MATH 214A Notes

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1 Lecture 1

Algebraic geometry is about shapes defined by polynomial equations. One may realize it is especially easier to understand algebraic sets over $\mathbb{C}$.

Example 1.1. $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 1\} \cong \mathbb{C}\backslash\{0\}$.

Algebraic geometry studies algebraic curves over $\mathbb{C}$, i.e., structure of dimension 1. Because the field $\mathbb{C}$ is algebraically closed, then every polynomial $f \in \mathbb{C}[x]$ can be factored into degree 1 polynomials, i.e., $f(x) = a(x - b_1) \cdots (x - b_n)$ for some $a \in \mathbb{C}$, $n \geq 0$, and $b_1, \ldots, b_n \in \mathbb{C}$. This would not happen over $\mathbb{R}$, for instance.

Algebraic geometry looks at equations with more variables, in general.

Example 1.2. Consider $\{x \in \mathbb{R} : x^3 + ax^2 + bx + c = 0\}$ for some $a, b, c \in \mathbb{R}$. Typically, the equation has 1 root or 3 roots, depending on the shape of the diagram. However, if we substitute $\mathbb{R}$ with $\mathbb{C}$, then we essentially always have 3 roots in this equation, even though sometimes there exists a double root.

To classify algebraic varieties, one key step for varieties over $\mathbb{C}$ is to look at them just as topological spaces.

Example 1.3. Consider $\{(x, y) \in \mathbb{C}^2 : x^d + y^d = 1\}$. This is a complex curve homeomorphic to a real 2-manifold of genus g minus a finite set. In this case, we have $g = \frac{(d-1)(d-2)}{2}$.

Theorem 1.4 (Faltings)**.** If an algebraic curve X over $\mathbb{Q}$ has genus $g \geq 2$, then the set of rational points $X(\mathbb{Q})$ is finite.

In some sense, complexity in algebra and topology are related.

Sometimes people also look at the connection between algebraic geometry and number theory.

Example 1.5. What is $\{(x, y, z) \in \mathbb{Z}^3 : x^5 + y^5 = z^5\}$? The only solution is $(0, 0, 0)$. Note that this set is equivalent to $\{(x, y) \in \mathbb{Q}^2 : x^5 + y^5 = 1\}$.

Number theory allows us to study numbers in finite fields. We can define numbers like the genus and topology even in finite characteristics.

Definition 1.6 (Affine Space)**.** Let k be an algebraically closed field. The *affine n-space over k* is

$$\mathbb{A}_k^n = k^n = \{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in k\}.$$

Let $R = k[x_1, \ldots, x_n]$. An element $f \in R$ determines a function $\mathbb{A}_k^n \to k$. For an element $f \in R$, its *zero set* is $\{f = 0\} \subseteq \mathbb{A}_k^n$, often defined by

$$Z(f) = \{f = 0\} := \{(a_1, \ldots, a_n) \in \mathbb{A}_k^n : f(a_1, \ldots, a_n) = 0\}.$$

Similarly, for a set T, its zero set is

$$Z(T) = \{a \in \mathbb{A}_k^n : f(a) = 0 \ \forall f \in T\}.$$

An *affine algebraic set* over k is a subset of $\mathbb{A}_k^n$ for some $n \geq 0$ of the form $Z(T)$ for some subset $T \subseteq R = k[x_1, \ldots, x_n]$.

Remark 1.7. Given a subset $T \subseteq R$, let $I \subseteq R$ be the ideal generated by T, then $Z(T) = Z(I)$.

Example 1.8. What is the algebraic set of the affine line $\mathbb{A}_k^1$? We want to find all subsets of $\mathbb{A}_k^1 \cong k$ defined by some ideal $I \subseteq k[x]$. If $I = \{0\}$, then $Z(I) = \mathbb{A}_k^1$. If not, then pick $f \neq 0$ in I, then $Z(I) \subseteq Z(f)$, and $f = a(x - b_1) \cdots (x - b_n)$, so $Z(f) = \{b_1, \ldots, b_n\}$.

We conclude that an affine set in $\mathbb{A}_k^1$ is either all of $\mathbb{A}_k^1$ or a finite set of points.

2 Lecture 2

Definition 2.1 (Zariski Topology)**.** Let k be an algebraically closed field and let $n \geq 0$. The *Zariski Topology* on $\mathbb{A}_k^n \cong k^n$ is defined by closed sets, which is defined as follows: a subset $S \subseteq \mathbb{A}_k^n$ is closed if and only if it is of the form $S = Z(I)$ for some ideal $I \subseteq R$ where $R = k[x_1, \ldots, x_n]$.

Example 2.2. The twisted cubic curve in $\mathbb{A}_k^3$ is defined as

$$\{(\mathcal{A}, \mathcal{A}^2, \mathcal{A}^3) : \mathcal{A} \in k\} \subseteq \mathbb{A}_k^3.$$

This is Zariski-closed in $\mathbb{A}_k^3$ since

$$S = \{y = x^2, z = x^3\} \subseteq \mathbb{A}_k^3$$

is equivalent to $Z(\{y - x^2, z - x^3\}$, which is just $Z(I)$ where $I \subseteq k[x, y, z]$ is just the ideal $(y - x^2, z - x^3)$.

Remark 2.3. If $k = \mathbb{C}$, then we also have the classical topology on $\mathbb{A}_{\mathbb{C}}^n = \mathbb{C}^n$, based on the usual metric on $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

It is easy to see that Zariski-closed in $\mathbb{A}_{\mathbb{C}}^n$ implies closure in the classical topology. The converse is obviously not true, for example consider the closed balls in $\mathbb{C}^3$.

Lemma 2.4. The Zariski topology in $\mathbb{A}_k^n$ is a well-defined topology.

Proof. By definition, a topological space is a set with a colletion of subsets called "the open subsets of X", such that

1. $\varnothing$ and X are open in X,
2. union of any collection of open sets is open,
3. intersection of finitely many open sets is open.

Equivalently, the closed subsets of X satisfy

1. $\varnothing$ and X are closed in X,
2. intersection of any collection of closed sets is closed,
3. union of finitely many closed sets is closed.

Indeed,

1. $\mathbb{A}_k^n = Z(0)$ and $\varnothing = Z(R)$.
2. Given a collection S_α of closed subsets of $X = \mathbb{A}_k^n$ where $\alpha \in I$ set, which could be infinite, the intersection of the collection is just the union of the zero sets.

   By definition, for each $\alpha \in I$, we can choose an ideal $I_\alpha \subseteq R$ with $S_\alpha = Z(I_\alpha) \subseteq \mathbb{A}_k^n$.

   Define $I = \sum_{\alpha \in I} I_\alpha \subseteq R$ (i.e., the set of all possible finite sums), then $Z(I) = \bigcap_{\alpha \in I} Z(I_\alpha) = \bigcap_{\alpha \in I} S_\alpha$, so it is closed.

3. Given closed sets $S, T \subseteq \mathbb{A}_k^n$, we want to show that $S \cup T$ is closed. By definition, choose I and J such that $S = Z(I)$ and $T = Z(J)$. Take $K = I \cap J$ or $J = IJ$ (i.e., finite sum of elements ab with $a \in I$ and $b \in J$), then it suffices to show that $Z(I \cap J) = Z(IJ) = Z(I) \cup (J)$.

   Example 2.5. Note that the two structures may not be equivalent. Let $R = k[x]$ and let $I = J = (x)$. Now $Z(I) = Z(J) = \{0\}$, then $I \cap J = (x)$, but $IJ = (x^2)$.

   Remark 2.6. Essentially, if $I = (f_1, \dots, f_r)$ and $J = (g_1, \dots, g_s)$, then $IJ = (f_i g_j : \forall i, j)$.

