MATH 227 Final Exam Review

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May 6, 2025

1 Gaussian Elimination

- Given a system of linear equations or a matrix equation, how do you write down the corresponding augmented matrix?
- How do you perform row reductions?
- How to identify the number of solutions from a given augmented matrix?

Question. *Let A be the augmented matrix*

$$\begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 1 & 3 & | & -2 \\ 2 & 5 & a & | & 4 \end{bmatrix}$$

where a is an unknown real number.

a. *Compute a row echelon form (REF) of matrix A.*

b. *Find the number of solutions of A. This should depend on values of a.*

c. *Suppose A has a unique solution. What is this solution?*

Solution.

a. *We will find the REF of the matrix by performing elementary row operations. We have*

$$\begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 1 & 3 & | & -2 \\ 2 & 5 & a & | & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 1 & 3 & | & -2 \\ 0 & 1 & a+2 & | & -2 \end{bmatrix}$$
$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & -1 & | & 3 \\ 0 & 1 & 3 & | & -2 \\ 0 & 0 & a-1 & | & 0 \end{bmatrix}$$

We are done because we have an upper triangular matrix.

Remark. *I want to comment that the REF of a given matrix is not unique, so it is okay if your answer differs from this.*

b. *A few things to notice:*

- *Let us first recall that the number of solutions (x_1, x_2, x_3) for an augmented matrix (or a system of linear equations) is either one, zero, or infinitely many.*
- *Since we have a REF, we notice that once we have a solution for x_3 using the last equation, x_2 is uniquely determined by the second equation and the value of x_3, then x_1 is uniquely determined by the first equation and the value of x_1 and x_2. Therefore, we just need to find the number of values that x_3 can take.*

- *If $a - 1 \neq 0$, then dividing both sides of the last equation by $a - 1$ gives a unique solution $x_3 = 0$. The discussion above tells us that in this case, the system has a unique solution.*
- *If $a - 1 = 0$, then we have an all-zero row, and we are left with solving three unknowns with two equations. This is impossible as there will always be dependencies between unknowns, therefore in this case there are infinitely many solutions.*

Remark. *We notice from this example that we do not have a case where there is no solution: such things only occur when we have a row of all-zero coefficients with non-zero value augmented, which never happens for this matrix.*

c. *From the discussion above, we know $x_3 = 0$ regardless what the value of a is. This gives $x_2 = -2$ and $x_1 = 7$. The unique solution is then $(x_1, x_2, x_3) = (7, -2, 0)$.*

2 Matrix Operations

- What are the operations you can do on matrices? What about vectors? How are they defined? What is the dimension of the resulting matrix/vector?
- What are true and not true about these operations?
- Using these operations, how to find a linear combination of vectors?

Question.

a. *Given a vector $\vec{v}$, how do you normalize it?*

b. *Write down the formula for the inverse of an invertible (2×2)-matrix.*

c. *Let A be a $(p \times q)$ matrix and B be a $(r \times s)$ matrix. Suppose we are able to compute the product AB and get a (3×2) matrix.*

   i. *What does this say about the dimensions of each matrix?*

   ii. *When is the product BA defined?*

d. *Suppose X, Y, Z are $(n \times n)$-matrices and let I_n be the identity matrix of dimension $(n \times n)$. For the statements below, identify the ones that are true, and create counterexamples for the false ones.*

   i. $XY = YX$;

   ii. $X(Y + Z) = XY + XZ$;

   iii. $(XY)Z = X(YZ)$;

   iv. *If X is invertible, then $X^{-1} = X^T$;*

   v. $X^T + Y^T = (X + Y)^T$;

   vi. *$X + I_n$ is invertible.*

Solution.

a. *The normalization of a vector $\vec{v}$ is just finding a vector of unit length that points in the same direction as $\vec{v}$. This is computed by $\frac{\vec{v}}{||\vec{v}||}$.*

b. *Let us write*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Remark. *Note that* $\det(A) = ad - bc$ *for* (2×2)*-matrices.*

c.

i. This forces $p = 3$, $s = 2$, *and* $q = r$.

ii. This is only defined when $s = p$, *and since* $p = 3$ *and* $s = 2$, *this is never defined.*

d. ii., iii., and v. are true. Let us list a counterexample for each of the rest.

i. Set

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then

$$XY = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = YX.$$

iv. Consider

$$X = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

which is invertible since $\det(A) = 2$, *then*

$$X^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = X^T.$$

vi. Suppose $n = 2$ *and set*

$$X = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

then

$$X + I_n = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This is not invertible, since $\det(X + I_n) = 0$.

