t, \alpha t)$. However, the problem is U as an open neighborhood of 1 in $S^1 \times S^1$ does not have a nice intersection $U \cap f(\mathbb{R})$.

6 Lecture 6

Definition 6.1. Let Y be a manifold and $X \subseteq Y$. We say X is regularly embedded in Y if given any $x \in X$, there exists an open neighborhood of x in Y and analytic functions $f_1, \ldots, f_r$ on Y such that

- $U \cap X = \{y \in U \mid f_1(y) = \cdots f_r(y) = 0\}$, and
- $df_1, \dots, df_r$ are linearly independent linear forms on $T_{Y,x}$.

Remark 6.2. If X is regularly embedded in Y, then X becomes a manifold. The maps $T^*_{X,x} \to T^*_{y,x}$ are injective with cokernel of dimension r. Locally, $X \hookrightarrow Y$ is isomorphic to $\mathbb{R}^{n-r} \hookrightarrow \mathbb{R}^n$ defined by $(v_1, \dots, v_{n-r}) \to (v_1, \dots, v_{n-r}, 0, \dots, 0)$.

Definition 6.3 (Lie Subgroup)**.** Let G be a Lie group and H be a subgroup of G. H is a Lie subgroup if H is regularly embedded in G.

Remark 6.4. H being closed in G implies H is a Lie subgroup of G.

Example 6.5.

- In dimension 1, there are two connected Lie groups, $\mathbb{R}$ and S^1.
- In dimension 2, there is one non-abelian connected Lie group: $\{x \mapsto ax + b \mid b \in \mathbb{R}, a > 0\}$ as maps on $\mathbb{R}$, denoted by the matrix

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

- In dimension 3, there are four non-abelian connected Lie groups:
  - The Heisenberg group given by $\mathbb{R}$-matrices of the form

    $$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

    where $ad > 0$.
  - $\mathrm{SL}_2(\mathbb{R})$,
  - $\mathrm{SO}(3)$,
  - $\mathrm{SU}(2)$. This is the group of matrices of the form

    $$\begin{pmatrix} a & b \\ -\bar{b} & -\bar{a} \end{pmatrix}$$

    where $|a|^2 + |b|^2 = 1$. This is isomorphic to the quaternion of group with norm 1, and is diffeomorphic to S^3 as manifolds.

Remark 6.6. There is a morphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Since $\mathrm{SU}(2) \subseteq \mathbb{H}^\times$, we know $\mathrm{SU}(2)$ acts by conjugation as an algebraic action of $\mathbb{H}$. Therefore, $\mathbb{H} = \mathbb{R} \cdot 1 + \oplus L$, where $L = \mathbb{R}i \oplus \mathbb{R}j\mathbb{R}k$. Now $\mathrm{SU}(2)$ stabilizes the decomposition above, then the morphism $\mathrm{SU}(2) \to \mathrm{GL}(L)$ restricts to the map $\mathrm{SU}(2) \to \mathrm{SO}(3)$. This is a surjection φ with kernel ± 1. Then φ is a covering (locally trivial fibration). Here $\mathrm{SO}(3)$ is not simply connected, and $\mathrm{SU}(2)$ is its universal covering space with $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2\mathbb{Z}$. We note that the two groups are locally isomorphic, i.e., $T_{\mathrm{SU}(2),1} \cong T_{\mathrm{SO}(3),1}$, and have isomorphic Lie algebras.

Example 6.7 (Complex Lie Groups)**.** $\mathrm{SL}_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ is regularly embedded with $f(g) = \deg(g) - 1$, which has $df \neq 0$ on the domain.

Similarly, $\mathrm{O}_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ is regularly embedded.

$\mathbb{C}F^{\times}$ is the only connected complex Lie group of dimension 1.

7 Lecture 7

Definition 7.1. A Lie algebra over k is a k-vector space $\mathfrak{g}$ together with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by $(a, b) \mapsto [a, b]$ such that

1. $[a, a] = 0$ for all $a \in \mathfrak{g}$,
2. $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ for all $a, b, c \in \mathfrak{g}$, called the Jacobi identity.

Remark 7.2. Note that (1) implies $0 = [a + b, a + b] = [a, b] + [b, a] + [a, a] + [b, b]$, so $[a, b] = -[b, a]$.

Remark 7.3. There is a functor from k-algebra to k-Lie algebra, given by sending A with multiplication to A with Lie bracket $[a, b] = ab - ba$. This functor has a left adjoint, the universal enveloping algebra, denoted $S(\mathfrak{g})$.

Example 7.4.

- $\mathfrak{gl}(k) = M_n(k)$ with $[a, b] = ab - ba$.
- $\mathfrak{sl}_n(k) = \{a \in M_n(k) \mid \mathrm{Tr}(a) = 0\}$ is a Lie subalgebra of $\mathfrak{gl}_n(k)$.

Example 7.5. Let A be a k-algebra. A derivation D of A is a map $A \to A$ such that $D(ab) = D(a)b + aD(b)$. The set of derivations $\mathrm{Der}(A)$ is a subset of $\mathfrak{gl}(A)$ as the Lie algebra of k-linear maps on A. This is now a Lie subalgebra.

Definition 7.6. An abelian Lie algebra is a Lie algebra with $[a, b] = 0$ for all a, b. Therefore, the data of abelian Lie algebra is just the data of vector space.

Let G be a Lie group (over $k = \mathbb{R}$ or $\mathbb{C}$). Let U be an open neighborhood of 1 in G with $f : U \hookrightarrow k^n$ that sends $1 \mapsto 0$ for some n. There exists another open neighborhood of 1 denoted $V \subseteq U$ such that given $g, g' \in V$, we have $gg' \in U$.

Let $V' = f(V)$ and $U' = f(U)$, then this induces a map $V' \times V' \to U'$ as a subset of $k^n \times k^n \to k^n$, such that $(x, y) \mapsto f(f^{-1}(x) \cdot f^{-1}(y))$. In general, for f_i analytic, we know this is a map

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (f_1(x_1, \ldots, x_n, y_1, \ldots, y_n), \ldots, f_n(x_1, \ldots, x_n, y_1, \ldots, y_n))$$

In particular, for any f_i, there is a power series F_i converging to it with respect to $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$.

Remark 7.7 (Properties).

- $F_i(X_1, \ldots, X_n, 0, \ldots, 0) = X_i$,
- $F_i(0, \ldots, 0, Y_1, \ldots, Y_n) = Y_i$,
- $F(X, F(Y, Z)) = F(F(X, Y), Z)$.

Therefore, this says that F_i's have no constant terms.

Definition 7.8. A formal group law in n variables is the data of

$$F \in (k\,[[X_1, \ldots, X_n, Y_1, \ldots, Y_n]])^n$$

satisfying these properties.

Remark 7.9. The existence of inverse is a consequence of the properties above. Moreover, there exists $\varphi \in (k\,[[X_1, \ldots, X_n]])^n$ such that $F(X, \varphi(X)) = F(\varphi(X), X) = 0$.

Example 7.10. For $n = 1$, we have the additive formal group $F(X, Y) = X + Y$ and the multiplicative formal group $F(X, Y) = X + Y + XY$.

If $k \supseteq \mathbb{Q}$, then the additive and multiplicative formal groups are isomorphic, with $\psi(X) = e^X - 1 \in k\,[[X]]$, such that $F_{\text{mult}}(\psi(X), \psi(Y)) = \psi(F_{\text{add}}(X, Y))$.

Example 7.11. The map $\mathrm{GL}_d(k) \to M_d(k) = k^{d^2}$ that sends $g \mapsto g - 1$ gives a formal group law in d^2-variables, given by $F(X, Y) = (X + 1)(Y + 1) - 1 = X + Y + XY$.

Let us denote F to be a formal group law and write

$$F(X, Y) = X + Y + \left(\sum_{a,b,i} \alpha X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_n^{b_n} \right)_{1 \le i \le n}$$

with $\sum a_J > 0$ and $\sum b_j > 0$. Define $B(X, Y)$ to be the part with $\sum a_i = \sum b_i = 1$, and define $B_i(X, Y)$ to be the linear combinations of $X_j Y_{j'}$'s. If we view B_i as a bilinear map $\beta_i : k^n \times k^n \to k$ and define $\mathfrak{g} = k^n$, then $[u, v] = (\beta_1(u, v) - \beta_1(v, u), \ldots, \beta_n(u, v) - \beta_n(v, u))$.

Proposition 7.12. $\mathfrak{g}$ is a Lie algebra.

Theorem 7.13. The map $F \mapsto \mathfrak{g}$ is an equivalence of categories from the category of formal group laws in n variables to the category of Lie algebras of dimension n.

8 Lecture 8

Recall that Lie groups give rises to formal group laws, which is categorically equivalent to Lie algebras. We now look for the correspondence regarding connected and simply connected Lie groups.

Given a formal group law in n variable, this is equivalent to a group structure on k^n, multi-variable given by the power series, without requiring convergence. Then $F(X,Y) \in k\left[[X_1,\dots,X_n,Y_1,\dots,Y_n]\right]^n$ such that $F(0,Y)=Y$ and $F(X,0)=X$, and $F(X,F(Y,Z)) = F(F(X,Y),Z)$.

Let F be in n variables and F' be in n'-variables, then a morphism $F \to F'$ is the data of $f \in k\left[[X_1,\dots,X_n]\right]^{n'}$ such that $f(F(X,Y)) = F'(f(X),f(Y))$, where we have $f(X) = (f_1(X),\dots,f_{n'}(X))$.

Example 8.1. Consider $G = \mathrm{GL}_n(k) \hookrightarrow M_n(k)$ mapping $a \mapsto a-1$, with the formal group law $F(X,Y) = X+Y+XY$. There is a Lie algebra structure on $M_n(k)$ given by $(U,V) \mapsto UV-VU$, and we obtain $\mathfrak{gl}_n(k)$.

To obtain a formal group law from a Lie algebra, we use the Baker-Campbell-Hausdorff formula to solve this quantization problem:

$$\exp(x)\exp(y) = \exp(x+y).$$

This does not hold for matrices unless $xy=yx$. In general, we have

$$\exp(x)\exp(y) = \exp\left(x+y+\frac{1}{2}[x,y]+\cdots\right)$$

where $[x,y] = xy-yx$, and the series involves sums of iterated brackets. This takes place in the (completion of) free Lie algebra on x and y.

Definition 8.2 (Enveloping Algebra)**.** Let k be a field. Consider

- a functor from the category of k-algebras to the category of k-Lie algebras, defined by $A \mapsto (\mathfrak{g}=A, [a,a'] = aa'-a'a)$.
- a Lie algebra $\mathfrak{g}$ with tensor algebra $T(\mathfrak{g}) = k \oplus \mathfrak{g} \oplus (\mathfrak{g}\otimes\mathfrak{g}) \oplus \cdots$. Therefore, if $\{e_i\}_{i\in I}$ gives a basis of $\mathfrak{g}$, then $T(\mathfrak{g}) = k\left\langle \{X_i\}_{i\in I}\right\rangle$ is the ring of non-commutative k-polynomials.

The enveloping algebra of $\mathfrak{g}$ is $U(\mathfrak{g}) = T(\mathfrak{g})/I$, where I is a two-sided ideal generated by $a\otimes b - b\otimes a - [a,b]$ (where $a\otimes b - b\otimes a \in \mathfrak{g}\otimes\mathfrak{g}$ and $[a,b]\in\mathfrak{g}$) for $a,b\in\mathfrak{g}$. $S(\mathfrak{g})$, the symmetric algebra, does not modulo out the commutator $[a,b]$. In an abelian Lie algebra, the symmetric algebra coincides with enveloping algebra.

Proposition 8.3. $\mathfrak{g} \mapsto U(\mathfrak{g})$ is the left adjoint to the forgetful functor from k-algebras to k-Lie algebras.

Proof. Let A be an algebra and $\mathfrak{g}$ be a Lie algebra. Then

$$\mathbf{Hom}_{\mathbf{Alg}}(U(\mathfrak{g}), A) \cong \mathbf{Hom}_{\mathbf{LieAlg}}(\mathfrak{g}, A).$$

Indeed, the left-hand side is $\{f \in \mathbf{Hom}_{\mathbf{Alg}}(T(\mathfrak{g}), A) \mid f(a \otimes b) = f(b \otimes a) = f([a, b]) \ \ \forall a, b \in \mathfrak{g}\}$, which is equivalent to $\mathbf{Hom}_{k\text{-}\mathbf{V.S.}}(g, \mathfrak{A})$. Also note that the right-hand side is $\{f' \in \mathbf{Hom}_{k\text{-}\mathbf{V.S.}}(\mathfrak{g}, A) \mid f'([a, b]) = f'(a)f'(b) - f'(b)f'(a) \ \ \forall a, b \in \mathfrak{g}\}$, but $f'([a, b]) = f'(a)f'(b) - f'(b)f'(a)$ implies $f'([a, b]) = f(ab) - f(ba)$, so by restriction the correspondence needed is just sending from left-hand side to right-hand side by sending f to f'. ☐

Theorem 8.4 (Poincaré-Birkhoff-Witt)**.** Fix a basis $\mathcal{B}$ of $\mathfrak{g}$, and a total order on $\mathcal{B}$. Then $U(\mathfrak{g})$ has a basis $\mathfrak{g}$ given by (the images of) $b_1 \otimes \cdots b_r$ for $r \geq 0$, $b_i \in \mathcal{B}$, and $b_1 \leq b_2 \leq \cdots \leq b_r$.

Remark 8.5. It is not obvious that the composition $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$ is injective. Note that $U(\mathfrak{g})$ is a filtered algebra, given by a filtration of subspaces $U(\mathfrak{g})^{\leq i}$ (as the images of $k \oplus \mathfrak{g} \oplus \cdots \mathfrak{g}^{\otimes i}$:

$$0 = U(\mathfrak{g})^{\leq -1} \subseteq U(\mathfrak{g})^{\leq 0} \subseteq \cdots \subseteq \mathfrak{g}$$

Therefore the filtered algebra is $\bigcup\limits_i U(\mathfrak{g})^{\leq i} = U(\mathfrak{g})$ and that for all $p \in U(\mathfrak{g})^{\leq i}$ and $q \in U(\mathfrak{g})^{\leq j}$ we have $U(\mathfrak{g})^{\leq i+j}$. Then the induced graded algebra is

$$(\mathrm{gr}(U(\mathfrak{g}))^i := U(\mathfrak{g})^{\leq i}/U(\mathfrak{g})^{\leq i-1}$$

for $i \geq 0$, with graded vector spaces

$$\mathrm{gr}(U(\mathfrak{g})) = \bigoplus_{i \geq 0} (\mathrm{gr}(U(\mathfrak{g})))^i.$$

Suppose $U = U(\mathfrak{g})$ and $\bar{U} = \mathrm{gr}(U(\mathfrak{g}))$, then for $u \in \bar{U}^i$ and $v \in \bar{U}^j$, take $a \in U^{\leq i}$ and $b \in U^{\leq j}$ such that $u = a + U^{\leq i-1}$ and $v = b + U^{\leq j-1}$, then $ab \in U^{\leq i+j}$. Put $uv = ab + U^{\leq i+j-1}$, then this only depends on u and v but not on a and b.

We have

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, hook] \arrow[d, hook] \arrow[rrd] \arrow[rrdd, two heads] & T(\mathfrak{g}) \arrow[r, two heads] & U(\mathfrak{g}) \\
S(\mathfrak{g}) \arrow[rrdd, dashed] & & U(\mathfrak{g})^{\leq i} = U(\mathfrak{g})^{\leq 0} + \mathrm{im}(\mathfrak{g}) \arrow[u, hook] \arrow[d, two heads] \\
 & & (\mathrm{gr}(U(\mathfrak{g})))^1 \arrow[d, hook] \\
 & & \mathrm{gr}(U(\mathfrak{g}))
\end{tikzcd}$$

Lemma 8.6.

- $\mathrm{gr}(U(\mathfrak{g}))$ is generated by $(\mathrm{gr}(U(\mathfrak{g})))^1$.
- $\mathrm{gr}(U(\mathfrak{g}))$ is a commutative algebra.

Proof.