   However, things look better if we look at their radicals.

   Exercise 2.7. Show that for any commutative R and ideals I and J, the radicals satisfy $\operatorname{rad}(I \cap J) = \operatorname{rad}(IJ)$.

   To finish the proof, we show that $Z(IJ) = Z(I) \cup Z(J)$. Indeed, we have $IJ \subseteq I$ and $IJ \subseteq J$, so $Z(IJ) \supseteq Z(I)$ and $Z(IJ) \supseteq Z(J)$, so $Z(I) \cup Z(J) \subseteq Z(IJ)$.

   Conversely, we want to show $Z(IJ) \subseteq Z(I) \cup Z(J) \subseteq \mathbb{A}_k^n$.

   Let $a = (a_1, \dots, a_n) \in k^n$ be a point in $Z(IJ)$. Suppose $a \notin Z(I)$ and $a \notin Z(J)$, so there exists $f \in I$ such that $f(a) \neq 0$, and there exists $g \in J$ such that $g(a) \neq 0$, then $(fg)(a) = f(a)g(a) = 0$, but $fg \in IJ$, $(fg)(a) \neq 0$, contradiction.

□

Remark 2.8. Note that $\mathbb{A}_k^n$ is not Hausdorff for $n > 1$. In fact, the intersection of any two non-empty open subsets is non-empty.

For $\mathbb{A}_k^1$, an open subset of $\mathbb{A}_k^1$ is either $\varnothing$ or a $\mathbb{A}_k^1$-finite set. Note that k is infinite since it is algebraically closed, so the intersection of two intervals on $\mathbb{A}_k^1$ (with finitely many isolated points excluded) should not be empty.

Definition 2.9 (Connected, Irreducible)**.** A topological space X is *connected* if $X \neq \varnothing$, and you cannot write X as the disjoint union of two non-empty closed subsets.

A topological space X is *irreducible* if $X \neq \varnothing$, and you cannot write X as the union of two proper closed subsets.

Example 2.10. For example, the set defined by two parallel lines is not connected; the set defined by the union of a circle and a line passing through the circle is connected, but not irreducible.

Remark 2.11. A Hausdorff space with at least 2 points is never irreducible.

Example 2.12. $[0,1]$ is not irreducible since $[0,1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$, but $\mathbb{A}_k^n$ is irreducible.

Theorem 2.13 (Hilbert's Nullstellensatz)**.** For an algebraically closed field k and $n \geq 0$, there is a one-to-one correspondence between radical ideals in $R = k[x_1, \ldots, x_n]$ and the Zariski closed subsets of $\mathbb{A}_k^n$. More precisely, this correspondence is given by the mapping $I \mapsto Z(I)$ for radical ideals I and the mapping $S \mapsto I(S) = \{f \in R : f(a) = 0 \ \forall a \in S\}$ for closed subset $S \subseteq \mathbb{A}_k^n$.

Definition 2.14 (Reduced Ring, Radical Ideal)**.** A commutative ring R is *reduced* if every nilpotent element is 0, i.e., if $a \in R$ such that $a^m = 0$ for some $m > 0$, then $a = 0$.

An ideal I in a commutative ring R is *radical* if the ring R/I is radical. In particular, $I \subseteq R$ is radical if and only if for any $a \in R$ with $a^m \in I$ for some $m > 0$, we know $a \in I$.

For any ideal I, $\operatorname{rad}(I) = \{a \in R : a^m \in I \text{ for some } m > 0\}$.

Lemma 2.15. An affine algebraic set $X \subseteq \mathbb{A}_k^n$ is irreducible if and only if $I(Y) \subseteq R$ is prime.

Proof. ($\Longrightarrow$): Let $Y \subseteq \mathbb{A}_k^n$ be an irreducible algebraic set.

We define the subspace topology on Y as follows: a subset of Y is closed in Y if it is the intersection of some closed subset (of X) and Y.

Therefore, since $Y \neq \varnothing$, so $I(Y) \neq R$ as $1 \in R$ is not in $I(Y)$.

Suppose $f, g \in R$ with $fg \in I(Y)$. We want to show that f or g is in $I(Y)$. Since $fg \in I(Y)$, $Y = (Y \cap \{f = 0\}) \cup (Y \cap \{g = 0\})$ is the union of two closed sets in Y. Therefore, since Y is irreducible, then either $Y = Y \cap \{f = 0\}$, or $Y = Y \cap \{g = 0\}$. That is, $f \in I(Y)$ or $g \in I(Y)$, as desired.

($\Longleftarrow$): Given an affine algebraic set $X \subseteq \mathbb{A}_k^n$ such that the ideal $I(X) \subseteq R$ is prime. That means $1 \notin I(X)$, and, if $f, g \in R$ such that $fg \in I(X)$, then $f \in I(X)$ or $g \in I(X)$. Note that if $X = \varnothing$, then $I(X)$ would be R, which is not prime. Therefore, $X \neq \varnothing$. Suppose $X = S_1 \cup S_2$ for closed subsets $S_1, S_2 \subsetneq X$. We pick $p \in S_2 \backslash S_1$ and $q \in S_1 \backslash S_2$. Since S_1 and S_2 are closed in $\mathbb{A}_k^n$, there is a polynomial $f \in I(S_1)$ and $f(q) \neq 0 \in k$. Similarly, there is a polynomial $g \in I(S_2)$ but with $g(p) \neq 0$. Then $fg \in I(X)$. Since $I(X)$ is prime, $f \in I(X)$ or $g \in I(X)$, contradiction. $\square$

3 Lecture 3

Remark 3.1. For any subset $X \subseteq \mathbb{A}_k^n$, $I(X) \subseteq R$ is radical.

Proof. If $f \in R$ has $f^m \in I(X)$ for some $m > 0$, then $f \in I(X)$. Therefore, at any $p \in X$, $f(p)^m = 0 \in k$. Hence, $f(p) = 0 \in k$. □

Remark 3.2. $Z(I) = Z(\mathrm{rad}(I))$ for ideal $I \subseteq R = k[x_1, \ldots, x_n]$.

Example 3.3. Affine n-space $\mathbb{A}_k^n$ is irreducible.

Proof. Think of $\mathbb{A}_k^n$ as a closed set in itself, then $I(\mathbb{A}_k^n) = 0$, and so $\mathbb{A}_k^n$ is irreducible if and only if $0 \subseteq k[x_1, \ldots, x_n]$ is prime, if and only if $k[x_1, \ldots, x_n]$ is a domain. □

Remark 3.4. For any irreducible topological space, the intersection of any two non-empty open subsets is non-empty. (So this holds in $\mathbb{A}_k$ per se.)

Definition 3.5 (Affine Variety)**.** An *affine variety* over k is an irreducible affine algebraic set in some $\mathbb{A}_k^n$.

Definition 3.6 (Irreducible)**.** Let R be a domain. Any element $f \in R$ is *irreducible* if $f \neq 0$ and for any $g, h \in R$ such that $f = gh$, either g or h must be a unit.

Remark 3.7. This concept is useless unless R is a UFD, where R admits a unique factorization.

Proposition 3.8. If R is a UFD, and $f \in R$ is irreducible, then (f) is a prime ideal. In particular, for any field k, the polynomial ring $k[x_1, \ldots, x_n]$ is a UFD.

We now have the notion of an irreducible polynomial $f \in k[x_1, \ldots, x_n]$ over k. In particular, the units in the polynomial ring $k[x_1, \ldots, x_n]$ is just k^*, i.e., the units in k.

Remark 3.9. The proposition implies that for any irreducible polynomial f over a field k, the ideal $(f) \subseteq R$ is prime.