3 Geometric Interpretation

- How to draw vectors? How to interpret operations of vectors?
- How to determine linear independence and span of vectors?
- How to define the basis and dimension of a vector space? Given a set of vectors, how do you check if the vectors form a basis?
- What is the geometric meaning of a vector space?
- Two important vector spaces associated to a matrix A: the null space $\mathrm{Nul}(A)$ and the column space $\mathrm{Col}(A)$. For each of them, how to find a basis and compute the dimension? For what values of d are they subspaces of $\mathbb{R}^d$?

Meta-question: using the concepts we have learned so far, what are the implications (and equivalent conditions) for a matrix to be invertible?

Question.

a. Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}.$$

i. *Describe the column space* $\mathrm{Col}(A)$ *as a span of vectors.*

ii. *Using the span you found in part i., find a basis for* $\mathrm{Col}(A)$.

iii. *What is the dimension of the null space* $\mathrm{Nul}(A)$*?*

iv. *Find a basis for* $\mathrm{Nul}(A)$.

b. *Think of* $\mathbb{R}^2$ *as a plane.*

i. *What do the subspaces of* $\mathbb{R}^2$ *look like?*

ii. *For each type of subspaces you identified, what dimension does it have? How many of them are there?*

iii. *Can you generalize this to* $\mathbb{R}^3$*?*

c. *For the statements below, identify the ones that are true, and create counterexamples for the false ones.*

i. *Let* B *be an* $(n \times m)$*-matrix. The dimension of* $\mathrm{Col}(B)$ *is at most* m.

ii. *Let* I_3 *be the* (3×3) *identity matrix, then the dimension of* $\mathrm{Nul}(I_3)$ *is* 0.

iii. *Any* n *linearly independent vectors in* $\mathbb{R}^n$ *form a basis.*

iv. *Any* n *vectors in* $\mathbb{R}^n$ *that span* $\mathbb{R}^n$ *form a basis.*

v. *Suppose* $\{\vec{a}_1, \vec{a}_2\}$ *and* $\{\vec{a}_3, \vec{a}_4\}$ *are both sets of linearly independent vectors, then so is* $\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}$.

Solution.

a.

i. *The column space* $\mathrm{Col}(A)$ *is the span of the columns of* A*, that is,*

$$\mathrm{Col}(A) = \textit{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} \right\}.$$

ii. *To find a basis using the span above, we just need to check if the span is linearly independent. The answer is no, since*

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}.$$

To produce a basis out of the these three vectors, we have to remove vectors that are linearly dependent of others. Removing

$$\begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix},$$

we get a set

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

This set still spans $\mathrm{Col}(A)$*, and is now linearly independent: for a set of two vectors, we just need to check if one of them is a multiple of the other, and the answer is no. Therefore, this is the basis of* $\mathrm{Col}(A)$.

iii. *From part ii., we deduce that* $\mathrm{Col}(A)$ *has dimension* 2*. By the rank-nullity-theorem, we find the dimension of the null space to be* $3 - 2 = 1$.

iv. *To do this the slow way, either solve for the augmented matrix*

$$\begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 2 & 4 & 6 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix},$$

or calculate a REF of A and then set up free variables. These methods are a bit time-costly so we will try the fast way instead. In this case, we just want to identify a non-trivial relation between the column vectors of the matrix A. From part iii., we know we just need one such relation. This is exactly given by the one we computed in part ii., that

$$1\begin{bmatrix}1\\2\\0\end{bmatrix}+1\begin{bmatrix}2\\4\\1\end{bmatrix}+(-1)\begin{bmatrix}3\\6\\1\end{bmatrix}=0.$$

Collecting the coefficients, this says that $(x_1,x_2,x_3)=(1,1,-1)$ is a vector in the null space, therefore the null space $\mathrm{Nul}(A)$ is the span of this vector. That is,

$$\mathrm{Nul}(A)=\mathrm{span}\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}.$$

b. *For $\mathbb{R}^2$, see the following table.*

Subspace of $\mathbb{R}^2$	***Dimension***	***How many***
$\{\vec{0}\}$ *(zero subspace)*	*0*	*1*
Lines through the origin	*1*	*Infinitely many*
$\mathbb{R}^2$ *itself*	*2*	*1*

For the generalization to $\mathbb{R}^3$, see the following table.