- $T(\mathfrak{g})$ is generated by $\mathfrak{g}$ as an algebra, therefore so is $U(\mathfrak{g})$, hence so is $\mathrm{gr}(U(\mathfrak{g}))$.
- For $a, b \in \mathfrak{g}$, we have $a \otimes b - b \otimes a - [a, b] \in I$. In $U(\mathfrak{g})$ we know $ab - ba = [a, b] \in U(\mathfrak{g})^{\leq 1}$, so in $\mathrm{gr}(U(\mathfrak{g}))$ we have $(a + U^{\leq 0})(b + U^{\leq 0}) - (b + U^{\leq 0})(a + U^{\leq 0}) \subseteq U(\mathfrak{g})^{\leq 1}$. Therefore, it is evaluated to be 0 in $\mathrm{gr}(U(\mathfrak{g}))^{\leq i}$.

☐

Theorem 8.7 (Poincaré-Birkhoff-Witt, Second Version)**.** The induced algebraic map $\rho : S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ is an isomorphism.

9 Lecture 9

Example 9.1. Let $\mathfrak{g}$ be abelian, $U(\mathfrak{g}) = S(\mathfrak{g}) = k[(X_b)_{b \in B}]$, where $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$.

Proof of Theorem 8.4. We want to show that $\mathcal{F} = \{f(b_1 \otimes \cdots \otimes b_r)\}_{b_1 \leq \cdots \leq b_r}$ is a basis of $U(\mathfrak{g})$. We first prove the generating property. Denote $U(\mathfrak{g})^{\leq n}$ be the image of $k \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{\otimes n}$. We prove by induction on n that $U(\mathfrak{g})^{\leq n}$ is contained in the subspace generated by $\mathcal{F}$.

Suppose $a_1, \ldots, a_{n+1} \in \mathfrak{g}$, then we want to show $a_1 \cdots a_{n+1} := f(a_1 \otimes \cdots \otimes a_{n+1}) \in \mathcal{F}$. By inductive hypothesis, $a_2 \cdots a_{n+1} \in \mathcal{F}$, then it suffices to show that $a \cdot b_1 \cdots b_r \in \mathcal{F}$ for $a \in \mathcal{B}$. If $a \leq b_1$, then $ab_1 \cdots b_r \in \mathcal{F}$, otherwise $ab_1 - b_1 a = [a, b_1]$ in $\mathfrak{g}$, therefore $ab_1 \cdots b_r = b_1 a b_2 \cdots b_r + [a, b_1] b_2 \cdots b_r$, where $[a, b_1] b_2 \cdots b_r \in I(\mathfrak{g})^{\leq r} \subseteq \langle \mathcal{F} \rangle$. We now repeat the process, as $b_1 a b_2 \cdots b_r = b_1 b_2 a b_3 \cdots b_r + b_1 [a, b_2] b_3 \cdots b_r$, where $b_1 [a, b_2] b_3 \cdots b_r \in U(\mathfrak{g})^{\leq r}$, and so on until $b_i \leq a \leq b_{i+1}$, then $b_1 \cdots b_i a b_{i+1} \cdots b_r \in \mathcal{F}$.

To show the linear independence, let $M = \bigoplus\limits_{b_1 \leq \cdots \leq b_r} k \cdot v(b_1, \ldots, b_r)$, and we construct a representation of $U(\mathfrak{g})$ on M. Because $U(\mathfrak{g})$ is generated by the image of $\mathfrak{g}$, then it is just generated by $\mathcal{B}$. We see that $b \in \mathcal{B}$ acts on $v(b_1, \ldots, b_r)$ as follows (by induction on r, and then on b):

- if $b \leq b_1$, then $b \cdot v(b_1, \ldots, b_r) = v(b, b_1, \ldots, b_r)$;
- if $b > b_1$, then proceed the induction on r via the same argument: $bb_1 = b_1 b + [b, b_1]$, so $bb_1 \cdots b_r = b_1 b b_2 \cdots b_r + [b, b_1] b_2 \cdots b_r$. Now we have $b \cdot v(b_1, \ldots, b_r) = b_1 \cdot (b \cdot v(b_2, \ldots, b_r)) + [b, b_1] \cdot v(b_2, \ldots, b_r)$, where $b_1 < b$, $b \cdot v(b_2, \ldots, b_r)$ is a linear combination of $v(b'_1, \ldots, b'_{r'})$ for $r' \leq r$, and that $v(b_2, \ldots, b_r)$ has less than $r - 1$ terms.

We need to check that given $b, b' \in \mathcal{B}$, $[b, b'] \cdot v(b_1, \dots, b_r) = b \cdot (b' \cdot v(b_1, \dots, b_r)) - b' \cdot (b \cdot v(b_1, \dots, b_r))$. (Involves induction on r.) Without loss of generality, assume $b > b'$. Now if $b' \leq b_1$, then this follows from definition; assume $b > b' > b_1$, then by reindexing we just need to show $[b, b'] \cdot b'' \cdot v = b \cdot b' \cdot b'' \cdot v - b' \cdot b \cdot b'' \cdot v$, (where we will set $b'' = b_1$ and $v = v(b_2, \dots, b_r)$). By the Jacobi identity, we know $[[b, b'], b''] + [[b', b''], b] + [[b'', b], b'] = 0$. By induction of r, this says that $[[b, b'], b''] \cdot v = [b, b'] \cdot b'' \cdot v - b'' \cdot [b, b'] \cdot v$, again, by induction, this is equivalent to $[b, b'] \cdot b'' \cdot v - b'' \cdot b \cdot b' \cdot v + b'' \cdot b' \cdot v \cdot v$. Therefore, the identity we want to show now is just equivalent to

$$[[b, b'], b''] \cdot v = b \cdot b' \cdot b'' \cdot v - b' \cdot b \cdot b'' \cdot v - b'' \cdot b \cdot b' \cdot v + b'' \cdot b' \cdot b \cdot v$$

which is true by induction on $\min\{b, b'\}$. (We index this statement as (b, b', b'').) So now $b > b' > b''$, then $\min\{b', b''\} = b'' < b$ and $\min\{b'', b\} = b'' < b$. Now we know the statement works on (b', b'', b) and (b'', b, b'). Therefore, by the Jacobi, we know the statement works on (b, b', b''). Hence, everything works, for some reason. ☐

Lemma 9.2. Let $\mathcal{B}$ be totally ordered, then $S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ is an isomorphism if and only if $\mathcal{F}$ is a basis of $U(\mathfrak{g})$.

Remark 9.3. $\mathcal{B}$ being totally order implies $\mathcal{F}$ is a basis of $U(\mathfrak{g})$, which implies $S(\mathfrak{g}) \cong \mathrm{gr}(U(\mathfrak{g}))$, which implies $\mathcal{F}$ is a basis of $U(\mathfrak{g})$ for any totally bounded basis $\mathfrak{B}$.

10 Lecture 10

Let k be a field and V a vector space over k. The filtration on V is the data of $0 = V^{\leq -1} \subseteq V^{\leq 0} \subseteq V^{\leq 1} \subseteq \cdots \subseteq V$ such that $\bigcup\limits_{n} V^{\leq n} = V$, with $\mathrm{gr}(V) = \bigoplus\limits_{n \geq 0} V^{\leq n}/V^{\leq n-1}$ be a $\mathbb{Z}_{\geq 0}$-graded vector space.

Let B be a family of elements of V, put $B^{\leq n} = B \cap V^{\leq n}$.

Lemma 10.1. Let $\gamma_n : V^{\leq n} \to \mathrm{gr}^n(V)$ be such that

$$\begin{tikzcd}
V^{\leq n} \arrow[rr] \arrow[dr, two heads] & & \mathrm{gr}^n(V) \\
& V^{\leq n}/V^{\leq n-1} \arrow[ur, equal] &
\end{tikzcd}$$

If $\gamma_n(B^{\leq n} \setminus B^{\leq n-1})$ is a basis of $\mathrm{gr}^n(V)$ for all n, then B is a basis of V.

Proof. Let $\mathfrak{g}$ be a Lie algebra over k, and let $U(\mathfrak{g})^{\leq n}$ be the linear combinations of $a_1, \dots, a_r$ where $r \leq n$ and $a_i \in \mathfrak{g}$, then this gives $\varphi : S(\mathfrak{g}) \twoheadrightarrow \mathrm{gr}(U(\mathfrak{g}))$, where $B_{\mathfrak{g}}$ is considered to be the basis of $\mathfrak{g}$ of total order. Then $B = \{a_1, \ldots, a_r \mid r \geq 0, a_1, \ldots, a_r \in B_{\mathfrak{g}}, a_1 \leq \cdots \leq a_r\}$, then $B^{\leq n} \setminus B^{\leq n-1} = \{a_1, \ldots, a_n \mid a_i \in B_{\mathfrak{g}}, a_1 \leq \cdots \leq a_n\}$, and this is a basis of $S^n(\mathfrak{g})$. Therefore, φ is an isomorphism if and only if $\{\varphi_n(B^{\leq n} \setminus B^{\leq n-1})\}$ is a basis of $\mathrm{gr}^n(U(\mathfrak{g}))$ for all n. Note that this is equivalent to having B as a basis of $U(\mathfrak{g})$. (We have shown one direction, the other direction is an exercise.) $\square$

Lemma 10.2. If $B^{\leq n}$ is a basis for $V^{\leq n}$ for all n, then $\gamma_n(B^{\leq n} \setminus B^{\leq n-1})$ is a basis of $\mathrm{gr}^n(V)$ for all n.

Remark 10.3. The quantization of a commutative algebra A is a non-commutative algebra B with a filtration such that $\mathrm{gr}(B) \cong A$.

Example 10.4. $A = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and Weyl algebra

$$B = k\left\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \right\rangle / (x_i x_j = x_j x_i, \partial_i \partial_j = \partial_j \partial_i, \partial_j x_i - x_i \partial_j = \delta_{ij}).$$

Then $B^{\leq i}$ is the linear span of $p(x)\partial_1^{a_1} \cdots \partial_n^{a_n}$ where $\sum a_n \leq i$. The basis of B is

$$\{x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}, \partial_1^{\beta_1}, \ldots, \partial_n^{\beta_n}\}.$$

Remark 10.5. $U(\mathfrak{g})$ has a Hopf algebra structure, with

- coproduct $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$,
- counit $\varepsilon : U(\mathfrak{g}) \to k$,
- antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ defined by $S(ab) = S(ba)$.

Example 10.6. Let G be a finite group, and $A = kG$ be its group algebra, then the structure is defined with $\Delta(g) = g \otimes g$, $\varepsilon(\mathfrak{g}) = 1$, and $S(\mathfrak{g}) = \mathfrak{g}^{-1}$. Note that kG does not determine G as an algebra, but determine it asaa Hopf algebra. In particular, we have $G = \{a \in kG \mid \Delta(a) = a \otimes a\}$.

Theorem 10.7. If k has characteristic 0, $\mathfrak{g}$ is the set of primitive elements of $U(\mathfrak{g})$, i.e., $b \in U(\mathfrak{g})$ such that $\Delta(b) = b \otimes 1 + 1 \otimes b$.

Proof. Assume that $\mathfrak{g}$ is abelian, then $U(\mathfrak{g}) = S(\mathfrak{g})$. Let $p \in S(\mathfrak{g})$, then there exists a subspace $\mathfrak{g}'$ of $\mathfrak{g}$ such that $p \in S(\mathfrak{g}')$. It suffices to consider the case where $\mathfrak{g}$ is finite-dimensional. Fix a basis. As $S(\mathfrak{g}) = k[x_1, \ldots, x_n]$, where $n = \dim(\mathfrak{g})$, then $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i = x_i + y_i$. There is now an isomorphism $k[x_1, \ldots, x_n] \otimes k[x_1, \ldots, x_n] \cong k[x_1, \ldots, x_n, y_1, \ldots, y_n]$, where $x_i \otimes 1 \mapsto x_i$ and $1 \otimes x_i \mapsto y_i$.

Now $\Delta(p(x_1, \ldots, x_n)) = p(x_1 + y_1, \ldots, x_n + y_n)$. Assume p to be primitive, then $p(x_1 + y_1, \ldots, x_n + y_n) = p(x_1, \ldots, x_n) + p(y_1, \ldots, y_n)$, then $p(2x_i) = 2p(x_i)$, hence this shows $2^n p_n(x_i) = p_n(2x_i) = 2p_n(x_i)$, where p_n is the part of degree n in P, so $p_n = 0$ if $n \neq 1$. Hence, $p \in kx_1 \oplus \cdots kx_n = \mathfrak{g}$.

For general $\mathfrak{g}$, Δ is compatible with the filtration of $U(\mathfrak{g})$, the induced map is $\mathrm{gr}(\Delta) : \mathrm{gr}(U(\mathfrak{g})) \to \mathrm{gr}(U(\mathfrak{g})) \otimes \mathrm{gr}(U(\mathfrak{g}))$, which is isomorphic to the map $\bar{\Delta} : S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes S(\mathfrak{g})$ which is Δ for $S(\mathfrak{g})$. For $a \in U(\mathfrak{g})$, we have $\Delta(a) = a \otimes 1 + 1 \otimes a$ for $a \neq 0$.

Let n be minimal such that $U(\mathfrak{g})^{\leq n} \twoheadrightarrow \mathrm{gr}^n(U(\mathfrak{g})) \cong S^n(\mathfrak{g})$ defined by $a \mapsto \bar{a}$. Then $\bar{\Delta}(\bar{a}) = \bar{a} \otimes 1 + 1 \otimes \bar{a}$, therefore $n = 1$, so $a \in U(\mathfrak{g})^{\leq 1} = k \oplus \mathfrak{g}$. In particular, $\Delta(\alpha + b) = \alpha\Delta(1) + b \otimes 1 + 1 \otimes b = \alpha(1 \otimes 1) + b \otimes 1 + 1 \otimes b$, but $\alpha + b$ is primitive for $\alpha \in k$ and $b \in \mathfrak{g}$, so $\alpha = 0$, hence $\alpha \in \mathfrak{g}$. □

11 Lecture 11

Consider $\mathfrak{g}$ to be a Lie algebra over k and $\mathrm{char}(k) = 0$. Suppose $U(\mathfrak{g})$ is filtered given by $U(\mathfrak{g})^{\leq i}$, which denotes the linear span of $a_1, \ldots, a_r$ for $a_i \in \mathfrak{g}$ and $0 \leq r \leq i$.

Note that if A and B are filtered vector space, then there is a filtration on $A \otimes B$ given by $(A \otimes B)^{\leq n} = \sum\limits_{i+j=n} (A^{\leq i} \otimes B^{\leq j})$ and a map $\mathrm{gr}(A \otimes B) \cong \mathrm{gr}(A) \otimes \mathrm{gr}(B)$. Similarly, there is a filtration on $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ given by

$$\Delta : U(\mathfrak{g})^{\leq n} \to (U(\mathfrak{g}) \otimes U(\mathfrak{g}))^{\leq n}.$$

The exponential map replaces $k[x]$ by $k\,[[x]]$, which is the limit of $k[x]/(x^n)$.

For Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, there is $U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) \cong U(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$, which follows from Theorem 8.4.

The forgetful functor from Lie algebras to vector spaces has a left adjoint $V \to L(V)$; the forgetful functor from algebras to vector spaces also has a left adjoint, given by $V \to T(V)$. There is also a functor from algebra to commutative algebra, given by $A \mapsto A/(aa' - a'a)$.

The question is, can we get a left adjoint from non-associated algebra to vector spaces? or from non-associated monoids to sets? For instance, let E be a set and $M(E)$ be a non-unital non-associative monoid, then $M(E) = \bigcup\limits_{n \geq 1} M_n(E)$, where each $M_n(E)$ is the rooted binary tree with n leaves an an element of E at each leaf. Therefore, $M_1(E) = E$, $M_2(E) = E \times E$, and the product is given by $M_n(E) \times M_m(E) \to M_{n+m}(E)$ by connecting the binary trees.

Therefore, $M(E)$ becomes a non-associative monoid and M is a non-associative monoid. This gives a restriction equivalence $\mathbf{Hom}_{\mathrm{Mon}}(M(E), M) \to \mathbf{Hom}_{\mathrm{Set}}(E, M)$. This induces $k^{M(E)}$ to be a vector space of basis $M(E)$ with multiplication, therefore as a non-associative, non-unital algebra.