Corollary 3.10. For an irreducible polynomial $f \in k[x_1, \ldots, x_n]$ over an algebraically closed field k, $\{f = 0\} \subseteq \mathbb{A}_k^n$ is an affine variety over k. This is called an *irreducible hypersurface* in $\mathbb{A}_k^n$.

For $n = 1$, an irreducible polynomial in $k[x]$ (with k algebraically closed) is of the form $c(x - a)$ for $a, c \in k$.

Recall the following exercise in homework:

Exercise 3.11. Let $g \in k[x_1, \ldots, x_{n-1}]$. Then $x_n^2 - g(x_1, \ldots, x_{n-1})$ is irreducible over k if and only if g is not a square in $k[x_1, \ldots, x_{n-1}]$.

For example, $x^2 - y^{17}$ is irreducible over $\mathbb{C}$, i.e., $\{x^2 = y^{17}\} \subseteq \mathbb{A}_{\mathbb{C}}^2$ is a variety.

Example 3.12. Over $\mathbb{R}$, x^2+y^2 is irreducible since $-y^2$ is not a square in $\mathbb{R}[y]$. Geometrically, we see that the set $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 0\} = \{(0,0)\}$.

Over $\mathbb{C}$, as $x^2+y^2 = (x+iy)(x-iy)$, then geometrically we see $\{(x,y) \in \mathbb{C}^2 : x^2+y^2 = 0\} = \{(x = iy)\} \cup \{(x = -iy)\}$.

Note that for $n \geq 3$, $x_1^2 + \cdot + x_n^2$ is irreducible over $\mathbb{C}$.

Definition 3.13 (Coordinate Ring)**.** For an affine algebraic set $X \subseteq \mathbb{A}_k^n$, the *coordinate ring* of X (or *ring of regular functions* on X) is $\mathcal{O}(X) := k[x_1, \ldots, x_n]/I(X)$. This is isomorphic to the image of mapping from $k[x_1, \ldots, x_n]$ to the ring of all functions $X \to k$.

Example 3.14. Consider $X = \{x^2 = y^3\} \subseteq \mathbb{A}_{\mathbb{C}}^2$. Then $x^5 - 7y$ is a regular function on X, and is equal to $x^5 - 7x + 8(x^2 - y^3)$ on X.

Remark 3.15. For an affine algebraic set X, $\mathcal{O}(X)$ is a finitely-generated commutative k-algebra. Also, for an affine variety $X \subseteq \mathbb{A}_k^n$, $\mathcal{O}(X)$ is a domain as well.

Conversely, for any finitely-generated commutative k-algebra R (which is a domain), $R \cong \mathcal{O}(X)$ for some affine variety $X \subseteq \mathbb{A}_k^n$ for some $n \geq 0$. Similar classification holds for general affine algebraic sets.

Proof. Let R be a finitely-generated k-algebra which is a domain, then $R = k[x_1, \ldots, x_n]/I$ for some $n \geq 0$ and some ideal I. Since R is a domain, I is prime. So $Z(I) \subseteq \mathbb{A}_k^n$ is an affine variety X.

We want to show that $R \cong \mathcal{O}(X)$ as k-algebras. Here $\mathcal{O}(X) \cong k[x_1, \ldots, x_n]/I(X)$, where we can denote $I(X) = I(Z(I))$. By Nullstellensatz, $I(Z(I))$ is just I if it is radical. Now since I is prime, then it is radical indeed, and we are done. □

Example 3.16. $\mathbb{A}_k^1$ and $X = \{y = x^2\} \subseteq \mathbb{A}_k^2$ have isomorphic coordinate rings (as k-algebras).

Proof. One would realize that $\mathcal{O}(\mathbb{A}_k^1) = k[x]$ and $\mathcal{O}(X) = k[x,y]/I(X)$. Note that $y - x^2$ is irreducible, so $(y - x^2) \subseteq k[x,y]$ is prime, then $I(X) = I(Z(y - x^2)) = (y - x^2)$. Therefore, $\mathcal{O}(X) = k[x,y]/I(X) \cong k[x,y]/(y - x^2) \cong k[x]$.

Geometrically, the two structures are just a horizontal line and a quadratic curve, respectively. The isomorphic is given by the projection of the quadratic curve onto the horizontal axis. □

4 Lecture 4

Definition 4.1 (Noetherian)**.** A topological space X is *Noetherian* if every descending sequence of closed subsets $X \supset Y_1 \supset Y_2 \supset \cdots$, there is some $N \in \mathbb{Z}^+$ such that $Y_N = Y_{N+1} =$

$\cdots$. This is essentially a DCC on X.

Remark 4.2. Note that $\mathbb{R}$ and $[0, 1]$ are not Noetherian with the classical topology.

Lemma 4.3. Every affine algebraic set over an algebraically closed field k is Noetherian (as a topological space).

Proof. We are given a closed subset $X \subseteq \mathbb{A}_k^n$ for some $n \geq 0$. Here $\mathcal{O}(X)$ is a finitely-generated (commutative) k-algebra (and a reduced ring). By the Nullstellensatz, we have a one-to-one correspondence between closed subsets of X and radical ideals of $\mathcal{O}(X)$. To see this, we know a one-to-one correspondence between closed subsets of $\mathbb{A}_k^n$ and radical ideals in $k(x_1, \ldots, x_n)$, then $\mathcal{O}(X) = k[x_1, \ldots, x_n]/I(X)$. By Hilbert's basis theorem, $\mathcal{O}(X)$ is a Noetherian ring, i.e., every ideal in $\mathcal{O}(X)$ is finitely-generated as an ideal, or equivalently, the ACC condition. Therefore, every decreasing sequence of closed subsets of X terminates, i.e., X is Noetherian as a topological space. □

Theorem 4.4. Every Noetherian topological space X can be written as a finite union of irreducible closed subsets, i.e., $X = Y_1 \cup \cdots \cup Y_n$ for some $n \geq 0$ and irreducible closed subsets Y_i of X.

Moreover, if we also require that Y_i is not contained in Y_j for all $i \neq j$, then this decomposition is unique up to reordering.

Remark 4.5. We call the Y_i's (with all the conditions above) the *irreducible component* of X.

Definition 4.6 (Dimension)**.** The *dimension* of a topological space X is $\dim(X) = \sup\{n \geq 0 :$ there is a chain of length n of irreducible closed subsets of X, $Y_0 \subsetneq Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_n\}$.

Exercise 4.7. Show that $\dim(\mathbb{R}^3) = 0$ for $\mathbb{R}^3$ with the classical topological space.

Example 4.8. $\dim(\mathbb{A}_k^1) = 1$ with the Zariski topology. Recall that any closed set on this space is either itself or a set of finitely many points. Therefore, the largest chain of irreducible closed subsets has length $\{a\} \subsetneq \mathbb{A}_k^1$ for any $a \in k$.

Definition 4.9 (Krull Dimension)**.** The *(Krull) dimension* of a commutative ring R is $\sup\{n \geq 0 :$ there is a chain of length n of prime ideals in $R : \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}$.

Lemma 4.10. Let X be an affine algebraic set over k. Then $\dim(X) = \dim(\mathcal{O}(X))$, i.e., the dimension of the topological space equals the (Krull) dimension of the ring.

Proof. We have a one-to-one correspondence between prime ideals in $\mathcal{O}(X)$ and irreducible closed subsets of X (containing whatever $I(X)$ we quotient out), reversing the directions of the inclusions. $\square$

Definition 4.11 (transcendence degree)**.** Let $k \subseteq E$ be a field extension (not necessarily finite, or even algebraic). There is a set I and a set of elements $x_i \in E$ for $i \in I$ such that $k \subseteq k(x_i : i \in I) \subseteq$, where $k(x_i : i \in I) = \operatorname{Frac}(k[x_i : i \in I])$ is the rational function field on a set of variables, such that E is algebraic over $k(x_i : i \in I)$. The *transcendence degree* of E over k is the cardinality $|I|$. This is well-defined.