Subspace of $\mathbb{R}^3$	***Dimension***	***How many***
$\{\vec{0}\}$ *(zero subspace)*	*0*	*1*
Lines through the origin	*1*	*Infinitely many*
Planes through the origin	*2*	*Infinitely many*
$\mathbb{R}^3$ *itself*	*3*	*1*

c. *Let us list the answer and reasoning for each of these.*

i. ***True**. We should notice that the dimension of $\mathrm{Col}(B)$ is not only at most m, it is also at most n. It is at most m because given m vectors, they can point in at most m independent directions, so the dimension is at most m; it is at most n because each vector has dimension n, so it is a vector in $\mathbb{R}^n$, which has dimension at most n.*

ii. ***True**. We see that the only solution of $I_3\vec{x}=0$ is $\vec{x}=0$, so $\mathrm{Nul}(I_3)$ is a point, with dimension 0.*

iii. ***True**. Each linearly independent vector should point in a "new direction" that the other vectors can't span to. $\mathbb{R}^n$ already has dimension n, so it has at most n such directions, so it must all be covered by these vectors.*

iv. ***True**. Having a set of n vectors that span $\mathbb{R}^n$, we should be able to remove $n-n=0$ vectors to get a basis of $\mathbb{R}^n$. This means the set of vectors is already a basis.*

v. ***False**. Set*

$$\vec{a}_1=\begin{bmatrix}1\\0\end{bmatrix},\vec{a}_2=\begin{bmatrix}0\\1\end{bmatrix},\vec{a}_3=\begin{bmatrix}2\\0\end{bmatrix},\vec{a}_4=\begin{bmatrix}0\\2\end{bmatrix}.$$

4 Linear Transformation

- What are the relations between linear transformations and matrices?
- Given a linear transformation described as an effect, how do we write down the corresponding matrix in terms of a given basis?
- Given a vector, what are the coordinates of the vector in terms of a basis $\mathcal{B}$? What about the other way around?

- Suppose T is a linear transformation and v and w are vectors, and suppose we know where T sends v and w to as vectors, respectively. Given a linear combination of v and w, where does T send this linear combination to?
- Given a linear transformation and two bases, how do the bases relate to each other? How do you find the change-of-basis matrix?

Question.

a. Find a matrix that describes the transformation that performs a clockwise rotation by 90 degrees.

b. Let T be a linear transformation such that $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}2\\5\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)=\begin{bmatrix}3\\4\end{bmatrix}$. Compute

$$T\left(\begin{bmatrix}0\\3\end{bmatrix}\right).$$

c. Consider the standard basis $\mathcal{E}=\left\{e_1=\begin{bmatrix}1\\0\end{bmatrix}, e_2=\begin{bmatrix}0\\1\end{bmatrix}\right\}$ and another basis $\mathcal{B}=\left\{b_1=\begin{bmatrix}3\\2\end{bmatrix}, b_2=\begin{bmatrix}2\\3\end{bmatrix}\right\}$.

i. Let $v=\begin{bmatrix}5\\8\end{bmatrix}$. Describe what $v_{\mathcal{B}}$ means, and calculate its value.

ii. Calculate the change-of-basis matrix that turns a vector in $\mathcal{B}$ into a vector in $\mathcal{E}$. What about the other way around?

Solution.

a. We need to know where the transformation sends each of the standard basis vector to. We know it sends $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}0\\-1\end{bmatrix}$, and sends $\begin{bmatrix}0\\1\end{bmatrix}$ to $\begin{bmatrix}1\\0\end{bmatrix}$, therefore the corresponding matrix is

$$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}.$$

b. By linearity of the linear transformation, if we can express $\begin{bmatrix}0\\3\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$, then we can express the answer as the same linear combination of $\begin{bmatrix}2\\5\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$. We see that

$$\begin{bmatrix}0\\3\end{bmatrix}=2\begin{bmatrix}1\\2\end{bmatrix}+(-1)\begin{bmatrix}2\\1\end{bmatrix},$$

so

$$T\left(\begin{bmatrix}0\\3\end{bmatrix}\right)=2\begin{bmatrix}2\\5\end{bmatrix}+(-1)\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}1\\6\end{bmatrix}.$$

c.

i. The vector $v_{\mathcal{B}}$ is the coordinate vector of v with respect to the basis $\mathcal{B}$. We should find scalars x_1, x_2 so that

$$\begin{bmatrix}5\\8\end{bmatrix}=v=x_1b_1+x_2b_2=x_1\begin{bmatrix}3\\2\end{bmatrix}+x_2\begin{bmatrix}2\\3\end{bmatrix}.$$

We solve that $x_1=1$ and $x_2=1$. Therefore $v_{\mathcal{B}}=\begin{bmatrix}1\\1\end{bmatrix}$.