Let I be a two-sided ideal of $k^{M(E)}$ generated by

- $(e_1e_2)e_3 + (e_2e_3)e_1 + (e_3e_1)e_2$ for $e_1, e_2, e_3 \in E$,
- ee for $e \in E$, and
- $ee' + e'e$ for all $e, e' \in E$.

This induces $\mathcal{L}(E) = k^{M(E)}/I$. Therefore, the forgetful functor from Lie algebras to sets has a left adjoint $E \mapsto \mathcal{L}(E)$.

Example 11.1. $E = \varnothing$ implies $\mathcal{L}(E) = 0$.

$E = \{e\}$ implies $\mathcal{L}(E) = k$. Therefore, for a Lie algebra $\mathfrak{g}$, we have $\mathbf{Hom}_{\text{Lie}}(k, \mathfrak{g}) \cong \mathfrak{g}$ by Yoneda Lemma.

For $E = \{e_1, e_2\}$, then in $\mathcal{L}(E)$ we have the bracket structure on $[e_1, e_2]$ This gives $\mathcal{L}_2(E) = k \cdot [e_1, e_2]$ and $\mathcal{L}_3(E) = k \cdot [[e_1, e_2], e_1] \oplus k[[e_1, e_2], e_2]$. Therefore, set $E = \{x, y\}$, we have $U(\mathfrak{L}(E)) \xrightarrow{\sim} k\langle x, y\rangle$.

One can define log and exp on this structure.

Let $A = k[[x]]$ be a local k algebra with maximal ideal $\mathfrak{m} = X \cdot k[[x]]$, then the exponential map is defined by

$$\begin{aligned} \mathfrak{m} &\to 1 + \mathfrak{m} \\ a &\mapsto \sum_{n \geq 0} \frac{1}{n!} a^n \end{aligned}$$

and logarithm map is defined by

$$\begin{aligned} 1 + \mathfrak{m} &\to \mathfrak{m} \\ 1 + a &\mapsto \sum_{n > 0} \frac{(-1)^{n+1}}{n} a^n \end{aligned}$$

Therefore, exponential and logarithm are inverse bijections.

12 Lecture 12

Let k be a field of characteristic 0. Let E be a set, then this induces a free Lie algebra $L(E)$ on E, and also induces $k\langle E\rangle = T(k^{(E)})$ as non-commutative polynomials. Therefore, we have

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[dr] & L(E) \arrow[r, hook] & U(L(E)) \\
& k\langle E\rangle \arrow[ur, dashed] &
\end{tikzcd}$$

We claim that the induced map is an isomorphism, and it suffices to construct the inverse. Consider the diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, hook] & k\left\langle E\right\rangle \\ L(E) \arrow[r, hook] \arrow[ur, dashed] & U(L(E)) \arrow[u, dashed] \end{tikzcd}$$

where $k\left\langle E\right\rangle$ is the Lie algebra with $[a,b] = ab - ba$, $U(L(E))$ is the universal enveloping algebra, and the induced map from $L(E)$ is the unique morphism of Lie algebra, and this induced the candidate inverse map as a morphism of algebra. To check that these are inverses, it suffices to consider the restrictions to E: since $k\left\langle E\right\rangle$ is generated by E as an algebra, $L(E)$ is generated by E as a Lie algebra, then $U(L(E))$ is generated by E.

In particular, if $E = \{x, y\}$, then $L(E) = \bigoplus_{n\geq 1} L_n(E)$. Let $L(E)^n$ be the span of iterated brackets of x and y involving n terms, i.e., $L(E)^1 = kx \oplus ky$, $L(E)^2 = k[x,y]$, $L(E)^3 = k\left[[x,y],x\right] \oplus k\left[[x,y],y\right]$. Therefore, $k\left\langle E\right\rangle^n$ becomes the set of homogeneous polynomials of degree n, and so $k\left\langle\right\rangle = \bigoplus_{n\geq 0} k\left\langle E\right\rangle^n$, and $k\left\langle\left\langle E\right\rangle\right\rangle = \prod_{n\geq 0} k\left\langle E\right\rangle^n$ is a local ring with maximal ideal $\mathfrak{m} = \prod_{n\geq 1} k\left\langle E\right\rangle^n$. Let $\tilde{L}(E) = \prod_{n\geq 1} L(E)^n$, then $L(E) \hookrightarrow k\left\langle E\right\rangle$ restricts to $L(E)^n \hookrightarrow k\left\langle E\right\rangle^n$, and we get $\hat{L}(E) \hookrightarrow \mathfrak{m}$.

The coproduct on $U(L(E))$ given by $\Delta(e) = e \otimes 1 + 1 \otimes e$ for $e \in E$ via isomorphism $U(L(E)) \xrightarrow{\sim} k\left\langle E\right\rangle$, obtain $\Delta : k\left\langle E\right\rangle \to k\left\langle E\right\rangle \otimes k\left\langle E\right\rangle$, then $\Delta(k\left\langle E\right\rangle)^n \subseteq \bigoplus_{r+s=n} k\left\langle E\right\rangle^r \otimes k\left\langle E\right\rangle^s$. Therefore, Δ gives a morphism of algebra $k\left\langle\left\langle E\right\rangle\right\rangle \to k\left\langle\left\langle E\right\rangle\right\rangle \hat{\otimes} k\left\langle\left\langle E\right\rangle\right\rangle$, which is isomorphic to $\lim_n (k\left\langle E\right\rangle \otimes k\left\langle E\right\rangle)/(Ek\left\langle E\right\rangle \otimes k\left\langle E\right\rangle + k\left\langle E\right\rangle Ek\left\langle E\right\rangle)^n$, known as the completed tensor product. (Here $k\left\langle\left\langle E\right\rangle\right\rangle = \lim_n k\left\langle E\right\rangle / (Ek\left\langle E\right\rangle)^n$.)

Lemma 12.1. $\hat{L}(E) = \{a \in k\left\langle\left\langle E\right\rangle\right\rangle \mid a \text{ primitive}\}$.

Proof. Let $a^{\leq n}$ be the part of a in $\prod_{1\leq i\leq n} k\left\langle E\right\rangle^n$, now a being primitive implies $a^{\leq n} \in k\left\langle E\right\rangle \cong U(L(E))$ is primitive, so $a^{\leq n} \in L(E)$, and therefore $a \in \hat{L}(E)$. □

Lemma 12.2. There exists inverse bijections $\exp : \mathfrak{m} \to \mathfrak{1} + m$ and $\log : 1 + \mathfrak{m} \to \mathfrak{m}$ that restricts to bijections between the set of primitive elements of $k\left\langle\left\langle E\right\rangle\right\rangle$ contained in $\mathfrak{m}$, and the set of group-like elements of $k\left\langle\left\langle E\right\rangle\right\rangle$ contained in $1 + \mathfrak{m}$.

We have

$$\begin{aligned}\Delta(\exp(a)) &= \sum_{n\geq 0} \frac{\Delta(a)^n}{n!} \\ &= \exp(\Delta(a)) \\ &= \exp(a \otimes 1 + 1 \otimes a) \\ &= (\exp(a) \otimes 1)(1 \otimes \exp(a))\end{aligned}$$

and similar results with log. Assuming a to be primitive, then

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$

Therefore, $\exp(a \otimes 1 + 1 \otimes a) = \exp(a \otimes 1)\exp(1 \otimes 1)$ because $a \otimes 1$ and $1 \otimes a$ commute as elements in $k\langle E\rangle \otimes k\langle E\rangle$.

Therefore, $\exp(x)\exp(y) \in 1+\mathfrak{m}$ is group-like, so $z = \log(\exp(x)\exp(y)) \in \mathfrak{m}$ is primitive, therefore $z \in \hat{L}(E)$.

Theorem 12.3 (Campbell-Hausdorf)**.** There is $z \in \hat{L}(E)$ such that $\exp(z) = \exp(x)\exp(y)$.

Proof. Write $z = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + \cdots$, so now in $k\langle\langle E\rangle\rangle$ we have

$$z = \log(1 + \sum_{p,q\geq 1} \frac{x^p y^q}{p!q!} = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}(\sum_{p,q\geq 1}\frac{x^p y^q}{p!q!})^n.$$

Therefore,

$$z = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sum_{\substack{p_1,\ldots,p_n\geq 1\\ q_1,\ldots,q_n\geq 1}} \frac{x^{p_1}y^{q_1}\cdots x^{p_n}y^{p_n}}{p_1!q_1!\cdots p_n!q_n!}.$$

The explicit formula is then given by

$$\begin{aligned} k\langle E\rangle^n &\to L(E)^n \\ e_1,\ldots,e_n &\mapsto \frac{1}{n}([e_1,[e_2,\cdots,[e_{n-1},e_n],\cdots]]\end{aligned}$$

for $e_i \in \{x,y\}$. Therefore, $e_1e_2 \mapsto \frac{1}{2}[e_1,e_2]$ and $e_1e_2e_3 \mapsto \frac{1}{3}[e_1,[e_2,e_3]]$, and so on. ☐

Lemma 12.4. Now given $\varphi : \mathfrak{m} \to \hat{L}(E)$, we have the identity composition $\hat{L}(E) \hookrightarrow \mathfrak{m} \xrightarrow{\varphi} \hat{L}(E)$ where $\mathfrak{m} \subseteq k\langle\langle E\rangle\rangle$.

For $z \in \hat{L}(E)$, we have $\varphi(2) = 2$.

13 Lecture 13

We constructed a formal group from the Lie group, via $\exp : \mathfrak{gl}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$.

Proposition 13.1. There exists an open neighborhood U of 0 in $\mathfrak{gl}_n(\mathbb{C})$ and an open neighborhood V of 1 in $\mathrm{GL}_n(\mathbb{C})$ such that exp restricts to a homeomorphism from U to V.

Proof. By the Jacobian criterion, $\exp((a_{ij})) = (\delta_{ij} + a_{ij} + \cdots)$ and so $\exp(U_1)\exp(U_2) = \exp(U_3)$ for some $U_3 \in U$. Let $U_1, U_2 \in U$ be close enough to 0. Therefore, we can take $U_3 = z(U_1, U_2)$ by Campbell-Hausdorff. $\square$

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k and let $e_1, \dots, e_n$ be a basis of $\mathfrak{g}$. We want to find $F \in k\left[[x_1, \dots, x_n, y_1, \dots, y_n]\right]^n$. There is now a morphism θ of Lie algebra given from the free Lie algebra on x, y, denoted $\mathcal{L}(x, y)$, to $k\left[[x_1, \dots, x_n, y_1, \dots, y_n]\right] \otimes \mathfrak{g} = k\left[[x_1, \dots, x_n, y_1, \dots, y_n]\right]^n$ which is the Lie algebra over $k\left[[x_1, \dots, x_n, y_1, \dots, y_n]\right]$. This is the map defined by sending $x \mapsto \sum x_i \otimes e_i$ and $y \mapsto \sum y_i \otimes e_i$.

Moreover, θ extends by continuity to the completion $\hat{\mathcal{L}}(x, y) \ni z(x, y)$. Now let $F = \theta(z(x, y))$. We need to check that

- F is a formal group law, and
- the bilinear part of F gives back the bracket on $\mathfrak{g}$.

Theorem 13.2. The functor from formal group laws to Lie algebra is essentially surjective.

Remark 13.3. Let F and F' be formal group laws in n and n' variables, respectively. A morphism $F \to F'$ is $\varphi \in k\left[[x_1, \dots, x_n]\right]^{n'}$ such that

$$F'(\varphi(x_1, \dots, x_n), \varphi(y_1, \dots, y_n)) = \varphi(F(x_1, \dots, x_n, y_1, \dots, y_n)).$$

Therefore, $F'(\varphi(x_1, \dots, x_n), \varphi(y_1, \dots, y_n)) = \varphi(F(x_1, \dots, x_n, y_1, \dots, y_n))$.

If F is a formal group law and $R = k\left[[x_1, \dots, x_n]\right]$, then $F : R \to R \hat{\otimes} R$ defined by $x_i \mapsto F_i$ is a map to the topological coproduct of R. If $\mathfrak{m}$ is a maximal ideal of R, then denote $\mathrm{Dist}_i(R) = (R/\mathfrak{m}^i)^*$ where $R/\mathfrak{m}^i$ is the set of polynomials of degree at most $i-1$, then this gives a canonical map $R/\mathfrak{m}^{i+1} \twoheadrightarrow R/\mathfrak{m}^i$, and induces $\mathrm{Dist}_i(R) \hookrightarrow \mathrm{Dist}_{i+1}(R)$. Therefore, $\mathrm{Dist}(R) := \bigcup\limits_{i \geq 0} \mathrm{Dist}(R)$ is the set of linear forms on R vanishing on $\mathfrak{m}^i$ for $i \gg 0$.

$$\begin{aligned} F : k\left[[z_1, \dots, z_n]\right] &\to k\left[[x_1, \dots, x_n, y_1, \dots, y_n]\right] \\ p(z_1, \dots, z_n) &\mapsto p(F_1(z_1, \dots, z_n), \cdots, F_n(z_1, \dots, z_n)). \end{aligned}$$

Therefore, F^* induces a map $\mathrm{Dist}(R) \otimes \mathrm{Dist}(R) \to \mathrm{Dist}(R)$ and therefore makes it into an algebra. Let $\tilde{R} = k\,[[x_1, \dots, x_n, y_1, \dots, y_n]]$, then it admits a maximal ideal $\mathfrak{n} = \mathfrak{n}' + \mathfrak{n}''$ where $\mathfrak{n}' = \sum x_i \tilde{R}$ and $\mathfrak{n}'' = \sum y_i \tilde{R}$. Therefore $\mathfrak{n}^i = \sum\limits_{j+l=i} (\mathfrak{n}')^j (\mathfrak{n}'')^l$, therefore we have an isomorphism

$$\tilde{R}/(\mathfrak{n}')^j(\mathfrak{n}'')^l \cong \mathrm{Dist}_j(k\,[[x_1, \dots, x_n]]) \otimes \mathrm{Dist}_l(k\,[[y_1, \dots, y_n]])$$

and therefore gives

$$\mathrm{Dist}_i(\tilde{R}) = \tilde{R}/\eta^i \cong \bigoplus_{j+l=i} \mathrm{Dist}_j(h\,[[x_1, \dots, x_n]] \otimes \mathrm{Dist}_l(h\,[[y_1, \dots, y_n]]\,.$$

There is now a universal property of F, such that for $F(\mathfrak{m}) \subseteq \mathfrak{n}$, then

$$\begin{tikzcd}
\tilde{R}^* \arrow[r, "F^*"] & R^* \\
(\tilde{R}/\eta^i)^* \arrow[u, hook] \arrow[r, dashed] & (R/\mathfrak{m}^i)^* \arrow[u, hook]
\end{tikzcd}$$

and F^* restricts to a diagram

$$\begin{tikzcd}
\mathrm{Dist}_i(\tilde{R}) \arrow[r] & \mathrm{Dist}_i(R) \arrow[r, hook] & \mathrm{Dist}(R) \\
\bigoplus\limits_{j+l=i} \mathrm{Dist}_j(R) \otimes \mathrm{Dist}_l(R) \arrow[u, "\sim"] \arrow[ur] \arrow[rr, hook] & & \mathrm{Dist}(R) \otimes \mathrm{Dist}(R) \arrow[u]
\end{tikzcd}$$

14 Lecture 14

Let $R = k\,[[x_1, \dots, x_n]] \supseteq \mathfrak{m}$ and let

$$\mathrm{Dist}(R) = \{l \in R^* \mid l(\mathfrak{m}^n) = 0, n \gg 0\}.$$

We saw that formal group law $F : R \to R \hat{\otimes} R$ in n variables correspondence with algebra structure on $\mathrm{Dist}(R)$.

$\mathrm{Dist}(R)$ has a basis $\{\partial_\alpha \mid \alpha \in (\mathbb{Z}_{\geq 0})^n\}$ where ∂_α sends $\sum\limits_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n}$ to $a_{\alpha_1, \dots, \alpha_n}$. Moreover, we have

$$\partial_\alpha(p) = \left(\frac{1}{\alpha_1! \cdots \alpha_n!} (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}(p) \right)(0)$$

We define $\mathrm{Dist}(R) = \bigcup\limits_{n \geq 0} \mathrm{Dist}_n(R)$ where $\mathrm{Dist}_n(R)$ vanishes on $\mathfrak{m}^n$. Then $\{\alpha_\alpha \mid \sum \alpha_i = n - 1\}$ is a basis of $\mathrm{Dist}_n(R)$.