Theorem 4.12. Let k be any field and let A be a domain which is also a finitely-generated (commutative) k-algebra. Then $\dim(A)$ is the transcendence degree of $\operatorname{Frac}(A)/k$, i.e., $\dim(A) = \operatorname{tr}\deg(\operatorname{Frac}(A)/k)$.

Corollary 4.13. For any $n \geq 0$ and algebraically closed field k, $\dim(\mathbb{A}_k^n) = n$.

Proof. We have $\dim(\mathbb{A}_k^n) = \dim(k[x_1, \ldots, x_n]) = \dim(\mathcal{O}(\mathbb{A}_k^n)) = \operatorname{tr}\deg(k(x_1, \ldots, x_n)/k) = n$. $\square$

Proposition 4.14 (Krull's Principal Ideal Theorem)**.** Let A be a Noetherian ring, and let $f \in A$ be an element which is neither zero divisor nor a unit, then every minimal prime ideal $\mathfrak{p}$ containing f has height 1.

Corollary 4.15. A variety in $\mathbb{A}_k^n$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single non-constant irreducible polynomial in $A = k[x_1, \ldots, x_n]$.

Proof. See Hartshorne Section I.1 Proposition 1.13. $\square$

In the classical topology, $\mathbb{C}P^n$ is a compact complex manifold, containing $\mathbb{C}^n$ as an open subset; note that $\mathbb{C}^n$ is not compact for $n \geq 1$.

Example 4.16. The 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is compact in the classical topology in $\mathbb{R}^3$. However, $S^2_{\mathbb{C}} = \{(x, y, z) \in \mathbb{A}^3_{\mathbb{C}} : x^2 + y^2 + z^2 = 1\}$ is not compact in the classical topology in $\mathbb{C}^3$.

Indeed, consider the function z desecending in $\mathbb{C}$. So we have an unbounded compact function on $S^2_{\mathbb{C}}$ with values decreasing in $\mathbb{C}$, so $S^2_{\mathbb{C}}$ is not compact.

Definition 4.17 (Projective Space)**.** For $n \geq 0$ and k algebraically closed, the *projective n-space over k P_k^n* is the set of one-dimensional k-linear subspaces of the k-vector space k^{n+1}.

Example 4.18. P_k^0 is just a point.

Definition 4.19 (Homogeneous Coordinates)**.** For $a_0, \ldots, a_n \in k$, not all zeros, we write $[a_0, \ldots, a_n] \in P_k^n$ to mean the line $k(a_0, \ldots, a_n) \subseteq k^{n+1}$.

Remark 4.20. Note that $[0, \ldots, 0]$ is not defined in P_k^n.

Clearly, $[a_0, \ldots, a_n] = [b_0, \ldots, b_n]$ if and only if there exists $c \in k^*$ such that $b_i = ca_i$ for all $0 \leq i \leq n$.

Example 4.21. We can define a bijection $P_k^1 \cong \mathbb{A}_k^1 \cup \{\infty\}$ by the following correspondence: every point in P_k^1, $[a_0, a_1]$ with coordinates not both 0, is either equal to $[0, 1]$ or to $[1, b]$ for some $b \in k$, and that is a unique way of writing the point.

Remark 4.22. By adding a point of infinity, we make sure parallel lines intersect at infinity.

5 Lecture 5

Remark 5.1. In fact, we can make a generalization: $P_k^1 := \mathbb{A}_k^1 \cup \{\infty\}$. Let k be an algebraically closed field and let $n \geq 0$, let $0 \leq i \leq n$, then $[x_0, \ldots, x_n] \in P^n(k)$. Note that there exists a bijective correspondence between $\{x_i \neq 0\}$ ($\subseteq P_k^n$) and $\mathbb{A}_k^n$, via $[x_0, \ldots, x_i, \ldots, x_n] \mapsto (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})$. Clearly P_k^n is covered by these $n+1$ "coordinate charts", as in $P_k^1 \cong (P_k^1 \backslash \{\infty\}) \cup (P_k^1 \backslash \{0\}) \cong \mathbb{A}_k^1 \cup \mathbb{A}_k^1$.

We can also see that $P_k^2 = \{x_0 \neq 0\} \cup P_k^1 \cong \mathbb{A}_k^2 \cup P_k^1 = \mathbb{A}_k^2 \cup \mathbb{A}_k^1 \cup \{*\}$, where $* = [0, x_1, x_2] \in P_k^2$.

Definition 5.2 (Homogeneous Polynomial)**.** A polynomial $f \in k[x_0, \ldots, x_n]$ is *homogeneous* of degree $d \geq 0$ if $f = \sum_{\text{finite sum}} a_{i_0,\ldots,i_n} x_0^{i_0} \ldots x_i^{i_n}$ with $a_I \in k$ and $i_0 + \ldots + i_n = d$.

Remark 5.3. Note that a polynomial f (homogeneous or not) does not give a well-defined function $f : P^n(k) \to k$: for a point $[b_0, \ldots, b_n] \in P^n(k)$, if there is another point in the same class (off by a scaling), the polynomial then produces a different value.

But, if f is homogeneous of degree d, then $f(ca_0, \ldots, ca_n) = c^d f(a_0, \ldots, a_n)$ for any $c \in k$.

Therefore, the zero set of a homogeneous polynomial f is a well-defined subset of P_k^n, $Z(f) = \{f = 0\} \subseteq P_k^n$, called a *hypersurface* in P_k^n.

Definition 5.4 (Projective Algebraic Set)**.** A *projective algebraic set* over k is a subset $X \subseteq P_k^n$ (for some $n \geq 0$) that equal to $Z(T) := \bigcap_{f \in T} Z(f)$ for some set T of homogeneous polynomials in $k[x_0, \ldots, x_n]$.

Remark 5.5. We will see later that this set T is defined as $T = Z(I)$ for a homogeneous ideal in $k[x_0, \ldots, x_n]$.

Definition 5.6 (Zariski Topology)**.** The *Zariski topology on* P_k^n (for $n \geq 0$) is the topology whose closed subsets are the projective algebraic sets in P_k^n.

Remark 5.7. This is a topology.

There is a correspondence $\mathbb{A}_k^{n+1} \backslash \{0\} \to P^n$ given by sending $(x_0, \ldots, x_n)$ to $[x_0, \ldots, x_n]$.

Definition 5.8 (Cone)**.** A *cone* in $\mathbb{A}_k^{n+1}$ is a closed subset that is a union of lines through 0.

Remark 5.9. The zero set of a homogeneous polynomial in $\mathbb{A}_k^{n+1}$ is a cone.

Definition 5.10 (Graded Ring)**.** A *graded ring* is a (commutative ring) $R = \bigoplus_{i \geq 0} R_i$ such that $R_i R_j \subseteq R_{i+j}$ for all i, j.

Example 5.11. $k[x_0, \ldots, x_n]$ is graded with $|x_i| = 1$ for each i.

Definition 5.12 (Homogeneous Ideal)**.** An ideal I in a graded ring R is *homogeneous* if

$$I = \sum_{d \geq 0} (I \cap R_d).$$

In particular, this implies that

$$I = \bigoplus_{d \geq 0} (I \cap R_d).$$

Definition 5.13 (Zero Set)**.** For a homogeneous ideal $I \subseteq k[x_0, \ldots, x_n]$, its zero set in P_k^n is $Z(I) = \bigcap_{f \in I \text{ homogeneous}} Z(f)$.