ii. The change-of-basis matrix from $\mathcal{B}$ to $\mathcal{E}$ is exactly given by the basis vectors of $\mathcal{B}$, written down in the matrix, i.e.,

$$[b_1\, b_2]=\begin{bmatrix}3 & 2\\2 & 3\end{bmatrix}.$$

The matrix from $\mathcal{E}$ to $\mathcal{B}$ is exactly its inverse, given by

$$\frac{1}{3\cdot 3-2\cdot 2}\begin{bmatrix}3 & -2\\-2 & 3\end{bmatrix}=\frac{1}{5}\begin{bmatrix}3 & -2\\-2 & 3\end{bmatrix}.$$

5 Orthogonality

- What does it mean for a set of vectors to be orthogonal? What does it mean for them to be orthonormal?
- How would you define an orthogonal matrix?
- How do you find the projection of a vector onto a vector subspace? How do you express the projection as a matrix?

Question.

a. *Let A be an $(n \times n)$-matrix. What are two ways of checking that A is an orthogonal matrix?*

b. *Find the shortest distance from the point $P = (2,6,0)$ to the line through the origin spanned by the vector $\vec{w} = (1,2,3)$.*

Solution.

a. *You can either check that $A^T = A^{-1}$ (or any equivalent form to this), or showing that the columns of A form an orthonormal set of vectors.*

b. *Let $\vec{p} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$ be the vector corresponding to the point P, and set $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. The shortest distance from $\vec{p}$ to the line spanned by $\vec{w}$ is the length of the vector from $\vec{p}$ to its projection onto $\vec{w}$, i.e., onto the line spanned by $\vec{w}$. The projection is given by*

$$\text{proj}_{\vec{w}}(\vec{p}) = \frac{\vec{p} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w} = \frac{14}{14}\vec{w} = \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The distance from point P to the line spanned by $\vec{w}$ is then the component perpendicular to the orthogonal projection, namely

$$\text{proj}_{\vec{w}}(\vec{p}) = \frac{14}{14}\vec{w} = \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The distance we want is the norm

$$\left\| \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \right\| = \sqrt{1^2 + 4^2 + (-3)^2} = \sqrt{1 + 16 + 9} = \sqrt{26}.$$

6 Eigenvalues, Eigenvectors, and Eigenspaces, and their Application

a. Basic Properties.

- How do you compute the eigenvalues of a matrix?
- Given an eigenvalue, how do you find the associated eigenvectors?
- Given an eigenvector, how do you find the associated eigenvalue?
- How do you find the diagonalization of a matrix? What are some necessary and sufficient conditions for a matrix to be diagonalizable?
- Given a diagonalizable matrix, how do you check the behavior of the matrix to large powers? For instance, can you compute $\lim_{n\to\infty} A^n$ for a diagonalizable matrix A?

Meta-question: what is the point of each of the applications below?

b. Graphs.

- Given a graph, how do you find its adjacency matrix? What is the difference between adjacency matrices of undirected and directed matrices?
- Given a graph, what are the differences between walks and paths?
- Given an adjacency matrix, how do you find the number of walks between two nodes using matrix power?
- Given a Markov chain, what is the transition matrix? How do you compute the steady-state vector?

c. Singular Value Decomposition (SVD).

- What are the definitions of different SVDs we have seen? Can you describe what each decomposition is consisted of?
- How do you perform low-rank approximation from a SVD?

d. Principal Component Analysis (PCA).

- What is the general procedure of PCA?

e. Least Square Solutions.

- What is the general procedure of least square method?
- How do you check if a vector is a least square solution to a system of linear equations?
- Given a table of data pairs (x, y), how do you find the least square solution for a given model? What about the other way around? Given a matrix equation of least square method, can you recover the data table?

Question.

a. Set

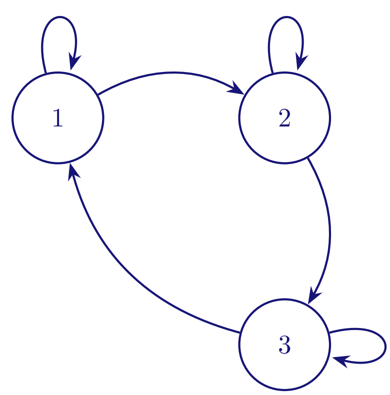
$$A = \begin{bmatrix} 2 & -3 \\ 1 & 6 \end{bmatrix}.$$

i. Compute the eigenvalues of A.

ii. For each eigenvalue of A you found in part i., compute its eigenvectors.

iii. Find the diagonalization of A.

b. Consider the directed graph below.



i. Write down the adjacency matrix B.

ii. Suppose you have computed

$$B^4 = \begin{bmatrix} 5 & 6 & 5 \\ 5 & 5 & 6 \\ 6 & 5 & 5 \end{bmatrix}$$

Find the total number of walks of length 4 that starts at node 3.