Since R is an algebra, we have $R \otimes R \to R$, and dually speaking there is $R^* \to (R \otimes R)^*$ which corresponds to This restricts to $\mathrm{Dist}(R) \xrightarrow{\Delta} \mathrm{Dist}(R) \otimes \mathrm{Dist}(R) \cong \mathrm{Dist}(R \otimes R)$.

Lemma 14.1. Δ makes $\mathrm{Dist}(R)$ into a bialgebra.

Let $\partial_i = \frac{\partial}{\partial x_i}$ be $\partial_{(0,\dots,0,\dots,0)}$, then $\Delta(\partial_i) = \partial_i \otimes 1 + 1 \otimes \partial_i$, and so $\Delta(\partial_i)(P \otimes Q) = \partial_i(PQ) = \partial_i(P)Q + P\partial_i(Q)$.

Lemma 14.2. $\{\partial_i\}_{1 \leq i \leq n}$ generates $\mathrm{Dist}(R)$ into an algebra.

Remark 14.3. $\mathrm{Dist}(R)$ is cocommutative and not commutative in general.

Lemma 14.4. $\partial_\alpha \cdot \partial_\beta = \frac{(\alpha+\beta)!}{\alpha!\beta!}\partial_{\alpha+\beta} + \cdots$ with terms in ∂_γ such that $|\gamma| < |\alpha| + |\beta|$.

For $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$, we have $\partial_\alpha \partial_\beta(P) = (\partial_\alpha \otimes \partial_\beta)(\varphi(P))$ where

$$\varphi(P) = P(F_1(x,y), \dots, F_n(x,y))$$

and $F_i(x,y) = x_i + y_i + \cdots$ with higher order terms. By the lemma we have induction on $|\alpha|$ we have $\partial_\alpha \in k\left\langle \partial_1, \dots, \partial_n \right\rangle$.

Let F be a formal group law with $F(x,y) = x + y + B(x,y) + \cdots$ where $B(x,y)$ is the bilinear form in x_i, y_i's. Let $\mathfrak{g} = ke_1 \oplus \cdots \oplus ke_n$, then there is a breacket structure on $B(x,y) - B(y,x)$. More instrinsically, $S(\mathfrak{g}^*) = k[x_1, \dots, x_n]$ gives $S(\mathfrak{g}^*)^\wedge = k\left[[x_1, \dots, x_n]\right] = R$. There is an embedding $\mathfrak{g} \hookrightarrow \mathrm{Dist}(R)$ with $e_i \mapsto \partial_i$, which is a Lie algebra map, where $\mathrm{Dist}(R)$ as a Lie algebra using commutator.

Theorem 14.5. The induced map $U(\mathfrak{g}) \to \mathrm{Dist}(R)$ is an isomorphism of bialgebras.

Proof. We see that this is a Lie algebra morphism because

$$P(F_1(x,y), \dots, F_n(x,y)) - P(F_1(y,x), \dots, F_n(y,x) = (\partial_i \otimes \partial_j - \partial_j \otimes \partial_i)(P(F_1(x,y), \dots, F_n(x,y))$$

where $F_i(x,y) - F_i(y,x) = B_i(x,y) - B_i(y,x) + \cdots$. Moreover, the algebraic morphism $U(\mathfrak{g}) \to \mathrm{Dist}(R)$ is compatible with coproduct, since it is enough to check on algebra generators of $U(\mathfrak{g})$: $e_1, \dots, e_n$ where $e_i \mapsto \partial_i$ and $\Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i$ and corresponds to $\partial_i \otimes 1 + 1 \otimes \partial_i$ when evaluated as ∂_i. Consider the filtration of $U(\mathfrak{g})^{\leq n}$, the linear span of $e_{i_1}, \dots, e_{i_n}$, to $\mathrm{Dist}_n(R)$, the linear span of ∂_α for $|\alpha| \leq n$, given by γ, then γ is compatible with the filtration, which makes it an isomorphism: $\gamma(e_{i_1}, \dots, e_{i_n} = \partial_{i_1} \cdots \partial_{i_n}$. $\square$

This induces a correspondence in $\mathrm{Dist}(R)^*$ and R.

15 Lecture 15

Let k be a field of characteristic 0.

Theorem 15.1. The functor from formal group laws to Lie algebras is an equivalence of categories.

Proof. Let F and F' be formal group laws and g and g' be corresponding Lie algebras. We have a correspondence

$$\mathbf{Hom}_{\text{bialg}}(U(g), U(g')) \xrightarrow{\sim} \mathbf{Hom}_{\text{Lie}}(g, g').$$

Let $\varphi : U(g) \to U(g')$, then φ is a morphism that sends primitives to primitives. Therefore, φ restricts to a linear map $g \to g'$. The morphism of Lie algebras restricts to a morphism of Lie algebras. Note that $\varphi|_g$ determines φ, because g generates $U(g)$ as an algebra. We start from $\psi : g \to g'$ as a morphism of Lie algebras, then we want to check $\Delta(\varphi(x)) = (\varphi \otimes \varphi)(\Delta(x))$, and it is enough to do it for $x \in g$, where we see $\varphi(x) = \psi(x) \in g'$, so $\Delta(\varphi(x)) = \varphi(x) \otimes 1 + 1 \otimes \varphi(x)$, so $(\varphi \otimes \varphi)(\Delta(n)) = (\varphi \otimes \varphi)(x \otimes 1 + 1 \otimes x$. We get $\mathbf{Hom}_{\text{Lie}}(g, g') \to \mathbf{Hom}_{\text{bialg}}(U(g), U(g'))$ which is the inverse to the previous map. Therefore, we have an isomorphism $U(g) \cong \text{Dist}(R)$ of bialgebras.

We now take $R = k\,[[x_1, \ldots, x_n]]$, then we need a map from formal group laws to products on $\text{Dist}(R)$. We have $\mathbf{Hom}_{\text{Lie}}(g, g') \xrightarrow{\sim} \mathbf{Hom}_{\text{bialg}}(\text{Dist}(R), \text{Dist}(R'))$. We want to show that $\mathbf{Hom}_{\text{bialg}}(\text{Dist}(R), \text{Dist}(R')) \xrightarrow{\sim} \mathbf{Hom}(F, F')$. We have

$$\mathbf{Hom}(F, F') = \{f \in k\,[[x_1, \ldots, x_n]]^{n'} \mid f(F_1(x, y), \ldots, F_n(x, y)) = F'(f_1, \ldots, f_n)\},$$

which is also equivalent to the set of maps $g : k\,[[x_1, \ldots, x_{n'}]] \to k\,[[x_1, \ldots, x_n]]$ such that $g(p) = p(g(x_1), \ldots, g(x_n))$, and that $g(1) = 1$ and $g(x_i)$ has no constant terms. We know such g is a morphism of algebras, so $g(m') \subseteq m$, and so it induces $R/m^d \cong \text{Dist}_d(R) \to R/m'd \cong \text{Dist}_d(R')$, therefore this induces $g^\vee : \text{Dist}(R) \to \text{Dist}(R')$ as a morphism of coalgebras. Note that $g^\vee$ determines g because $g = (g^\vee)^*$ and $\text{Dist}(R)^* \cong R$. Therefore, g swaps F and F' and gives a morphism of topological coalgebras, then $g^\vee$ is also a morphism of algebras. ☐

Remark 15.2. If g is a Lie algebra of dimension n, then $U(g)^* \cong k\,[[x_1, \ldots, x_n]]$ as an isomorphism of algebras.

Remark 15.3. There is a correspondence between Lie groups and global objects of G:

- the convergent formal group laws correspond to neighborhoods of 1 in G,
- the formal group laws correspond to formal neighborhoods of 1 in G,

- the Lie algebras correspond to the tangent spaces at 1 in G.

We saw the correspondence between Lie algebras and formal group laws. We will see how Lie groups fit in this picture. Consider $G = (k, +)$ and $g = k$, then we have an equivalence between the Lie algebras between g and g' and the algebra g' itself. Now for $F(x, y) = x + y$, there is a corresponding morphism $F \to F'$ given by $f \in k\left[[x]\right]^{n'}$ such that $f(x + y) = F'(f_1(x), \ldots, f_{n'}(x), f_1(y), \ldots, f_{n'}(y))$.

Lemma 15.4. Consider a differential equation $\frac{df}{dx} = A(f)$ where A is an analytic transformation, then any formal solution is convergent.

16 Lecture 16

Lemma 16.1. Let G be a Lie group with F a family of formal group laws $F = (F_i(x_1, \ldots, x_n, y_1, \ldots, y_n))_{1\leq i\leq}$ The map $T : F_{\text{add}} \to F$ as a morphism of formal group laws $T \in k\left[[x]\right]^n$ is convergent.

We obtain a differential equation $\frac{d\tau}{dx} = A(\tau)$ where $A \in k\left[[x_1, \ldots, x_n]\right]^n$.

Lemma 16.2. If A is convergent, then the differential equation above with $T(0) = 0$ has a unique solution and that solution is convergent.

Consider G' another Lie group and $T : F_G \to F_{G'}$ morphism of formal group laws.

Lemma 16.3. T is convergent.

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g} = \mathrm{Lie}(G)$, and let $t_i : \mathbb{C} \to \mathfrak{g}$ sending $1 \mapsto x_i$ be a morphism of Lie algebras. Then there now exists $\tau_i : F_{\text{add}} : F_G$ extending t_i. One can see that τ_i is convergent. Indeed, there exists $0 \in U_i \subseteq k$ an open neighborhood such that τ converges on U_i, then this defines $\tilde{\tau}_i : U_i \to G$. Let $\tilde{\tau} : U_1 \times \cdots U_n \to G$ be defined by $(u_1, \ldots, u_n) \mapsto \tilde{\tau}_1(u_1) \cdots \tilde{\tau}_n(u_n)$. Then $\mathrm{Lie}(\tilde{\tau}_i) = t_i$, so $\mathrm{Lie}(\tilde{\tau}) : k^n \simeq \mathfrak{g} \xrightarrow{\mathrm{id}} \mathrm{Lie}(G)$. This map takes $(0, \ldots, 0, 1, 0, \ldots, 0)$ to x_i on tangent spaces at $(0, , \ldots, 0)$ and $1 \in G$, so $\tilde{\tau}$ is an isomorphism, therefore it is locally an isomorphism. Therefore, $\tau_0 \tau_i : F_{\text{add}} \to F_{G'}$ is convergent, so there exists $V_i \subseteq U_i$ open neighborhood of 0 in k such that $\tau_0 \tau_i$ converges on V_i. Therefore, we have a local isomorphism $U_1 \times \cdots U_n \to G$, an open inclusion map $V_1 \times \cdots \times V_n$ to $U_1 \times \cdots \times U_n$, and a map $\tilde{\tau}' : V_1 \times \cdots \times V_n \to G'$ by sending $(v_1, \ldots, v_n) \mapsto \tau_1'(x_1) \cdots \tau_n'(v_n)$. This induces a local isomorphism $\psi : V_1 \times \cdots \times V_n \to G$. In a neighborhood of $1 \in G$, this defines a local isomorphism $\tilde{\tau}' \circ \psi^{-1} : U \to G'$, which is just τ as we require. $\square$

The morphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$ of Lie groups is locally an isomorphism but not an isomorphism in general.

To study universal covering maps, we need to consider connected, path-connected, locally path-connected, locally simply connected topological spaces.

Let $f : Y \to X$ be continuous, then it is a covering if given any $x \in X$, there exists $U \ni x$ open neighborhood such that $f^{-1}(U) = \coprod_i U_i$ and $f : U_i \to U$ homeomorphism. In particular, a universal covering $\tilde{X}$ of X is such that given any covering Y of X, there is a unique covering map $\tilde{X} \to Y$ such that the diagram commutes. Note that this is unique with the basepoint.

Proposition 16.4. Let G be a connected topological group. Fix $\tilde{e} \in \tilde{G}$ such that $p(\tilde{e}) = 1$, then there exists a unique topological group structure on $\tilde{G}$ with identity $\tilde{e}$ and such that p is a morphism of groups.

Proof. Consider the diagram

$$\begin{tikzcd}
\tilde{G} \times \tilde{G} \arrow[r, dashed] \arrow[rr, dashed, bend left] \arrow[dr, "\tilde{p}\times\tilde{p}"'] & (G \times G) \times_G \tilde{G} \arrow[r] \arrow[d, "p"] & \tilde{G} \arrow[d, "p"] \\
& G \times G \arrow[r, "\times"] & G
\end{tikzcd}$$

where p is the universal covering. □

Proposition 16.5. If $p : \tilde{G} \to G$ is surjective, then $\pi_1 : (G, 1) \cong \ker(p) \subseteq Z(\tilde{G})$.

Proof. Note that $\pi_1(G, 1) \simeq p^{-1}(G)$ as topological groups. Let $x \in \ker(p)$, we see that the conjugation action by x, denoted c_x, satisfies the deck transformation, and so $c_x = \mathrm{id}_{\tilde{G}}$. If $G, \tilde{G}$ are both connected Lie groups, then $p : \tilde{G} \to G$ is a surjection, so $\ker(p) = \pi_1(G)$ is a discrete subgroup. □

Example 16.6. Let $G = S^1$ and $k = \mathbb{R}$, then $p : \tilde{G} = \mathbb{R} \to G$ is the exponential map with kernel $\mathbb{Z}$.

17 Lecture 17

Let G be a topological group and $p : \tilde{G} \to G$ be a universal cover. If G is a Lie group, then there is a unique analytic manifold structure on $\tilde{G}$ making p analytic and therefore $\tilde{G}$ becomes a Lie group.

Theorem 17.1 (Lie Correspondence)**.** There is an equivalence of categories between the connected, simply connected Lie groups and the Lie algebras.

Let γ be a map from the connected, simply connected Lie groups to Lie groups, let ψ be the map from connected, simply connected Lie groups to the Lie algebras, and let Lie be the map from Lie groups to Lie algebras, then $\psi = \mathrm{Lie} \circ \gamma$.

Theorem 17.2. $\gamma \circ \psi^{-1}$ is a left adjoint to Lie, i.e., $\mathbf{Hom}(\psi^{-1}(\mathfrak{g}), H) \cong \mathbf{Hom}(\mathfrak{g}, \mathrm{Lie}(H))$ where $\mathfrak{g}$ is a Lie algebra and H is a Lie group.

There are inclusion maps from connected, simply connected Lie groups to connected Lie groups, and then to Lie groups. Correspondingly, there are left adjoints backwards.

A morphism $G \to H$ where G is a connected topological group is determined by its restriction to any open neighborhood of 1. If both G and H are Lie groups, then the restriction is just the morphism of formal group laws to the data of morphisms of Lie algebras.

For connected Lie groups, ψ is faithful.

Remark 17.3. For ψ full, we need to extend morphisms defined only on a neighborhood of 1. Moreover, ψ is essentially surjective.

Theorem 17.4. Every compact connected and simply connected real Lie group is isomorphic to a product of the following compact, connected, simply connected Lie groups:

- $\mathrm{SU}(n)$ for $n \geq 3$,
- $\mathrm{Sp}(n)$ for $n \geq 1$,
- $\mathrm{Spin}(n)$ for $n \geq 7$, and
- G_2, F_4, E_6, E_7, E_8.

Remark 17.5 (Lie Subgroups as Stabilizers)**.** Let G be a Lie group and X be a manifold. The action of G on X can be given by analytic map $\gamma : G \times X \to X$ such that $\gamma(1, x) = x$ for all $x \in X$, and that $\gamma(g_1, \gamma(g_2, x)) = \gamma(g_1 g_2, x)$ for all $g_1, g_2 \in G$.

Proposition 17.6. The stabilizer G_{x_0} of $x_0 \in X$ is a Lie subgroup.

Proof. Let $\varphi : G \to X$ be defined by $g \mapsto \gamma(g, x_0) := g \cdot x_0$, then for $h \in G$, there is a commutative diagram between $g \mapsto hg$ on G and $x \mapsto h \cdot x$ on X, via the action. Note that the properties of φ in a neighborhood of $1 \in G$ is the same as the properties of φ in a neighborhood of $g \in G$, so the rank of the Jacobian $T(\varphi)$ is constant. Therefore, φ looks locally like a linear map between vector spaces, and so φ is a regular embedding. □

Proposition 17.7. If G acts transitively on X, then $G/G_{x_0} \cong X$ defined by $gG_{x_0} \mapsto g \cdot x_0$ is an isomorphism of manifolds.