Remark 5.14. If $I = (g_1, \ldots, g_r)$ with $g_1, \ldots, g_r$ homogeneous, then $Z(I) = Z(g_1) \cap \cdots \cap Z(g_r)$.

Definition 5.15 (Projective Algebraic Variety)**.** A *projective algebraic variety* is an irreducible projective algebraic set $X \subseteq P_k^n$ for some $n \geq 0$.

Remark 5.16. A projective algebraic set over k is a Noetherian topological space. So it is a finite union of its irreducible components.

Remark 5.17. Given an affine algebraic set $X \subseteq \mathbb{A}_k^n$, we can think of $\mathbb{A}_k^n$ as an open subset of P_k^n, and therefore produces a bijective correspondence between $\{x_0 \neq 0\} (\subseteq CP^n) \Leftrightarrow \mathbb{A}_k^n$.

Note that

1. The bijection above is a homeomorphis.
2. $\{x_0 \neq 0\} \subseteq P_k^n$ is open.

We can then consider its *projective closure*, i.e., its closure in P_k^n.

Remark 5.18. How would we usually calculate that closure?

Given as set of polynomials with $X = \{f(x_1, \dots, x_n) = 0, \dots\} \subseteq \mathbb{A}_k^n$, then say that f_i has degree at most d, then we can write down an "associated" homogeneous polynomial $g_i(x_1, \dots, x_n)$ with degree d by $x_1^{i_1} \dots x_n^{i_n} \mapsto x_0^{d-i_1-\dots-i_n} x_1^{i_1} \dots x_n^{i_n}$.

The correspondence is now given by

$$[1, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n] \in P_k^n \Longleftrightarrow (x_1, \dots, x_n) \in \mathbb{A}_k^n.$$

Therefore,

$$\{g_1 = 0, \dots, g_r = 0\}(\subseteq P_k^n) \cap \{x_0 \neq 0\}(\cong \mathbb{A}_k^n) = \{f_1 = 0, \dots, f_r = 0\} \subseteq \mathbb{A}_k^n.$$

The subtlety is that the set on the left might be bigger than the precise closure in P_k^n of the set in the right. (That is, the calculation from right to left may not be well-defined.)

Definition 5.19 (Regular Function)**.** Let X be an affine algebraic set over algebraically closed field k. (That is, $X \subseteq \mathbb{A}_k^n$ is closed.) Let $U \subseteq X$ be an open subset, then a function $f : U \to K$ is called *regular* if for every $x \in U$ there exists an open neighborhood $x \in V \subseteq U$ on which we can write $f = \frac{g}{h}$ where g and h are polynomials in $k[x_1, \dots, x_n]$ such that $h \neq 0$ at all points of V.

Remark 5.20. This is a locally defined class of functions. That is, the expression may not be the same in different neighborhoods.

Example 5.21. $\frac{1}{x}$ is a regular function on $\mathbb{A}_k^1 \backslash \{0\}$. In fact, as we will see, the ring of all regular functions $\mathcal{O}(\mathbb{A}_k^1 \backslash \{0\}) \cong k[x][\frac{1}{x}]$, i.e., the ring of Laurent polynomials.

Remark 5.22. Note that for a function to be regular on the entire affine variety, this is equivalent to the following: a function is *regular* on the entire affine variety if it can globally be written as a polynomial.

Therefore, it is not so interesting to define regularity on an affine algebraic set with the same definition: one can just take the definition on the entire affine variety and restrict its domain. Our alternative definition essentially looks for the localization on open subsets.

6 Lecture 6

Definition 6.1 (Quasi-affine Algebraic Set)**.** A *quasi-affine algebraic set* over k an algebraically closed field is an open subset of an affine algebraic (closed) set $X \subseteq \mathbb{A}_k^n$. That is, $X \cap U$ where U is open in $\mathbb{A}_k^n$, i.e., $X - Y$ where Y is closed in $\mathbb{A}_k^n$, i.e., $X - Y$ where Y is a closed in X. This describes the idea of "a solution set minus another solution set".

Lemma 6.2. A regular function $f : U \to k$ on a quasi-affine algebraic set U is continuous as a mapping $f : U \to \mathbb{A}^1_k$ (with the Zariski topology).

Proof. We have to show that for every closed $S \subseteq \mathbb{A}^k_1$, $f^{-1}(U)$ is closed in U. By our knowledge of the closed subset of $\mathbb{A}^1_k$, it suffices to prove this for $S = \{a\}$ for some $a \in k$. By assumption, U is covered by open set $V \subseteq U$, on which $f = \frac{g}{h}$ with $g, h \in x[k_1 \ldots, x_n]$ with $h\mid_V \neq 0$ everywhere on V.

Lemma 6.3. For a topological space X with an open covering by open V_α, a subset S is closed in X if and only if $S \cap V_\alpha$ is closed in V_α for all α, and likewise for open subsets.

Subproof. Left as an exercise. ■

So it suffices to show that $f^{-1}(a) \cap V$, for each open $V \subseteq U$ as above. Now $f^{-1}(a) \cap V = \{x \in V : f(x) = a\} = \{x \in V : \frac{g(x)}{h(x)} = a\} = \{x \in V : g(x) - ah(x) = 0\}$, but this is a polynomial function on $\mathbb{A}^n_k$, restricted to V, and therefore this is a closed subset of V. □

Definition 6.4 (Quasi-projective Algebraic Set)**.** A *quasi-projective algebraic set* V over k is an open subset V of some projective algebraic set $X \subseteq P^n_k$ for some $n \geq 0$.

Remark 6.5. A quasi-affine algebraic set can be viewed as a quasi-projective algebraic set in P^n_k by the inclusion $\mathbb{A}^n_k \subseteq P^n_k$ as $\mathbb{A}^n_k = \{x_i \neq 0\} \subseteq P^n_k$ for any $0 \leq i \leq n$.

Definition 6.6 (Morphism of Quasi-projective Algebraic Set)**.** Let X and Y be quasi-projective algebraic sets over k. A *morphism* $f : X \to Y$ is a continuous function such that for every open $U \subseteq Y$ and every regular function g on U, the composition $g \circ f : f^{-1}(U) \to k$ is a regular function open in X.

Definition 6.7 (Regular functions on Quasi-projective Algebraic Set)**.** Let U be a quasi-projective algebraic set over k. A function $f : U \to k$ is *regular* if and only if for every point $x \in U$, there is an open $x \in V \subseteq U$ and $g, h \in k[x_0, \ldots, x_n]$ homogeneous of the same degree d such that

1. $h \neq 0$ at every point of V, and
2. $f = \frac{g}{h}$ on V.

Remark 6.8. Note that for homogeneous polynomial g, h of the same degree d,

$$\frac{g(ca_0, \ldots, ca_n)}{h(ca_0, \ldots, ca_n)} = \frac{c^d g(a_0, \ldots, a_n)}{c^d h(a_0, \ldots, a_n)} = \frac{g(a_0, \ldots, a_n)}{h(a_0, \ldots, a_n)}.$$

Remark 6.9. In defining a morphism, it is not enough to take $U = Y$ in the definition.

Example 6.10. The ring of regular functions on P_k^1 is just k, i.e., the constant functions.

Remark 6.11. Note that $P_k^1 \backslash \{\infty\} \cong P_k^1 \backslash \{0\} \cong \mathbb{A}_k^1$.