Let us turn the directed graph into a Markov chain by considering the following scenario: for each of the node 1, 2, *and* 3, *there is a probability* $\frac{1}{2}$, $\frac{1}{3}$, *and* $\frac{1}{4}$, *respectively, for you to stay there at the next time step if you are actually at that node.*

iii. *Suppose you are at a node: you do not know which one, but it is equally likely for you to be at each of them. Compute the probability of you being at each node at the next time step, respectively.*

c. *Let*

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

i. *Without plugging in any numbers, write down the formula for standard and compact SVD, and label the dimension of each matrix in the decomposition.*

ii. *Compute standard SVD and compact SVD for* C.

iii. *Write down the rank-*1 *decomposition of* C.

Solution.

a.

i. *To compute the eigenvalues, we need to solve for* $\det(A - \lambda I) = 0$, *that is, values* λ *such that*

$$\det\left(\begin{bmatrix} 2-\lambda & -3 \\ 1 & 6-\lambda \end{bmatrix}\right) = 0.$$

This gives

$$\begin{aligned} 0 &= (2-\lambda)(6-\lambda) + 3 \\ &= \lambda^2 - 8\lambda + 15 \\ &= (\lambda - 3)(\lambda - 5). \end{aligned}$$

Therefore, the eigenvalues are $\lambda = 3, 5$.

ii. *For each* λ *above, we now solve for the null space of* $A - \lambda I$. *For* $\lambda_1 = 3$,

$$A - 3I = \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix}.$$

The null space is spanned by the vector $\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. *For* $\lambda_2 = 5$,

$$A - 5I = \begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix}.$$

The null space is spanned by the vector $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

iii. *Set*

$$P = \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix},$$

then

$$A = PDP^{-1}.$$

b.

i. *This is*

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

ii. *The number of walks of length* 4 *starting at node* 3 *and ending at node* i *are given by the entry in the* i*th row and third column of the matrix* B^4*. The total number of such walks is then the sum of entries on that column, namely*

$$5 + 6 + 5 = 16.$$

iii. *The original probability vector is*

$$\vec{p} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Using the weights on each arrow, we find the transition matrix to be

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{4} \end{bmatrix}.$$

The probability vector at the next time step is then

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{12} \\ \frac{5}{18} \\ \frac{11}{36} \end{bmatrix}.$$

c.

i. *For an* $(n \times m)$*-matrix* C,

- *the SVD gives* $C = U\Sigma V^T$ *where* U *has dimension* 3×3, Σ *has dimension* 3×2, *and* V *has dimension* 2×2;
- *the compact SVD gives* $C = U_c \Sigma_c V_c^T$ *where* U_c *has dimension* 3×2, Σ *has dimension* 2×2, *and* V *has dimension* 2×2.

ii. *Compute that*

$$C^T C = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 6$ *and* $\lambda_2 = 1$, *with singular values* $\sigma_1 = \sqrt{6}$ *and* $\sigma_2 = 1$, *respectively. The corresponding eigenvectors are*

$$v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

respectively. We also have

$$CC^T = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Recall that the matrix $C^T C$ *and* CC^T *share the same non-zero eigenvalues, so for the* (3×3)*-matrix* CC^T, *we should add an eigenvalue* 0. *For eigenvalues* $\lambda_1 = 6$ *and* $\lambda_2 = 1$, *this time the eigenvectors are*

$$u_1 = \frac{1}{\sqrt{30}} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

For eigenvalue $\lambda_3 = 0$, *we have an eigenvector*

$$u_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}.$$

For standard SVD, we have

$$C = [u_1 \; u_2 \; u_3] \begin{bmatrix} 6 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}.$$

For compact SVD, we will eliminate the last row of Σ *to get* Σ_c*, meaning we will eliminate the contributions of eigenvalue* 0*, therefore we are also eliminating the corresponding eigenvectors. There is no such eigenvectors in* V^T*, but there is* u_3 *in* U*, which we delete, and we get a compact SVD*

$$C = [u_1 \; u_2] \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}.$$

iii. *Since* C *has rank* 2*, it is a* 2*-term sum of rank-*1 *decomposition. That is,*

$$C = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T.$$