A representation of G on a finite-dimensional k-vector space V is a morphism of Lie groups $\rho : G \to \mathrm{GL}(V)$. This induces an action of G on V, and gives $\mathrm{Lie}(\rho) : \mathrm{Lie}(G) \to \mathfrak{gl}(V)$.

We have $\mathrm{GL}(V) \cong \mathrm{GL}(V^*)$ defined by $a \mapsto ({}^t a)^* {-1}$. Then the dual ρ^* of ρ is the map $\rho^* : G \to \mathrm{GL}(V) \to \mathrm{GL}(V^*)$.

Lemma 17.8. $\mathrm{Lie}(\rho^*)$ is $\mathrm{Lie}(G) \xrightarrow{\mathrm{Lie}(\rho)} \mathfrak{gl}(V) \xrightarrow{\sim} \mathfrak{gl}(V^*)$ where the isomorphism is given by $a \mapsto -{}^t a$.

Note that $a : V \to V$ induces ${}^t a : V^* \to V^*$. Therefore, ${}^t[a, b] = [-{}^t a, {}^t b]$ and ${}^t(ab) = {}^t b \cdot {}^t a$ with $[a, b] = ab - ba$ in $\mathfrak{gl}(V)$.

Proof. $({}^t(1 + x))^{-1} = 1 - {}^t x$ up to higher terms in x_{ij}'s. ☐

18 Lecture 18

Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\rho : G \to \mathrm{GL}(V)$ be the representation of G. This induces the map $\mathrm{Lie}(\rho) : \mathfrak{g} \to \mathfrak{gl}(V)$ and therefore $\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}(V^*)$ defined by $g \mapsto ({}^t g)^{-1}$, and therefore induces $\mathfrak{gl}(V) \to \mathfrak{gl}(V^*)$ defined by $a \mapsto -{}^t a$.

Note that there is

$$\begin{tikzcd}
\mathrm{GL}_n(k) \arrow[r, hook, "i"] \arrow[d, "f"] & M_n(k) & U = \{a \mid ||a|| < \varepsilon\} \arrow[l, hook', "i'"'] \\
\mathrm{GL}_n(k) & &
\end{tikzcd}$$

with mappings $i : 1 \mapsto 0$ with $g \mapsto g - 1$, $f : g \mapsto ({}^t g)^{-1}$, $i' : a \mapsto (a + {}^t a)^{-1} - 1$. Therefore, for $a \in M_n(k)$ such that $||a||$ is small enough, we know ${}^t(1 + a)^{-1} = 1 - {}^t a + b$ and therefore $({}^t(1 + a))^{-1} - 1 = -{}^t a + b$. In particular, this induces a map on tangent space by $a \mapsto -{}^t a$.

For vector spaces $V_1, \ldots, V_r$, with $V = V_1 \otimes \cdots \otimes_V r$, we have $\rho : \mathrm{GL}(V_1) \times \cdots \mathrm{GL}(V_r) \to \mathrm{GL}(V)$. For $i = 1, \ldots, r$, let $a_i \in \mathrm{End}(V_i)$ be close to 0, then $(1 + a_1) \otimes \cdots \otimes (1 + a_r) - 1 \in \mathrm{End}(V)$.

Over $\mathrm{Lie}(\rho)$, we have

$$\begin{aligned}
\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_r) &\to \mathfrak{gl}(V) \\
(a_1, \ldots, a_r) &\mapsto a_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes a_r
\end{aligned}$$

Therefore, for $\rho : G = \mathrm{GL}(V) \to \mathrm{GL}(V^*)$, if we fix $\zeta \in V^*$, then let $G_\zeta = \{g \in \mathrm{GL}(V) \mid \rho(g)(\zeta) = \zeta\}$.

Lemma 18.1. $\mathrm{Lie}(G_\zeta) \cong \{a \in \mathfrak{gl}(V) \mid \zeta \circ a = 0\}$.

Proof. $\mathrm{Lie}(G_\zeta) \cong \{a \in \mathfrak{gl}(V) \mid (-{}^t a)(\zeta) = 0\}$. □

For $\rho : G \to \mathrm{GL}(V)$ and $v \in V$, $G_v = \{g \in G \mid \rho(g)(v) = v\}$. For $f : G \to V$ such that $g \mapsto \rho(g)(v)$, then $T_1(f) : \mathfrak{g} \to T_v V = V$, then $\mathrm{Lie}(G_v) \cong \ker(T_1(f))$. Here $T_1(f) : \mathfrak{g} \to \mathfrak{gl}(V) \to V$ with $a \mapsto a(v)$, and the kernel is $\{a \in \mathfrak{g} \mid \mathrm{Lie}(\rho)(a)(v) = 0\}$.

Example 18.2. $\mathrm{GL}(V) \xrightarrow{\det} k^* = \mathrm{GL}(k)$. Then $\det(1 + a) = 1 + \mathrm{tr}(a) + \cdots$ and therefore $\mathrm{Lie}(\det) = \mathrm{tr} : \mathfrak{gl}(V) \to k$. For $0 \neq l \in L = k$, we have $\mathrm{GL}(V)_l = \mathrm{SL}(V)$, and so $\mathrm{Lie}(\mathrm{SL}(V)) = \{a \in \mathfrak{gl}(V) \mid \mathrm{tr}(a) = 0\} = \mathfrak{sl}(V)$.

We can think of

$$\rho : \mathrm{GL}(V) \to \mathrm{GL}(V \otimes V^*)$$
$$\mathfrak{g} \mapsto \mathfrak{g} \otimes {}^t\mathfrak{g}^{-1}$$

as a composition of

$$\mathrm{GL}(V) \xrightarrow{\Delta} \mathrm{GL}(V) \times \mathrm{GL}(V) \to \mathrm{GL}(V) \times \mathrm{GL}(V^*) \to \mathrm{GL}(V \otimes V^*)$$

and similarly, we have

$$\mathrm{Lie}(\rho) : \mathfrak{gl}(V) \to \mathfrak{gl}(V \otimes V^*)$$
$$a \mapsto a \otimes 1 - 1 \otimes {}^t a$$

as a composition of

$$\mathfrak{gl}(V) \to \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to \mathfrak{gl}(V) \times \mathfrak{gl}(V^*) \to \mathfrak{gl}(V \otimes V^*)$$

by

$$a \mapsto (a, a) \mapsto (a, -{}^t a) \mapsto a \otimes 1 - 1 \otimes {}^t a$$

Let $\beta; V \times V \to k$ be the bilinear form corresponding to element β' in $V \otimes V^*$, then for $G = \mathrm{GL}(V)$, let $G_{\beta'}$ be the stabilizer of β' in the bilinear form, then $G_{\beta'} = \{g \in \mathrm{GL}(V) \mid \beta(gv_1, gv_2) = \beta(v_1, v_2)\}$. So we have a bijection $\mathrm{Lie}(G_{\beta'}) \cong \{a \in \mathfrak{gl}(V) \mid (a \otimes 1 - 1 \otimes {}^t a)(\beta) = 0\}$.

Theorem 18.3. Let G be a connected compact Lie group, then $G \cong (G_1 \times \cdots \times G_r \times (S)^n)/Z$, where $G_1, \ldots, G_r$ are simple connected and simply connected Lie groups, i.e., of type A, B, C, D, corresponding to $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, and (the last two) $\mathrm{Spin}(n)$, as well as exceptional groups G_2, F_4, E_6, E_7, E_8. Moreover, Z is a finite central subgroup.

In particular, $\mathrm{Lie}(G) \cong \mathrm{Lie}(G_1) \times \cdots \times \mathrm{Lie}(G_r) \times \mathbb{R}^n$, where $\mathrm{Lie}(G_1) \times \cdots \times \mathrm{Lie}(G_r))$ is semisimple and the whole group is reductive.

19 Lecture 19

Let G be a Lie group and $\rho : G \to \mathrm{GL}(V)$ is a representation. Then there is a bijection between $\mathbf{Hom}_k(V \otimes V, k)$, the bilinear forms on V, and $V^* \otimes V^*$. Correspondingly, there is $\mathfrak{g} = \mathrm{Lie}(G) \to \mathfrak{gl}(V^* \otimes V^*)$ mapping $a \mapsto -{}^t a \otimes 1 - 1 \otimes {}^t a$.

Let β be a bilinear form on V as elements of $V^* \otimes V^*$. Then $\mathrm{Lie}(G_\beta) = \{a \in \mathfrak{g} \mid (-{}^t a \otimes 1 - 1 \otimes {}^t a)(\beta) = 0\}$, i.e., $-\beta(av, v') - \beta(v, av') = 0$ for all $v, v' \in V$.

Example 19.1. $\mathrm{O}_n(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^t g \cdot g = 1\} = \mathrm{GL}_n(\mathbb{C})_\beta$ where $\beta = \sum x_i \varphi_i$. Then $\mathrm{Lie}(O_n(\mathbb{C})) = \{a \in \mathfrak{gl}_n(\mathbb{C}) \mid {}^t a + a = 0\}$.

Here $\beta(ab, b') = \beta(v, {}^t a \cdot v')$, and $\beta(av, v') + \beta(v, av') = 0$ if and only if $\beta(v, (a + {}^t a)v') = 0$.

Example 19.2. $\mathrm{Sp}_n(\mathbb{C}) = \mathrm{GL}_{2n}(\mathbb{C})_\beta$ for $\beta = \sum\limits_{i=1}^{n}(x_i y_{n+i} - x_{n+i} y_i)$.

Let $\mathrm{Spin}(n)$ be the double cover of $\mathrm{SO}(n)$ for $n > 2$. Then $\mathrm{Spin}(1) = O(1) = \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Spin}(2) = U(1) = \mathrm{SO}(2)$, $\mathrm{Spin}(3) = \mathrm{Sp}(1) = \mathrm{SU}(2)$, $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, $\mathrm{Spin}(5) = \mathrm{Sp}(2)$, and $\mathrm{Spin}(6) = \mathrm{SU}(4)$.

Lemma 19.3. *Let H and G be Lie groups, where H is connected and simply connected. Let U be an open neighborhood of 1 in H, and let $f : U \to G$ be analytic such that $f(uu') = f(u)f(u')$ given $u, u' \in U$ and $uu' \in U$. Then f extends uniquely to a morphism of Lie groups $H \to G$.*

Proof. $\Gamma \subseteq U \times G \subseteq H \times G$ is a graph of f. Then Γ is a "subgroup chunk" of $H \times G$ that contains 1. Therefore, there exists an open neighborhood $1 \in V \subseteq \Gamma$ that satisfies the said property.

Now consider the connected Lie subgroup $\langle \Gamma \rangle$ in $H \times G$, which is unique, such that $\Gamma \subseteq \langle \Gamma \rangle$ is an open neighborhood of 1. Then note that $\varphi : \langle \Gamma \rangle \to H$ gives a cover that is locally an isomorphisma round the identity, and since $\langle \Gamma \rangle$ is connected and H is simply connected, we know that φ is an isomorphism. Therefore, the composition

$$H \xrightarrow{\varphi^{-1}} \langle \Gamma \rangle \hookrightarrow H \times G \xrightarrow{\pi_2} G$$

is a morphism of Lie groups extending f. □

Theorem 19.4. *Let H be connected and simply connected, then*

$$\mathbf{Hom}(H, G) \cong \mathbf{Hom}(\mathrm{Lie}(H), \mathrm{Lie}(G)).$$

Proof. By Lemma 19.3, $\mathbf{Hom}(H, G)$ is isomorphic to the hom of an open neighborhood of 1 in H to G, which is isomorphic to $\mathbf{Hom}(F_H, F_G)$, which is isomorphic to $\mathbf{Hom}(\mathrm{Lie}(H), \mathrm{Lie}(G))$. □

Theorem 19.5. Let k be $\mathbb{R}$ or $\mathbb{C}$, then there is an equivalence of categories between connected simply connected Lie groups and the finite-dimensional Lie algebras.

Proof. Note that $\mathfrak{g}$ being Lie algebra gives a formal group law F by the Baker-Campbell-Hausdorff formula, then one can prove that this is convergent. By Ado's theorem, $\mathfrak{g}$ has a faithful finite-dimensional linear representation, then there exists $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n(k)$ which is an embedding of Lie algebras, which sends F to F_{GL_n}, and therefore this gives an embedding $U \hookrightarrow \mathrm{GL}_n(k)$. Let $G = \langle U \rangle$, then this is a connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$, then the universal cover of G, denoted $\tilde{G}$, is connected and simply connected, and $\mathrm{Lie}(\tilde{G}) = \mathfrak{g}$. ☐

20 Lecture 20

Let k be a field, $\mathfrak{g}$ be a Lie algebra over k, $L, L' \subseteq \mathfrak{g}$ and $[L, L']$ be the k-span of $[x, x']$ for $x \in L$ and $x' \in L'$. Now $[\mathfrak{g}, \mathfrak{g}]$ is a k-subspace of $\mathfrak{g}$.

We say $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal if $\mathfrak{h}$ is a k-subspace of $\mathfrak{g}$, and $[x, h] \in \mathfrak{h}$ for all $x \in \mathfrak{g}, h \in \mathfrak{h}$. In particular, suppose $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

Example 20.1.

- For instance, $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$, called the derived subalgebra of $\mathfrak{g}$.
- The center $Z(\mathfrak{g}) = \{z \in \mathfrak{g} \mid [x, z] = 0 \ \forall g \in \mathfrak{g}\}$ is an ideal of $\mathfrak{g}$.
- The map

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = \mathrm{End}_k(g)$$
$$x \mapsto \mathrm{ad}_x : y \mapsto [x, y]$$

  has kernel $\ker(\mathrm{ad}) = Z(g)$.

Let $L \subseteq \mathfrak{g}$, then $n_{\mathfrak{g}}(L) = \{x \in \mathfrak{g} \mid \mathrm{ad}_x(L) \subseteq L\}$. Let $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $[\mathfrak{g}, \mathfrak{h}]$ is an ideal of $\mathfrak{g}$. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra.

Defining $C^1(\mathfrak{g}) = \mathfrak{g}$ and $C^{n+1}(\mathfrak{g}) = [\mathfrak{g}, C^n(\mathfrak{g})]$. Then $C^n(\mathfrak{g})$ is an ideal of $\mathfrak{g}$. We say $\mathfrak{g}$ is nilpotent if there exists n such that $C^n(\mathfrak{g}) = 0$.

Lemma 20.2. The following are equivalent:

1. $\mathfrak{g}$ is nilpotent,
2. there exists a chain of ideals $0 = a_m \subseteq a_{m-1} \subseteq \cdots \subseteq a_i = \mathfrak{g}$ of $\mathfrak{g}$ such that $a_i/a_{i+1} \subseteq Z(\mathfrak{g}/a_{i+1})$ for all $i \geq 1$.
3. there exists r such that given $x_1, \ldots, x_r \in \mathfrak{g}$, we have $\mathrm{ad}_{x_1} \cdots \mathrm{ad}_{x_r} = 0$.

Proof. (1) $\Rightarrow$ (2): Let $a_i = C^i(g)$, let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$, then $\mathfrak{h}/[\mathfrak{g},\mathfrak{h}] \subseteq Z(\mathfrak{g}/[\mathfrak{g},\mathfrak{h}])$.

(2) $\Rightarrow$ (3): Let $x \in \mathfrak{g}$ and $y \in a_i$, then $\mathrm{ad}_x(y) = [x,y] \in a_{i+1}$. For $x_1, \dots, x_{m-1} \in \mathfrak{g}$, $\mathrm{ad}_{x_{m-1}}(a_1) \subseteq a_2$, $\mathrm{ad}_{x_{m-2}}(a_2) \subseteq a_3$, and so on. Therefore, $\mathrm{ad}_{x_1} \cdots \mathrm{ad}_{x_{m-1}}(\mathfrak{g}) = 0$.