Proof Sketch. We will see that $\mathcal{O}(\mathbb{A}_k^1) = k[x]$, even by our new definition. So a regular function$f : P_k^1 \to k$ would restrict to regular functions on $V_0 = \{x_0 \neq 0\} \cong \mathbb{A}_k^1$ but also in $V_1 = \{x_1 \neq 0\} \cong \mathbb{A}_k^1$, and as $[x_0, x_1] \in P_k^1$, therefore f would be in $k[x]$ and also $k[y]$. But $[1, a] = [\frac{1}{a}, 1]$, so f is both a polynomial in x and in $\frac{1}{x}$, which forces f to be a constant. $\square$

Example 6.12. For a quasi-projective algebraic set X, a morphism $f : X \to \mathbb{A}_k^n$ is of the form $f(x) = (f_1(x), \ldots, f_n(x))$ where $f_1, \ldots, f_n$ are regular functions on X, and the converse is true.

Corollary 6.13. If X is a quasi-projective variety (meaning that it is irreducible), and f is a regular function on X that is not identically zero, then every irreducible component of the closed subset $\{f = 0\} \subseteq X$ has dimension $\dim(X) - 1$.

Proof. This is a corollary of Krull's Principal Ideal Theorem. $\square$

Theorem 6.14. Let $X \subseteq \mathbb{A}_k^n$ be a closed subset (i.e. an affine algebraic set), then the definition of the ring $\mathcal{O}(X)$ of regular functions agrees with our old definition $k[x_1, \ldots, x_n]/I(X)$.

Proof.

Definition 6.15. For an affine algebraic set $X \subseteq \mathbb{A}_k^n$, a *standard open subset* of X is a subset of the form $\{g \neq 0\} \subseteq X$, where $g \in k[x_1, \ldots, x_n]$.

Lemma 6.16. The standard open subsets of X form a basis for the topology of X, for X an affine algebraic set.

Subproof. We have to show that every open subset of X is a union of standard ones. By definition, an open set $U \subseteq X$ is $X - \{g_1 = 0, \ldots, g_r = 0\}$ for some $g_1, \ldots, g_r \in k[x_1, \ldots, x_n]$, and this is just the set $\bigcup_{1 \leq i \leq r} \{g_i \neq 0\}$. $\blacksquare$

Write $\mathcal{O}(X)$ for our new descriptions of regular functions. Clearly there is a homomorphism of k-algebras

$$\varphi : k[x_1, \ldots, x_n]/I(X) \to \mathcal{O}(X),$$

and clearly φ is injective. We now show that it is surjective. Let $f \in \mathcal{O}(X)$, we know we can cover X by open sets $U_\alpha \subseteq X$ on which $f = \frac{g_\alpha}{h_\alpha}$ with g_α, h_α as polynomials in $k[x_1, \ldots, x_n]$, and $h_\alpha \neq 0$ everywhere on U_α. By Lemma 6.16, we can assume that each U_α is a standard open subset in X, i.e., $U_\alpha = \{k_\alpha \neq 0\} \subseteq X$ for some $k_\alpha \in k[x_1, \ldots, x_n]$. Note that on U_α,

$$f = \frac{g_\alpha}{h_\alpha} = \frac{g_\alpha k_\alpha}{h_\alpha k_\alpha},$$

and this is still well-defined. Note that $\{k_\alpha \neq 0\} = \{h_\alpha k_\alpha \neq 0\} \subseteq X$. Therefore, we can replace h_α and k_α by $h_\alpha k_\alpha$ in our discussion. We now have polynomials g_α and h_α such that

$$X = \bigcup_\alpha \{h_\alpha \neq 0\}$$

and, on $\{h_\alpha \neq 0\}$, $f = \frac{g_\alpha}{h_\alpha}$. Note that $h_\alpha^2 \cdot f = g_\alpha h_\alpha$ on $\{h_\alpha \neq 0\} \subseteq X$, and also on $\{h_\alpha = 0\} \subseteq X$. Therefore, the equation is true on all of X.

Because $X = \bigcup_\alpha \{h_\alpha \neq 0\}$, we have $Z(h_\alpha^2 : \alpha \in \zeta) \subseteq X$ as the empty set $\varnothing$. By the Nullstellensatz, let $I = (h_\alpha : \alpha \in \zeta) \subseteq k[x_1, \ldots, x_n]/I(X) = R/I(X)$, then it has $\operatorname{rad}(I) = R$. In particular, $I = R$. Therefore, 1 can be expressed as some finite sum of the forms $r_\alpha h_\alpha^2$ for some $r_\alpha \in R$. Hence, on all of X, $1 \cdot f = (\sum r_\alpha h_\alpha^2) \cdot f = \sum r_\alpha h_\alpha^2 f = \sum r_\alpha g_\alpha h_\alpha \in R = k[x_1, \ldots, x_n]/I(X)$. □

7 Lecture 7

Lemma 7.1. Let X be a quasi-projective algebraic set over k algebraically closed. $\mathcal{O}(X)$ is a ring, in fact a commutative reduced k-algebra.

Proof. The main point is to show that the sum and product of regular functions are still regular. Call our set U, then given functions $f_1, f_2 : U \to k$ that locally are of the form $\frac{g}{h}$ with $g, h \in k[x_1, \ldots, x_n]$, both homogeneous of same degree d, with $h \neq 0$ of the given point p. Then say $f_1 = \frac{g_1}{h_1}$ near p and $f_2 = \frac{g_2}{h_2}$ near p. Obviously, $f_1 f_2 = \frac{g_1 g_2}{h_1 h_2}$ where the numerator and the denominator are homogeneous of the same degree, and the denominator is still non-zero at this point. The sum is similar: $\frac{g_1}{h_1} + \frac{g_2}{h_2} = \frac{g_1 h_2 + h_1 g_2}{h_1 h_2}$, and therefore we have the same argument. □

Lemma 7.2. For a quasi-projective algebraic set X over k, a morphism $f : X \to \mathbb{A}_k^n$ is equivalent to a list of n regular functions $f_1, \ldots, f_n$ on X.

Proof. Clearly, a function $U \to \mathbb{A}_k^n = k^n$ is equivalent to a list of n functions $U \to k$, i.e., $f(x) = (f_1(x), \ldots, f_n(x))$. If f is a morphism, then the pullbacks of the n regular functions, $x_1, \ldots, x_n \in \mathcal{O}(\mathbb{A}_k^n) = k[x_1, \ldots, x_n]$, so $f_1, \ldots, f_n$ are regular functions on X.

Conversely, suppose $f_1, \ldots, f_n$ are regular functions on $X = U$. To show that $f(x) = (f(x_1), \ldots, f(x_n))$ is a morphism $U \to \mathbb{A}_k^n$ over k, let $V \subseteq \mathbb{A}_k^n$ be open and let $g \in \mathcal{O}(U)$.

(One can check that f is indeed continuous.) To show that $i(g) = g \circ f$ is regular on $f^{-1}(V)$, here g can be written locally as $\frac{h}{k}$, with h, k polynomials near each point $p \in U$ with $k(p) \neq 0$. We want to show that $\frac{h(f_1,\dots,f_n)}{k(f_1,\dots,f_n)}$ is regular on $f^{-1}(V)$, so one has to write this as a ratio of homogeneous polynomials of the same degree, using that each function is of that form (near p). □

Remark 7.3. For a quasi-affine algebraic set $Y \subseteq \mathbb{A}^n_k$ and X a quasi-projective algebraic set X over k, a morphism $f : X \to Y$ is equal to n regular functions $f_1, \dots, f_n \in \mathcal{O}(X)$ such that $(f_1(x), \dots, f_n(x)) \in Y$ for every $x \in X$.

Remark 7.4. The morphisms of quasi-projective algebraic sets over k form a category.