(3) $\Rightarrow$ (1): Note that $C^i(\mathfrak{g})$ is the k-linear span of ad_{x_1}
cdots $\mathrm{ad}_{x_{i-1}}(\mathfrak{g})$ for $x_1, \dots, x_{i-1} \in \mathfrak{g}$, therefore $C^{r+1}(\mathfrak{g}) = 0$. ☐

Example 20.3. Let $\mathfrak{g}$ be the set of up-right triangular matrices, then $\mathfrak{gl}_n(k) \supseteq \mathfrak{g}$. Then $C^n(\mathfrak{g}) = 0$.

Definition 20.4 (Full Flag)**.** Let V be a n-dimensional vector space. A full flag in V is a sequence of subspaces

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

with $\dim(V_i) = i$.

Example 20.5. Let $V = ke_1 \oplus \cdots \oplus ke_n$, then $V_i = ke_1 \oplus \cdots \oplus ke_i$ gives the standard flag.

Theorem 20.6 (Engel)**.** Let $\mathfrak{g}$ be a Lie algebra and let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation where $\dim(V) < \infty$. If $\rho(x)$ is nilpotent for all $x \in \mathfrak{g}$, then there exists a full flag $0 = V_0 \subseteq \cdots \subseteq V_n = V$ such that $\rho(x)(V_i) \subseteq V_{i-1}$ for all $x \in \mathfrak{g}$.

Therefore, take $V_1 = ke_1$, $V_2 = ke_1 \oplus ke_2$, and so on, then $\rho(\mathfrak{g})$ is a subset of the up-right triangular matrices.

Proof. We can replace $\mathfrak{g}$ by $\rho(\mathfrak{g})$, a Lie subalgebra of $\mathfrak{gl}(V)$. We assume that $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, and proceed by induction on $\dim(V)$: it suffices to find $v \in V$ such that $x(v) = 0$ for all $x \in \mathfrak{g}$.

Let $\mathfrak{g} \to \mathfrak{gl}(V/V_1)$ for $V_1 = kV$. Then it suffices to have $\dim(\mathfrak{g}) < \infty$. We proceed by induction on $\dim(\mathfrak{g})$ instead. Let $\mathfrak{h}$ be a maximal proper Lie subalgebra of $\mathfrak{g}$. We claim that $\dim(\mathfrak{g}/\mathfrak{h}) = 1$, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Therefore, the theorem holds for $\mathfrak{h}$. Let $x \in \mathfrak{g}$, we view x as an endomorphism of $\mathfrak{gl}(V)$ in two ways: for $a \in \mathfrak{gl}(V)$, we think of it as left multiplication by x or right multiplication by x, hence this shows l_x and/or r_x is nilpotent, because x is nilpotent. We also have a third endomorphism given by $\mathrm{ad}_x : a \mapsto xa - ax$ which is also nilpotent because the two multiplications commute, i.e., $\mathrm{ad}_x = l_x - r_x$.

Consider

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "ad"] \arrow[dr, "\rho'"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d] \\
 & \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})
\end{tikzcd}$$

Then proceed by induction on $\dim(\mathfrak{h}) < \dim(\mathfrak{g})$, then the theorem holds for $\mathfrak{h}$ and ρ', where $\mathfrak{h} \subseteq \mathfrak{g}$ is a stable subspace for the adjoint action of $\mathfrak{h}$.

Theorem 20.7. There exists $0 \neq y \in \mathfrak{g}/\mathfrak{h}$ such that $\rho'(h)(y) = 0$ for all $h \in \mathfrak{h}$. For $x \in \mathfrak{g}$, $x + \mathfrak{h} = y$, then $[h, x] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$. Then $x \in n_{\mathfrak{g}}(\mathfrak{h})$.

Now $\mathfrak{h} \subsetneq kx \oplus \mathfrak{h} \subseteq n_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}$ as a series of Lie subalgebras, then since $\mathfrak{h}$ is maximal and proper, then $kx \oplus \mathfrak{h} = \mathfrak{g} = n_{\mathfrak{g}}(\mathfrak{h})$, hence proves the claim. By the theorem for $\mathfrak{h} \to \mathfrak{gl}(V)$, then $M = \{v \in V \mid hv = 0 \ \forall h \in \mathfrak{h}\} \neq 0$, and since $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then M is stable under action of $\mathfrak{g}$.

We now have $x(M) \subseteq M$ where x is nilpotent, then there exists $0 \neq v \in M$ such that $xv = 0$, so $av = 0$for all $a \in \mathfrak{g}$. □

Corollary 20.8. Assume $\dim(\mathfrak{g}) < \infty$, then $\mathfrak{g}$ is nilpotent if and only if ad_x is nilpotent for all $x \in \mathfrak{g}$.

Proof. ($\Rightarrow$) is clear. For ($\Leftarrow$), consider $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. By theorem, for all $x_1, \dots, x_n \in \mathfrak{g}$, we have $\mathrm{ad}_{x_1} \cdots \mathrm{ad}_{x_n} = 0$. □

21 Lecture 21

Let $\mathfrak{g}$ be a Lie algebra. Let $D^1(\mathfrak{g}) = \mathfrak{g}$ and $D^{n+1}(\mathfrak{g}) = [D^n(\mathfrak{g}), D^n(\mathfrak{g})]$, therefore $D^n(\mathfrak{g}) \subseteq C^n(\mathfrak{g})$.

We say $\mathfrak{g}$ is solvable if $D^n(\mathfrak{g}) = 0$ for some n. Therefore, $\mathfrak{g}$ being nilpotent, implies $\mathfrak{g}$ is solvable.

Lemma 21.1. $\mathfrak{g}$ is solvable if and only if there exists ideals of $\mathfrak{g}$ as $0 = a_m \subseteq a_{m-1} \subseteq \cdots \subseteq a_1 = \mathfrak{g}$ such that a_i/a_{i-1} is abelian for $m > i \geq 1$. Therefore, $[\mathfrak{g}, \mathfrak{g}] = 1$ if and only if $\mathfrak{g}$ is abelian.

Example 21.2. Let $\mathfrak{g}$ be the set of up-right triangular matrices, so $\mathfrak{g} \subseteq \mathfrak{gl}_n(k)$ is solvable.

Theorem 21.3 (Lie)**.** Let k be algebraically closed and of characteristic 0. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation with $\dim(V) < \infty$, and assume that $\mathfrak{g}$ is solvable. There exists a full flag $0 = V_0 \subseteq \cdots \subseteq V_n = V$ such that $\rho(x)(V_i) \subseteq V_i$ for all i and for all $x \in \mathfrak{g}$. Therefore, $\rho(g)$ is a subset of the up-right triangular matrices.

Proof. We can assume ρ to be injective, and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. We proceed by induction on $\dim(\mathfrak{g})$. Therefore, $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, and since $\mathfrak{g}$ is solvable, then $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Let H be a hyperplane in $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ as a Lie subalgebra, and let $\mathfrak{h}$ be the inverse image in $\mathfrak{g}$. $\mathfrak{h}$ acts as an ideal in $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{h} \oplus kx_0$ where $x_0 \in \mathfrak{g} \setminus \mathfrak{h}$, then by induction on V, we need to show that there exists $0 \neq v \in V$ acting as an eigenvector for $\rho(x)$ for all $x \in \mathfrak{g}$, then use V/kV.

We note that theorem holds for $\mathfrak{h}$: there exists $0 \neq v_0 \in V$ such that v_0 is an eigenvector for $\rho(x)$ for $x \in \mathfrak{h}$. Define $\chi : \mathfrak{h} \to k$ by $\rho(x)(v_0) = \chi(x)v_0$ for $x \in \mathfrak{h}$. Let $0 \neq L = \{v \in V \mid \rho(x) \cdot v = \chi(x)v \ \forall x \in \mathfrak{h}\}$.

We claim that $x_0(L) \subseteq L$. Assume this: x_0 has a non-zero eigenvector $v \in L$, then v is an eigenvector for all $x \in \mathfrak{g}$, and we are done. Take $0 \neq l \in L$, then M is the linear span of $l, x_0(l), x_0^2(l), \cdots, x_0^r(l)$, which are linear independence and gives a basis of M. For $h \in \mathfrak{h}$, let $i \geq 1$, then $h \cdot x_0^i(l) = (hx_0 - x_0h)x_0^{i-1}(l) + x_0hx_0^{i-1}(l) = [h, x_0]x_0^{i-1}(l) + x_0hx_0^{i-1}(l)$. For $h(l) = x(h)l$, we have $hx_0(l) = [h, x_0](l) + x_0h(l) = x([h, x_0])l + x(h)x_0(l)$. So h acts on M by a up-right triangular matrix with $x(h)$'s on the diagonal. Therefore the trace of this action is just $\dim(M) \cdot x(h)$, so $hx_0(l) = x(h)x_0(l) + x([x_0, h]) \cdot l$, with $x_0(l) \in L$ if and only if $x([x_0, h']) = 0$ for all $h' \in \mathfrak{h}'$. Let $h = [x_0, h'] \in \mathfrak{h}$, then the trace of $[x_0, h']$ acting on M, where M is stable under $\mathfrak{h}$ and x_0, is just 0. Hence, $x([x_0, h']) = 0$. ☐

Corollary 21.4. Let $\mathfrak{g}$ be solvable and $\dim(\mathfrak{g}) < \infty$, and $\mathrm{char}(k) = 0$, then $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof. Consider $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}([\mathfrak{g}, \mathfrak{g}])$, then $0 = V_0 \subseteq \cdots \subseteq V_n = [\mathfrak{g}, \mathfrak{g}]$, with $\mathrm{ad}_x(V_i) \subseteq V$ for $x \in \mathfrak{g}$. Let V_i's be ideals of $\mathfrak{g}$, then V_i/V_{i-1} has dimension 1, so it is abelian. For $g \in [\mathfrak{g}, \mathfrak{g}]$, $\mathrm{ad}_y(V_i) \subseteq V_{i-1}$, and we have a basis for $\rho = \mathrm{ad}$ with $[\rho(x), \rho(x')] = \rho[x, x']$, and $\rho([\mathfrak{g}, \mathfrak{g}])$ being nilpotent. Therefore, $\ker(\rho) = Z(\mathfrak{g})$, with $[\mathfrak{g}, \mathfrak{g}]/(Z\mathfrak{g} \cap [\mathfrak{g}, \mathfrak{g}]$ is nilpotent. ☐

Lemma 21.5. Let $\mathfrak{g}$ be a Lie algebra and $Z \subseteq Z(\mathfrak{g})$ is a subspace, and if $\mathfrak{g}/Z$ is nilpotent, then $\mathfrak{g}$ is nilpotent.

Lemma 21.6. Let $\mathfrak{g}$ be a Lie algebra, and a_1, a_2 are ideals of $\mathfrak{g}$. If a_1, a_2 are solvable, then $a_1 + a_2$ is solvable.

$\mathrm{rad}(\mathfrak{g})$ is the largest solvable ideal of $\mathfrak{g}$, as the sum of all solvable ideals.

Definition 21.7. Let $\mathfrak{g}$ be finite-dimensional, then $\mathfrak{g}$ is semisimple if $\mathrm{rad}(\mathfrak{g}) = 0$.

Example 21.8. $\mathfrak{sl}_n(k)$ is semisimple. $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple.

Theorem 21.9. Let k be characteristic 0. There exists a Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ such that $\mathfrak{g}' \cong \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$.

As a vector space, we have $\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}'$, often denoted by $\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \rtimes \mathfrak{g}'$.

22 Lecture 22

Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation with $\dim(V) < \infty$. Let $\beta : V \times V \to k$ be a bilinear form.

Definition 22.1. β is $\mathfrak{g}$-invariant if $\beta(\rho(x)v, v')+\beta(v, \rho(x)v') = 0$ for all $x \in \mathfrak{g}$ and $v, v' \in V$.

Example 22.2. Let $\rho = \text{ad}$ mapping $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, then $\beta(x, y) = \text{Tr}(\text{ad}_x \cdot \text{ad}_y)$ is $\mathfrak{g}$-invariant for $x, y \in \mathfrak{g} = V$. This is the Killing form.

If $L \subseteq \mathfrak{g}$ is an ideal, then $L^\perp = \{x \in \mathfrak{g} \mid \beta(x, l) = 0 \ \forall l \in L\}$ is also an ideal.

Theorem 22.3. $\mathfrak{g}$ is semisimple (no non-zero solvable ideal) if and only if β is non-degenerate.

Proof. Let $\text{rad}(\mathfrak{g})$ be the largest solvable ideal of $\mathfrak{g}$, if $\text{rad}(\mathfrak{g}) \neq 0$, consider n to be the maximal number such that $D^n(\text{rad}(\mathfrak{g})) \neq 0$, then $D^n(\text{rad}(\mathfrak{g}))$ is an abelian ideal. Now semisimple is equivalent to no non-zero abelian ideal. Let $\mathfrak{a}$ be an abelian ideal, with $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$, and we conclude that $\text{Tr}(\text{ad}_x \,\text{ad}_y) = 0)$, so $x \in \mathfrak{g}^\perp$.

Assume $\mathfrak{g}$ is semisimple, then $\mathfrak{g}^\perp$ is an ideal, therefore $\text{Tr}(\text{ad}_x \,\text{ad}_y) = 0$ for all $x \in \mathfrak{g}^\perp$ and $y \in \mathfrak{g}$. The next lemma will show that $\mathfrak{g}^\perp$ is solvable, so $\mathfrak{g}^\perp = 0$. ☐

Lemma 22.4 (Cartan's Criterion)**.** Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ with $\dim(V) < \infty$. Now $\mathfrak{g}$ is solvable if and only if $\text{Tr}_V(xy) = 0$ for all $x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}]$.

First Half. ($\Rightarrow$): Given a basis of V, $\mathfrak{g}$ is a subspace of the up-right diagonal matrices which contains x, and $[\mathfrak{g}, \mathfrak{g}]$ is a subspace of the up-right diagonal matrices with zeroes on diagonal, which contains y, so the trace of xy is 0. ☐

Let V be a vector space with $a \in \text{End}_k(V)$. The Jordan decomposition gives $a = s + u$ where s is semisimple, i.e., diagonalizable, and u is nilpotent, such that s and u commute. In particular, there exists $p \in k[x]$ such that $p(a) = s$ with $p(0) = 0$.

Fix $L \subseteq M \subseteq \text{End}_k(V)$. Let $T = \{a \in \text{End}_k(V) \mid [a, M] \subseteq L\}$.

Lemma 22.5. Let $t \in T$. If $\text{Tr}(at) = 0$ for all $t \in T$, then a is nilpotent.

Proof. Let e_i give a basis, then $s(e_i) = \lambda_i e_i$ over $k = \mathbb{C}$. Define s' to be $s'(e_i) = \bar{\lambda}_i e_i$, so there exists $Q \in k[x]$ such that $Q(\lambda_i) = \bar{\lambda}_i$ for all i, and $Q(0) = 0$. Therefore, $Q(s) = s'$. Since $a \in T$, then $s = p(a) \in T$, so $s' = Q(s) \in T$. In particular, the trace of as' is the sum of squares of λ_i's, therefore is zero. Hence, $s = 0$, and a is nilpotent. For any k or characteristic 0, we define the above concepts in $k' \subseteq k$, so there is an embedding $k' \hookrightarrow \mathbb{C}$ of finite transcendental extension over $\mathbb{Q}$. ☐

We now prove the other half of Cartan's Criterion.

Second Half. ($\Leftarrow$): Let $L = [\mathfrak{g}, \mathfrak{g}] \subseteq M = \mathfrak{g} \subseteq \text{End}_k(V)$. Let $T = \{a \in \text{End}_k(V) \mid [a, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]\}$. We claim that if $y \in [\mathfrak{g}, \mathfrak{g}]$, then ad_y is nilpotent. This implies $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent and so $\mathfrak{g}$ is solvable.

Let $t \in T$, then we want $\text{Tr}_V(ty) = 0$. Take $y = [x_1, x_2]$ where $x_1, x_2 \in \mathfrak{g}$, then $\text{Tr}_V(t[x_1, x_2]) = \text{Tr}(tx_1x_2 - tx_2x_1) = \text{Tr}(tx_1x_2 - x_1tx_2) = \text{Tr}([t, x_1] \cdot x_2) = 0$. ☐

23 Lecture 23

Let $\mathfrak{g}$ be a Lie algebra and $\operatorname{rad}(\mathfrak{g})$ be the largest solvable ideal. Let

$$\begin{aligned}\beta : \mathfrak{g} \times \mathfrak{g} &\to k \\ (x, y) &\mapsto \operatorname{Tr}(\operatorname{ad}_x \operatorname{ad}_y)\end{aligned}$$

Example 23.1. Let $\mathfrak{g}$ be k-matrices of the form $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$, then $\beta \neq 0$ and $\mathfrak{g}$ is solvable.

Theorem 23.2. $\ker(\beta(-, \mathfrak{g})) \subseteq \operatorname{rad}(\mathfrak{g})$.

Proof. Let $\mathfrak{a} = \ker(\beta)$ be an ideal, then $\beta(x, y) = 0$ if $x \in \mathfrak{a}$ and $y \in [\mathfrak{a}, \mathfrak{a}]$. By Cartan, $\mathfrak{a}$ is solvable. ☐

Exercise 23.3. Let β be non-degenerate for $\mathfrak{sl}_n(k)$ where $n \geq 2$, then $\mathfrak{sl}_n(k)$ is semisimple.

Lemma 23.4. Let $\mathfrak{g}$ be semisimple and $\mathfrak{a}$ be ideal, then $\mathfrak{a}^\perp$ is an ideal and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$.

Proof. By Cartan, $\mathfrak{a} \cap \mathfrak{a}^\perp$ is solvable. ☐

Definition 23.5. $\mathfrak{g}$ is solvable if 0 and $\mathfrak{g}$ are the only ideals and $\mathfrak{g} \neq k$.

Theorem 23.6. Every semisimple Lie algebra is a direct sum of simple Lie algebras.

Proof. Use the lemma above and proceed by induction on dimension. ☐

Theorem 23.7. Every finite-dimensional representation of a semisimple Lie algebra is semisimple (completely reducible).

Let $\mathfrak{g}$ be semisimple, then the representation of $\mathfrak{g}$ is isomorphic to the representation of $U(\mathfrak{g})$. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation, and let $\beta_V : \mathfrak{g} \times \mathfrak{g} \to k$ be the corresponding Killing form of trace map. If ρ is injective, then β_V is non-degenerate. Therefore, a basis $\{e_i\}$ of $\mathfrak{g}$ gives an orthogonal basis $\{e_i'\}$ of β_V since $\beta_V(e_i', e_j') = \delta_{ij}$. Therefore, $\sum_i e_i \otimes e_i' \in \mathfrak{g} \otimes \mathfrak{g}$ is mapped to $C_v \in U(\mathfrak{g})$.

Lemma 23.8. $C_v \in Z(U(\mathfrak{g}))$.

Indeed, note that β is $\mathfrak{g}$-invariant.

Now ρ extends to a morphism of algebra $U(\mathfrak{g}) \to \operatorname{End}_k(V)$, with $\rho(C_v)$ the endomorphism of V as a representation of $U(\mathfrak{g})$. Then its trace is just $\dim(\mathfrak{g})$. Assuming V is simple representation of $\mathfrak{g}$ hence of $U(\mathfrak{g})$, then by Schur's lemma we know $\rho(C_v) = \lambda \cdot \operatorname{id}_V$ for some $\lambda \in k$, so $\lambda = \frac{\dim(\mathfrak{g})}{\dim(V)}$.

Proof of Theorem. Assume V has a simple subrepresentation W and $V/W = k$ is the trivial representation. We claim that there exists L subrepresentation of V with $V = W \oplus L$. We can replace $\mathfrak{g}$ by $\rho(\mathfrak{g}) \subseteq \mathfrak{gl}(V)$, and can assume that ρ is injective. Then $\rho|_W$ is injective for W simpler, so we can consider $C_W \in Z(U(\mathfrak{g}))$ which acts on W by $\frac{\dim(\mathfrak{g})}{\dim(W)}$. Now let $\mathfrak{a}$ be the kernel of this map, then $\mathfrak{a} = 0$ by simplicity. Now C_W act by 0 on V/W, so W is the kernel of $C_W - \frac{\dim(\mathfrak{g})}{\dim(W)} \operatorname{id}$, and so $L = \ker(C_W)$ is stable under the action of $\mathfrak{g}$. Therefore, V is semisimple.

We now look at the categorical properties. For representation V and subrepresentation W, we proceed by induction and assume $L = V/W$ is simple. Then taking the short exact sequences we have

$$\begin{tikzcd}
0 \arrow[r] & W \otimes L^* \arrow[d, equal] & V \otimes L^* \arrow[r] & L \otimes L^* \arrow[r] & 0 \\
0 \arrow[r] & W \otimes L^* \arrow[r] & M \arrow[u, hook] \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Now if the bottom row splits, then $0 \to W \to V \to V/W \to 0$ splits. Now $M = V \otimes L^* \oplus T$ with $T \cong k$. This gives a mapping as splitting. $\square$

24 Lecture 24

Theorem 24.1. Let $\mathfrak{g}$ be a Lie algebra, then there exists a Lie algebra $\mathfrak{h}$ such that $\mathfrak{g} = \operatorname{rad}(\mathfrak{g}) \oplus \mathfrak{h}$.

Theorem 24.2. Let $\mathfrak{g}$ be a Lie algebra, $\alpha \subseteq \mathfrak{g}$ be an ideal. If $\mathfrak{g}/\mathfrak{a}$ is semisimple, then there exists Lie algebra $\mathfrak{h}$ with $\mathfrak{g} \cong \alpha \oplus \mathfrak{h}$.

Proof. First consider the case where α is abelian. Fix a k-linear section $\lambda : \mathfrak{g}/\alpha \to \mathfrak{g}$, define

$$\begin{aligned} \varphi : \mathfrak{g}/\alpha \times \mathfrak{g}/\alpha &\to \alpha \\ (x, y) &\mapsto \lambda([x, y]) - [\lambda(x), \lambda(y)] \end{aligned}$$

Note that α is a $\mathfrak{g}/\alpha$-module via action by Lie bracket. Since α is semisimple, for every finite-dimensional $\mathfrak{g}/\alpha$-module V, we have $H^i(\mathfrak{g}/\alpha, V) = 0$ for all $i > 0$. Therefore, $H^2(\mathfrak{g}/\alpha, \alpha) = 0$. Note that φ is a 2-cycle, so $\varphi = d\psi$ for some ψ, and we may modify λ by ψ to make it a Lie algebra homomorphism.

Now suppose α is non-abelian, then by induction on $\dim(\alpha)$, let $\alpha' \subsetneq \alpha$ be an ideal of $\mathfrak{g}$, then $\mathfrak{g}/\alpha' = \alpha/\alpha/ \oplus \mathfrak{h}'$ for some lie algebra $\mathfrak{h}'$, by the inductive hypothesis. Let $\tilde{\mathfrak{h}}'$ be the inverse image of $\mathfrak{h}'$ in $\mathfrak{g}$, then there exists $\mathfrak{h} \subseteq \tilde{\mathfrak{h}}'$ such that $\tilde{\mathfrak{h}}' \cong \alpha' \oplus \mathfrak{h}$, and we have $\mathfrak{g} \cong \alpha \oplus \mathfrak{h}$. $\square$

Definition 24.3. $\mathfrak{g}$ is reductive if $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g})$.

Theorem 24.4. Let $\mathfrak{g}$ be a Lie algebra, then the following are equivalent:

1. $\mathfrak{g}$ is reductive,
2. $\mathfrak{g} = \alpha \oplus \mathfrak{h}$ for some semisimple α and abelian $\mathfrak{h}$,
3. $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a semisimple representation,
4. $\mathfrak{g}$ has a faithful semisimple representation.

Proof. (4) $\Rightarrow$ (1): Let V be the representation, since $\mathrm{rad}(\mathfrak{g})$ is solvable, then there exists a basis $\{e_i\}$ of V such that $\mathrm{rad}(\mathfrak{g})$ are upper-triangular matrices. Let $\lambda_i : \mathrm{rad}(\mathfrak{g}) \to k$ such that $\alpha(e_i) - \lambda_i(a)e_i \in \bigoplus_{j<i} k \cdot e_j$ for every $\alpha \in \mathrm{rad}(\mathfrak{g})$. Then λ_i vanishes on $[\mathrm{rad}(\mathfrak{g}), \mathrm{rad}(\mathfrak{g})]$, so λ_i is a map from $\mathfrak{b} = \mathfrak{g}/[\mathrm{rad}(\mathfrak{g}), \mathrm{rad}(\mathfrak{g})]$ to k. Let $E = \{\lambda \in \mathfrak{b}^* \mid \exists i, \lambda_i = \lambda\}$, $V_\lambda = \{v \in V \mid \alpha(v) = \lambda(\alpha)v, \forall \alpha \in \mathrm{rad}(\mathfrak{g})\}$, and $V' = \bigoplus_{\lambda \in E} V_\lambda$. Since V is semismiple, $V = V' \oplus V$" for some subrepresentation V'' of $\mathfrak{g}$. If $V' \neq V$, then $V'' \neq 0$, so V'' contains an eigenvector of $\mathrm{rad}(\mathfrak{g})$, then there exists λ such that $V''_\lambda \neq 0$ and $V''_\lambda \subseteq V_\lambda$, contradiction. Therefore, $V' = V$. Hence, all of $\mathrm{rad}(\mathfrak{g})$ are diagonal matrices in V, and so $\mathrm{rad}(\mathfrak{g}) \subseteq Z(\mathfrak{g})$, and on the other hand $Z(\mathfrak{g})$ is solvable and hence the other inclusion.

(1) $\Rightarrow$ (2): $\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{h} = Z(\mathfrak{g}) \oplus \mathfrak{h}$ where $\mathfrak{h} = mathfrakg/\mathrm{rad}(\mathfrak{g})$ is semisimple.

(1) $\Rightarrow$ (3): Suppose $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{h}$ where $\mathfrak{h}$ is semisimple, then $\mathrm{ad}_\mathfrak{g} = \mathrm{ad}_\mathfrak{h}$, and by Weyl's Theorem, this is semisimple.

(3) $\Rightarrow$ (4): trivial.

(2) $\Rightarrow$ (3): $Z(\mathfrak{g}) \subseteq \mathfrak{g}$ is stable under adjoint action, so there exists ideal $\mathfrak{a} \subseteq \mathfrak{g}$ that is stable under adjoint action with $\mathfrak{g} = \mathfrak{a} \oplus Z(\mathfrak{g})$, so $\mathfrak{g} = \mathfrak{a} \times Z(\mathfrak{g})$ as Lie algebra, then $\rho : Z(\mathfrak{g}) \to \mathfrak{gl}_n(\mathfrak{h})$ with $\dim(Z(\mathfrak{g})) = n$, and we obtain $\mathrm{ad} \oplus \rho$ as a faithful representation of $\mathfrak{g}$, and since both are smemisimple, we are done. ☐

25 Lecture 25

Example 25.1. Let $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$ with $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $[h, e] = 2e$, $[g, f] = 2f$, $[e, f] = h$. Let V be a representation of $\mathfrak{g}$, then $v \in V$ is a weight vector of weight $\lambda \in \mathbb{C}$ if $h \cdot v = \lambda v$, and $v \in V$ is a highest weight vector of weight λ if $h \cdot v = \lambda v$ and $ev = 0$.

Example 25.2. Let $L = \mathbb{C}^2 = \mathbb{C}b_1 \oplus \mathbb{C}b_2$ be a representation of $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}) \hookrightarrow \mathfrak{gl}_2(\mathbb{C}) = \mathfrak{gl}(L)$, then $S^n(L) = (L^{\otimes n})^{S_n}$ be a representation of $\mathfrak{g}$, and $\mathbb{C}[x_1, x_2] = S(L)$ be a representation of $\mathfrak{g}$, and let $S^n(L)$ be homogeneous polynomial of degree n. Let b_1, b_2 be weight vectors of weight 1 and -1, respectively, now b_1 is the highest weight vector of weight 1. For $x_i = b_i$, we have $e \cdot (x_i x_j) = e(x_i)x_j + x_i e(x_j)$, and $e(x_1^2) = 0$, now $h(x_1^2) = h(x_1)x_1 + x_1 h(x_1) = 2x_1^2$. Then x_1^2 is a highest weight vector with weight 2 in $S^2(L)$. In general, x_1^n is the highest weight vector of weight n in $S^n(L)$.

Theorem 25.3. *Every finite-dimensional representation V of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to a direct sum of $S^n(L)$'s. The multiplicity of $S^n(L)$ in V is the dimension of the space of highest weight vectors of weight n.*

Lemma 25.4. *Let V be a finite-dimensional representation of $\mathfrak{g}$, then V has a non-zero highest weight vector.*

Proof. Let V_λ be the λ-eigenspace of $\mathfrak{h}$, so $\bigoplus V_\lambda \neq 0$. Take λ maximal with $V_\lambda \neq 0$ and $v \in V_\lambda$ where $v \neq 0$, then $V_{\lambda+2} = 0$. Now $h \cdot e(v) = [h, e](v) + eh(v) = 2e(v) + \lambda e(v) = (\lambda + 2)e(v)$, so $e(v) \in V_{\lambda+2}$, thus $e(v) = 0$. □

Lemma 25.5. *Assume V is irreducible, and let $0 \neq v \in V$ be a highest weight vector and let n be its weight. Then n is a non-negative integer, and if $v_i = \frac{1}{i!} f^i(v)$ for $0 \leq i \leq n$, then v_i's form a basis of V, and $e(v_i) = (n - i + 1)v_{i-1}$ for $i > 1$, and $e(v_1) = 0$; $f(v_i) = (i+1)v_{i+1}$ for $i < n$, and $f(v_n) = 0$, and $h(v_1) = (n - 2i)v_i$.*

Therefore, V is determined by n.

Proof. Formula for $f(v_i)$: by definition, $i < n$, and consider $h(v_i)$ by induction on i. Then

$$\begin{aligned} h(v_{i+1}) &= \frac{1}{i+1} hf(v_i) \\ &= \frac{1}{i+1}([h, f](v_i) + fh(v_i)) \\ &= \frac{1}{i+1}(-2f(v_i) + (n - 2i)f(v_i)) \\ &= (n - 2(i+1))v_{i+1} \end{aligned}$$

and since $[e, f] = h$, then $(n - 2i)v_i = [e, f](v_i) = (i + 1)e(v_{i+1}) - f(e(v_i))$.

With induction, note $e(v_0) = 0$, and $(i+1)e(v_{i+1}) = (i(n-1+1)+n-2i)v_i = (i+1)(n-i)v_i$. The non-zero v_i's are linearly independent as different eigenvalues for $\mathfrak{h}$. Therefore, there exists r such that $v_r = 0$ but $v_{r-1} \neq 0$ for $r \leq \dim(V)$, then $f(v_{r-1}) = 0$. Now $e(v_r) = (n - r + 1)v_{r-1}$, so $n = r - 1$ and n is a non-negative integer. Now $\{v_0, \ldots, v_n\}$ generates a subspace W invariant under e, f, and h, then this is a subrepresentation and so $W = V$. □

Remark 25.6. $\mathfrak{h}$ is diagonalizable on V, and $V \cong S^n(L)$.

26 Lecture 26

Suppose $k = \mathbb{C}$ and let $\mathfrak{g}$ be a Lie algebra.

Definition 26.1. A Cartan subalgebra is a nilpotent subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\{x \in \mathfrak{g} \mid \operatorname{ad}_x(\mathfrak{h}) \subseteq \mathfrak{h}\} =: n_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Lemma 26.2. Suppose $\mathfrak{h}' \subseteq \mathfrak{h}$ is a Cartan subaglebra, then $\mathfrak{h}' = \mathfrak{h}$.

Proof. Since $\mathfrak{h}$ is nilpotent, then ad_x acts nilpotently on $\mathfrak{h}$ for $x \in \mathfrak{h}$, hence for $x \in \mathfrak{h}'$, so $\mathfrak{h}'$ acts nilpotently on $\mathfrak{h}/\mathfrak{h}'$. There exists $0 \neq v \in \mathfrak{h}/\mathfrak{h}'$ killed by $\mathfrak{h}'$ (by Engel), then let $\tilde{v} \in \mathfrak{h}$ be such that $\tilde{v} + \mathfrak{h}' = v$, then $[\tilde{v}, x] \in \mathfrak{h}'$ for all $x \in \mathfrak{h}'$. So $\tilde{v} \in n_{\mathfrak{g}}(\mathfrak{h}') = \mathfrak{h}'$, so $\mathfrak{h} = \mathfrak{h}'$. ☐

Fix $x \in \mathfrak{g}$ and $\lambda \in \mathbb{C}$, then $\mathfrak{g}_x^\lambda$ is called the generalized λ-eigenspace of ad_x, so $\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_x^\lambda$.