Definition 7.5 (Isomorphism)**.** An *isomorphism* $f : X \to Y$ of quasi-projective algebraic set over k is a morphism $f : X \to Y$ that has a two-sided inverse.

Example 7.6. $X = \mathbb{A}^1_k \backslash \{0\} \cong \{xy = 1\} \subseteq \mathbb{A}^2_k = Y$. Note that X is quasi-affine and Y is affine.

Proof. Use the morphism $Y \to X$ by $(x, y) \mapsto x$ and $X \to Y$ by $x \mapsto (x, x^{-1})$, and this is well-defined since $x^{-1} \in \mathcal{O}(\mathbb{A}^1_k \backslash \{0\})$. □

Remark 7.7. Sometimes we say that a quasi-projective algebraic set is affine if it is isomorphic to an affine algebraic set, i.e., a closed subset of some $\mathbb{A}^n_k$.

Example 7.8. The hypersurface $\{x_n = f(x_1, \dots, x_{n-1})\} \subseteq \mathbb{A}^n_k$ is isomorphic to $\mathbb{A}^{n-1}_k$, where f is any polynomial in $k[x_1, \dots, x_{n-1}]$.

Example 7.9. Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic set over k (i.e., a closed subset of $\mathbb{A}^n_k$). Let $g \in \mathcal{O}(X)$, then the standard open subset $\{g \neq 0\}$ is affine, in fact it is isomorphic to $\{(x_1, \dots, x_n, x_{n+1}) : x_{n+1} g(x_1, \dots, x_n) = 1\} \subseteq \mathbb{A}^{n+1}_k$.

Proof. Map $U = \{g \neq 0\} \subseteq X$ by $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, g(a_1, \dots, a_n)^{-1}) \in Y$, then this is a morphism. The inverse morphism is given by $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n) \in U = \{g \neq 0\}$. □

Example 7.10. $\mathbb{A}^2_k \backslash \{0\} = \{x_1 = 0\} \cup \{x_2 = 0\}$ is a quasi-affine algebraic set which is not affine.

Corollary 7.11. Let $X \subseteq \mathbb{A}^n_k$ be an affine algebraic set (i.e., closed in $\mathbb{A}^n_k$), and let $g \in \mathcal{O}(X)$, then $\mathcal{O}(\{g \neq 0\}) = \mathcal{O}(X)[\frac{1}{g}]$.

Proof. A morphism $f : X \to Y$ of quasi-projective algebraic sets induces a k-algebraic homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Therefore, an isomorphism $f : X \to Y$ induces an isomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ of k-algebras. Therefore,

$$\begin{aligned}\mathcal{O}(\{g \neq 0\}) &= \mathcal{O}(\{x_{n+1}g(x_1, \ldots, x_n) = 1\} \subseteq \mathbb{A}_k^{n+1}) \\ &= k[x_1, \ldots, x_{n+1}]/(f_1(x_1, \ldots, x_n), \ldots, f_r(x_1, \ldots, x_n), x_{n+1}g(x_1, \ldots, x_n) - 1) \\ &= \mathcal{O}(X)[x_{n+1}]/(x_{n+1}g(x_1, \ldots, x_n) - 1) \\ &\cong \mathcal{O}(X)[\frac{1}{g}].\end{aligned}$$

□

Theorem 7.12. The correspondence mentioned in the proof above can be formalized. Let $f : X \to Y$ be a morphism of quasi-projective algebraic sets over an algebraically closed field k. f determines a pullback homomorphism of k-algebras $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Moreover, if Y is affine (i.e., isomorphic to a closed subset of some $\mathbb{A}_k^n$), then this construction gives a one-to-one correspondence between morphisms $X \to Y$ and k-algebra homomorphisms $\mathcal{O}(Y) \to \mathcal{O}(X)$. It follows that if both X and Y are affine, then X and Y are isomorphic if and only if the k-algebras $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are isomorphic.

8 Lecture 8

Lemma 8.1. Let $X \subseteq \mathbb{A}_k^{n+1}$ be a cone (that is, X is closed and is a union of lines through 0), then the ideal $I(X) \subseteq k[x_0, \ldots, x_n]$ is homogeneous.

Proof. We have to say: for any $f \in I(X)$, if we write $f = f_0 + \ldots + f_d$ with f_i homogeneous of degree i, then f_i should be in $I(X)$.

Let $(a_0, \ldots, a_n)$ be a point in X, then we know that (because X is a cone and $f \in I(X)$) $f(ca_0, \ldots, ca_n) = 0$ for all $c \in k$. In particular, $f_0(a_0, \ldots, a_n) + cf_1(a_0, \ldots, a_n) + \cdots + c^d f_d(a_0, \ldots, a_n)$. Note that every term is in k, but as polynomial in c, this polynomial $g(c) \in k[c]$ such that $g(c) = 0$ for all $c \in k$. Hence, all its coefficients are 0.

Since k is algebraically closed, it is infinite. So $g = 0 \in k[c]$, that is, $f_i(a_0, \ldots, a_n) = 0$ for each $0 \leq i \leq d$. Since $(a_0, \ldots, a_n) \in X$ are arbitrary, $f_i \in I(X)$, so the ideal $I(X)$ is homogeneous. □

Remark 8.2. Note that the zero set in P^n of the ideal $(x_0, \ldots, x_n)$ in $k[x_0, \ldots, x_n]$ since $[0, \ldots, 0]$ is not a point in P^n. We get a one-to-one correspondence between homogenous prime ideals that are not $(x_0, \ldots, x_n)$ (called the *irrelevant ideal*), and irreducible closed subsets of P_k^n.

Definition 8.3 (Local Ring)**.** Let X be a quasi-projective algebraic set over k algebraically closed. Then for a point $p \in X$, the *local ring* of X at p is

1. an equivalence class of pairs (U, f) with open $p \in U \subseteq X$ and $f \in \mathcal{O}(U)$, with $(U, f) \sim (V, g)$ if there is an open neighborhood $p \in W \subseteq U \cap V$ such that $f|_W = g|_W$. (That is, an element of $\mathcal{O}_{X,p}$ is a germ of regular functions at p.)

2. The direct limit $\varinjlim_{p \in U \subseteq X} \mathcal{O}(U)$, i.e., with $p \in U \subseteq V \subseteq X$, there is a restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$.

Lemma 8.4. $\mathcal{O}_{X,p}$ is a local ring.

Proof. That is, we want to show that $\mathcal{O}_{X,p}$ has exactly one maximal ideal. Equivalently, $\mathcal{O}_{X,p}$ has a maximal ideal $\mathfrak{m}$ such that for all $f \in \mathcal{O}_{X,p} \backslash \mathfrak{m}_{X,p}$, then $f \in \mathcal{O}^*_{X,p}$. Let $\mathfrak{m} = \ker(\mathcal{O}_{X,p} \to k)$, i.e., the kernel of the evaluation at p. One can see this is surjective (using constant functions), then let $f \in \mathcal{O}_{X,p} \backslash \mathfrak{m}$, then we can view $f \in \mathcal{O}(U)$ for some open set $p \in U \subseteq X$. Then $\{f \neq 0\} \subseteq U$ is an open subset of X containing p, so $\frac{1}{f} \in \mathcal{O}(V)$, hence $\frac{1}{f} \in \mathcal{O}_{X,p}$. □

Lemma 8.5. Let X be an affine algebraic set over k, then for a point $p \in X$ with $\mathfrak{m} = \ker(\mathcal{O}_{X,p} \to k)$ as the evaluation map at p, then $\mathcal{O}_{X,p} = \mathcal{O}(X)_{\mathfrak{m}}$ as the localization.