Lemma 26.3.

- $\lambda, \mu \in \mathbb{C}$, $[\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu] \subseteq \mathfrak{g}_x^{\lambda+\mu}$,
- $\mathfrak{g}_x^0$ is a Lie subalgebra of $\mathfrak{g}$ containing x.
- $n_{\mathfrak{g}}(\mathfrak{g}_x^0) = \mathfrak{g}_x^0$.

Proof. $(\operatorname{ad}_x - (\lambda+\mu)\operatorname{id})^n([y,z]) = \sum_{i=0}^{n} \binom{n}{i} [(\operatorname{ad}_x - \lambda)^i(y), (\operatorname{ad}_x - \mu)^{n-i}(z)]$ over induction on n, so we have the inclusion.

Let $y \in n_{\mathfrak{g}}(\mathfrak{g}_x^0)$, then $[y, x] \in \mathfrak{g}_x^0$, so $\operatorname{ad}_x^n([y,x]) = 0$ for some n, now $\operatorname{ad}_x^{n+1}(y) = \operatorname{ad}_x^n([x,y]) = 0$, so $y \in \mathfrak{g}_x^0$. ☐

Definition 26.4. The rank of $\mathfrak{g}$ is $\min\{\dim(\mathfrak{g}_x^0) \mid x \in \mathfrak{g}\}$. We say $\mathfrak{g}$ is regular if $\dim(\mathfrak{g}_x^0) = \operatorname{rank}(\mathfrak{g})$.

Theorem 26.5.

1. If x is regular, then $\mathfrak{g}_x^0$ is a Cartan subalgebra of $\mathfrak{g}$.
2. Given $\mathfrak{h}$ a Cartan subalgebra, there exists regular x such that $\mathfrak{h} = \mathfrak{g}_x^0$.

Example 26.6. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, and $\mathfrak{h}$ be the set of diagonal matrix, which is Cartan, then x is a diagonal matrix with distinct eigenvalue, so x is regular. One can show that $\mathfrak{g}_x^0 = \mathfrak{h}$, and $\mathfrak{g}_x^\lambda = \mathbb{C}e_{ij}$ for some i, j or 0 when $\lambda \neq 0$.

Proof. Let U be the set of $y \in \mathfrak{g}_x^0$ such that ad_y is not nilpotent on $\mathfrak{g}_x^0$. Now U is open in $\mathfrak{g}_x^0$, and let U' be the set of $y \in \mathfrak{g}_x^0$ where ad_y is invertible on $\mathfrak{g}/\mathfrak{g}_x^0$, then this contains x. Therefore, U' is Zariski open in $\mathfrak{g}_x^0$, and U' is not empty, so U' is dense. Now $U \neq \varnothing$, so $y \in U \cap U' \neq \varnothing$, and $\mathfrak{g}_y^0 \subseteq \mathfrak{g}_x^0$. Therefore $\mathfrak{g}_y^0$ is a proper subspace of $\mathfrak{g}_x 60$, contradicting the fact that x is regular. ☐

Proposition 26.7. Suppose $\mathfrak{h}$ and $\mathfrak{h}'$ are Cartan subalgebras. Let G be the adjoint group of $\mathfrak{h}$, then there exists $u \in G$ such that $\mathrm{ad}_u(\mathfrak{h}) = \mathfrak{h}'$.

Now assume $\mathfrak{g}$ is semisimple, we fix $\mathfrak{h}$ to be a Cartan subalgebra.

Proposition 26.8. $\mathfrak{h}$ is abelian.

For $\lambda \in \mathfrak{h}^*$, we have $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid \mathrm{ad}_y(x) = \lambda(y)x \ \forall y \in \mathfrak{h}\}$.

Theorem 26.9.

- $\mathfrak{h} = \mathfrak{g}_0$,
- $\dim(\mathfrak{g}_\lambda) = 1$ if $\lambda \neq 0$,
- $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus\limits_{\lambda \in R}$, where $R = \{\lambda \in \mathfrak{h}^* \setminus \{0\} \mid g_\lambda \neq 0\}$ is the set of roots.

Example 26.10. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{h}$ is the set of diagonal matrices in $\mathbb{C}^n$, then $\mathfrak{h}^* = \mathbb{C}^n/\mathbb{C}(1, \dots, 1) = \bigoplus\limits_{i=1}^{n} \mathbb{C}\varepsilon_i$. Then $R = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, so $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}e_{ij}$.

27 Lecture 27

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$.

Definition 27.1. We say $x \in \mathfrak{g}$ is regular if the dimension of the generalized 0-eigenspace of ad_x is minimal.

We say $\mathfrak{h} \subseteq \mathfrak{g}$ is the Cartan subalgebra if $\mathfrak{h}$ is nilpotent and $n_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Theorem 27.2. Assume $\mathfrak{g}$ is semisimple,

1. $\mathfrak{h}$ is abelian, i.e., $\mathfrak{h}$ is Cartan subalgebra,
2. elements of $\mathfrak{h}$ are semisimple,
3. the restriction of the Killing form to $\mathfrak{h}$ is non-degenerate.

Definition 27.3. Let $\mathfrak{g}$ be semisimple. We say $x \in \mathfrak{g}$ is semisimple if ad_x is diagonalizable. We say $x \in \mathfrak{g}$ is nilpotent if ad_x is nilpotent.

Theorem 27.4. Let $x \in \mathfrak{g}$. There exists unique x_s, x_n in $\mathfrak{g}$ such that

1. x_s is semisimple and x_n is nilpotent,
2. $x = x_s + x_g$,
3. $[x_s, x_g] = 0$.

Proof. Let $y = \mathrm{ad}_x$, then we have a decomposition $y = y_s + y_n$ in $\mathrm{End}_k(\mathfrak{g})$. Because $\mathfrak{g}$ is semisimple, then $Z(\mathfrak{g}) = 0$, we have $\mathrm{ad} : \mathfrak{g} \hookrightarrow \mathrm{End}_k(\mathfrak{g})$. Consider the decomposition $\mathfrak{g} = \bigoplus_{\lambda\in\mathbb{C}} \mathfrak{g}_x^\lambda$ where $\mathfrak{g}_x^\lambda$ is the generalized λ-eigenspace of ad_x. If we view $\mathfrak{g}$ as a Lie subalgebra of $\mathrm{End}_k(\mathfrak{g})$, then let $y = x$ and put $x_n = y_n$ and $x_s = y_s$, so $[x_s, x] = 0$, therefore $\mathrm{ad}_{x_s}, \mathrm{ad}_{x_n}$ preserve $\mathfrak{g}_x^\lambda$, therefore $(\mathrm{ad}_{x_s} - \lambda)^d = 0$ on $\mathfrak{g}_x^\lambda$ for $d \gg 0$. Therefore $\mathrm{ad}_{x_s} = \lambda \cdot \mathrm{id}$ is the action by λ. Therefore,

$$\begin{aligned}\mathrm{ad}_{x_s}([a,b]) &= (\lambda+\mu)[a,b] \\ &= [a, \mathrm{ad}_{x_s}(b)] + [\mathrm{ad}_{x_s}(a), b]\end{aligned}$$

so ad_{x_s} is a derivation of $\mathfrak{g}$, hence $\mathrm{ad}_{x_s} \in \mathrm{Der}(\mathfrak{g})$. ☐

Lemma 27.5. Let $\mathfrak{g}$ be semisimple, then $\mathrm{ad} : \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}(\mathfrak{g})$.

Proof. Given $\mathfrak{g} \hookrightarrow \mathrm{Der}(\mathfrak{g})$, the image $\mathrm{ad}(\mathfrak{g})$ is semisimple and an ideal, therefore there exists ideal $\mathfrak{a}$ of $\mathrm{Der}(\mathfrak{g})$ such that $\mathrm{Der}(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{g}$. Therefore $[z, \mathrm{ad}_x] = 0$ for all $z \in \mathfrak{a}$ and $x \in \mathfrak{g}$, therefore z acts by zero on $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, so $z = 0$. ☐

Remark 27.6. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{b}$ be a semisimple ideal, then there exists an ideal $\mathfrak{a} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. ($\mathrm{rad}(\mathfrak{g}) \cap \mathfrak{b} = 0$, and $\mathfrak{b}$ has a complement in $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$.)

Proof of Theorem. Let $\mathfrak{h} = \mathfrak{g}_x^0$ for some regular x, then regular elements are semisimple, then the first two points follow. ☐

Let $\lambda \in \mathfrak{h}^*$, and define $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [y,x] = \lambda(y)x \ \forall y \in \mathfrak{h}\}$, then the set of ad_y for $y \in \mathfrak{h}$ are commuating diagonalizable. Therefore $\mathfrak{g} = \bigoplus_{\lambda\in\mathbb{C}} \mathfrak{g}_\lambda = \mathfrak{h} \oplus \bigoplus_{\lambda\in R} \mathfrak{g}_\lambda$. Let $y \in \mathfrak{h}$ be regular, then $\mathfrak{h} \subseteq \mathfrak{g}_0 \subseteq \mathfrak{g}_y^0 = \mathfrak{h}$, so $\mathfrak{g}_0 = \mathfrak{h}$. Here $R = \{\lambda \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\lambda \neq 0\}$.

Theorem 27.7.

1. R is a root system in $\sum_{\alpha\in R} \mathbb{R}\alpha \subseteq \mathfrak{h}^*$,
2. $\dim(\mathfrak{g}_\alpha) = 1$ for $\alpha \in R$,
3. given a basis Δ in R, let $n_+ = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$ and $n_- = \bigoplus_{\alpha\in R^-} \mathfrak{g}_\alpha$, then n_+ and n_- are nilpotent subalgebras of $\mathfrak{g}$, and $\mathfrak{b} = n_+ \oplus \mathfrak{h}$ is a solvable subalgebra of $\mathfrak{g}$, with $[\mathfrak{b},\mathfrak{b}] = n_+$.

Definition 27.8. Let V be a Euclidean vector space and $R \subseteq V \setminus \{0\}$ is a finite subset generating V. We say that R is a root system in V if

1. $s_\alpha(R) \subseteq R$ where $s_\alpha(v) = v - 2\frac{(\alpha,v)}{(\alpha,\alpha)}\alpha$,
2. $s_\alpha(\beta) - \beta \in \mathbb{Z} \cdot \alpha$ for all $\alpha, \beta \in R$,

3. $\mathbb{R}\alpha \cap R = \{\alpha, -\alpha\}$ for all $\alpha \in R$.

Definition 27.9. A basis of R is a subset Δ such that $R = R^+ \coprod R^-$ and Δ becomes a basis of V, where $R^+ = R \cap \bigoplus\limits_{\alpha\in\Delta} \mathbb{R}_{>0}\alpha$ and R^- is defined similarly.

28 Lecture 28

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra, therefore $\mathfrak{h}$ is abelian and elements of $\mathfrak{h}$ are semisimple, with $n_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. Therefore,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha,$$

and let $R = \{0 \neq \lambda \in \mathfrak{h}^* \mid \mathfrak{g}_\lambda \neq 0\}$, where $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [y, x] = \lambda(y)x, \ \forall y \in \mathfrak{h}\}$.

Suppose $V = \sum\limits_{\alpha\in R} \mathbb{R}\alpha \subseteq \mathfrak{h}^*$, then

- $\mathbb{C} \otimes_{\mathbb{R}} V = \mathfrak{h}^*$, so let $x = \mathfrak{g}_\alpha$ and let $y \in \mathfrak{h}$ such that $\alpha(y) = 0$ for all $\alpha \in R$, then $[y, x] = \alpha(y)x = 0$, therefore $y \in Z(\mathfrak{g}) = 0$, therefore $y = 0$,
- let $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, let $h_\alpha \in \mathfrak{h}$ such that $\alpha = (h_\alpha, -)$ as the Killing form, we have $\mathfrak{h}_\alpha = \mathbb{C} \cdot h_\alpha$.

Let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, and $z \in \mathfrak{g}_\gamma$, then as $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ gives $\mathrm{ad}_x \mathrm{ad}_y(z) \in \mathfrak{g}_{\alpha+\beta+\gamma}$, and $\mathrm{ad}_x \mathrm{ad}_y(\mathfrak{g}_\gamma) \subseteq \mathfrak{g}_{\alpha+\beta+\gamma}$, so if $\alpha + \beta \neq 0$, then the trace would be zero, and the Killing form restricts to a non-degenerate form on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$.

For $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = (x, y)h_\alpha$, and for $a \in \mathfrak{h}$ there is $([x, y], a) = (x, [y, a]) = \alpha(a)(x, y) = ((x, y)h_\alpha, a)$, so $[x, y] = (x, y)h_\alpha$. Let $z = [x, y]$, then $[z, x] = (x, y)[h_\alpha, x] = (x, y)\alpha(h_\alpha)x$, and similar results for $[z, y]$, therefore $s = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z$ is a Lie subalgebra of $\mathfrak{g}$. Assume x, y are non-zero, and $\alpha(z) = 0$, then $z \in Z(s)$ and s is solvable, then the adjoint action shows that ad_z is nilpotent on $\mathfrak{g}$, but $z \in \mathfrak{h}$, so ad_z is diagonalizable, therefore $\mathrm{ad}_z = 0$, so $z = 0$, so $[x, y] = 0$, and $(x, y) = 0$. If $(x, y) \neq 0$, then $\alpha(h_\alpha) \neq 0$, and we can find such x and y, so $\alpha(h_\alpha) \neq 0$.

Define $H_\alpha \in \mathfrak{h}_\alpha$ by $\alpha(H_\alpha) = 2$. Fix $0 \neq X_\alpha \in \mathfrak{g}_\alpha$, let $y_\alpha \in \mathfrak{g}_{-\alpha}$ with $[X_\alpha, Y_\alpha] = H_\alpha$, so $[H_\alpha, X_\alpha] = 2X_\alpha$ and $[H_\alpha, Y_\alpha] = -2Y_\alpha$. Let $s_\alpha = \mathbb{C}X_\alpha \oplus \mathbb{C}Y_\alpha \oplus \mathbb{C}H_\alpha$, then this is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

We now show that $\dim(\mathfrak{g}_\alpha) = \dim(\mathfrak{g}_{-\alpha}) = 1$.

Assume $\dim(\mathfrak{g}_\alpha) > 1$, then there is $0 \neq y \in \mathfrak{g}_{-\alpha}$ with $(X_\alpha, y) = 0$, so $[X_\alpha, y] = 0$ and $[H_\alpha, y] = -2y$, so y is a highest weight vector of weight -2. But the representation of s_α on $\mathfrak{g}$ is finite-dimensional, so the highest weight are non-negative, contradiction, so $\dim(\mathfrak{g}_\alpha) = 1$.

29 Lecture 29

Fix a basis of the root system R in $\mathfrak{h}^*$.

Example 29.1. Let $\mathfrak{g} = \mathfrak{sl}_n$ and $R = \{\varepsilon_i - \varepsilon_j\}_{1 \leq i \neq j \leq n}$, and this has a basis

$$\Delta = \{\alpha_1 = \varepsilon_2 - \varepsilon_1, \cdots, \alpha_{n-1} = \varepsilon_n - \varepsilon_{n-1}\}$$

is a basis.

Theorem 29.2. There is a bijection between isomorphism classes of (simple) and semisimple Lie algebra over $\mathbb{C}$ and isomorphism of classes of (irreducible) root systems.

Theorem 29.3. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Fix $\mathfrak{h}$ and Δ. We have $H_\alpha \in \mathfrak{h}$ for $\alpha \in \Delta$. There are $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that there is an isomorphism of Lie algebra $\mathrm{Lie}(C) \cong \mathfrak{g}$, where C is the Cartan matrix of $\mathfrak{g}$, given by $e\alpha \mapsto X_\alpha$, $f\alpha \mapsto Y_\alpha$, and $h\alpha \mapsto H_\alpha$.