Proof. For a commutative ring R and prime ideal $\mathfrak{p} \subseteq R$, an element of the localization $R_{\mathfrak{p}}$ can be written as $\frac{a}{b}$ with $a \in R$ and $b \in R \backslash \mathfrak{p}$. So an element of $\mathcal{O}(X)_{\mathfrak{m}}$ is a fraction $\frac{a}{b}$ with $a \in \mathcal{O}(X)$ and $b \in \mathcal{O}(X)$ with $b(p) \neq 0$. Therefore $\frac{a}{b} \in \mathcal{O}(\{b \neq 0\})$ hence is contained in $\mathcal{O}_{X,p}$. □

Remark 8.6. Recall that $\mathcal{O}(\{g \neq 0\}) = \mathcal{O}[x][\frac{1}{g}]$.

Remark 8.7. An isomorphism $f : X \to Y$ of quasi-projective algebraic sets over k induces an isomorphism of local rings $\mathcal{O}_{Y,f(p)} \cong \mathcal{O}_{X,p}$.

Definition 8.8 (Dimension Near a Point)**.** Let $X \subseteq \mathbb{A}^n_k$ be a closed subset, write $I(X) = (f_1, \ldots, f_r) \in k[x_1, \ldots, x_n]$, and let $p \in X$. Let m be the dimension of X near p, i.e., the dimension of U for all small enough open neighborhoods of p.

Remark 8.9. If X is irreducible, then it has the same dimension near every point. Note that we can define derivatives of polynomials manually:

$$\frac{\partial}{\partial x_j}(x_1^{i_1}, \ldots, x_n^{i_n}) = i_j x_1^{i_1} \ldots x_j^{i_j - 1} \ldots x_n^{i_n}$$

Note that we have a unique ring homomorphism $\mathbb{Z} \to k$, and can be viewed as a polynomial in $k[x_1, \ldots, x_n]$.

We have

$$\frac{\partial}{\partial x}(fg) = f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g$$

and etc.

Remark 8.10. If k has characteristic $p > 0$, then $p = 0 \in k$, so $\frac{\partial}{\partial x}(x^p) = px^{p-1} = 0 \in k[x]$. We now get a $n \times r$ matrix in k, of the form $\left(\frac{\partial f_i}{\partial x_j} \mid_p\right)$, and therefore a map $A^n \to A^r$.

Definition 8.11 (Smooth)**.** $X \subseteq \mathbb{A}_k^n$ is *smooth* over k at $p \in X(k)$ if the matrix $D_p = \left(\frac{\partial f_i}{\partial x_j} \mid_p\right)$ has rank $n - m$ where m is the dimension of X near p.

Definition 8.12 (Zariski Tangent Space)**.** The *Zariski tangent space* is defined to be $T_{X,p} = \ker(D_p : k^n \to k^r)$. The smoothness of X at p means that (X, p) has dimension $\dim(X)$ near p. Note that we always have a $\geq$ relation.

Example 8.13. Let $X = \{xy = 0\} \subseteq \mathbb{A}_k^2$. Where is X smooth? Let $(a, b) \in X(k)$, then the matrix $D_p = \left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right) \mid_{(a,b)}= (y\ x) \mid_{a,b}= (b\ a) \in M_{1\times 2}(k)$. Therefore, X is smooth if and only if this matrix has rank 1 (note that it always has rank at most 1), if and only if $a \neq 0$ or $b \neq 0$.

Thus, X is smooth (of dimension 1) everywhere except $(0, 0)$.

Example 8.14. Where is the curve $X = \{xy = 1\} \subseteq \mathbb{A}_K^2$ smooth?

The matrix of derivatives is (write $f = xy - 1$) $(y\ x)$, and so X is smooth at (x, y) if and only if $(x, y) \neq (0, 0)$. But $(0, 0)$ is not on the curve, so X is smooth everywhere.

9 Lecture 9

Remark 9.1.

1. Smoothness does not depend on the choice of generators $g_1, \ldots, g_r$.
2. This "commutes with localization".
3. Smoothness is preserved by isomorphisms.

Example 9.2 (Zariski Tangent Space)**.** Consider $X = \{xy = 0\} \subseteq \mathbb{A}_k^2$, then at every point $x \in X$, we define a vector space $T_pX \subseteq k^n$ for $X \subseteq \mathbb{A}_k^n$. The tangent space is two-dimensional at the origin, and is one-dimensional everywhere else.

Definition 9.3 (Presheaf)**.** Let X be a topological space. A *presheaf* of Abelian groups on X is an Abelian group $A(U)$ for every open set $U \subseteq X$, together with restriction homomorphisms $r_U^V : A(V) \to A(U)$ for every open $U \subseteq V \subseteq X$, such that

- $r_U^U = 1_{A(U)}$ for every $U \subseteq X$,
- $r_U^W = r_U^V r_V^W$ for open $U \subseteq V \subseteq W \subseteq X$ as homomorphism $A(W) \to A(U)$.

Example 9.4. Let X be a topological space. Let $C(U)$ be the presheaf of continuous $\mathbb{R}$-valued functions.

Example 9.5. Let X be C^∞-manifold, then we have the presheaf of C^∞ (smooth) $\mathbb{R}$-valued functions.

Example 9.6. Let X be a complex manifold. We have the presheaf $\mathcal{O}_{an}$ of $\mathbb{C}$-analytic functions (on open subsets of X). For instance, if $X = \mathbb{C}P^1$, then $\mathcal{O}_{an}(X) = \mathbb{C}$.

Example 9.7. Let X be a quasi-projective algebraic set over k algebraically closed, then we have the presheaf $\mathcal{O}_X$ of regular functions.

Remark 9.8. We may call $A(U)$ the Abelian group of section of A on U.

Remark 9.9. Let X be a topological space. Define a category $\mathbf{Top}(X)$ with objects the open subsets of X, and $\mathbf{Hom}_{\mathbf{Top}(X)}(U, V) = \begin{cases} *, & \text{if } U \subseteq V \\ \varnothing, & \text{if } U \not\subseteq V \end{cases}$. A presheaf of Abelian groups on X is exactly a contravariant functor $\mathbf{Top}(X) \to \mathbf{Ab}$.

Definition 9.10 (Sheaf)**.** A *sheaf* of Abelian groups on a topological space X is a presheaf A of Abelian groups such that

- for every open set $U \subseteq X$ and every open cover $\{U_\alpha\}_{\alpha \in I}$ of U if $a, b \in A(U)$ such that $a \mid_{U_\alpha} = b \mid_{U_\alpha}$ for every $\alpha \in I$, then $a = b \in A(U)$,
- for every open set $U \subseteq X$ and every open cover $\{U_\alpha\}_{\alpha \in I}$ of U, for any collection of $a_\alpha \in A(U_\alpha)$ for all $\alpha \in I$, if $a_\alpha \mid_{U_\alpha \cap U_\beta} = a_\beta \mid_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in I$, then there is an $a \in A(U)$ such that $a \mid_{U_\alpha} = a_\alpha$ for all $\alpha \in I$.

Remark 9.11. If A is a sheaf, then the $a \in A(U)$ described in the second property is unique, given by the first property.

Example 9.12. The presheaves described above are sheaves.

Remark 9.13. If A is a sheaf, then $A(\varnothing) = 0$ is the trivial Abelian group.

Proof. Take $U = \varnothing$, notice that U is covered by no open subsets. □

Example 9.14. Let A be an Abelian group and X be a topological space. The constant presheaf T_A on X is defined by $T_A(U) = A$ for every open $U \subseteq X$. This is not a sheaf if $A \neq 0$, since $T_A(\varnothing) = A$, not 0.

Example 9.15. Let A be an Abelian group on a space X. Define a presheaf S_A on X by $S_A(U) = \begin{cases} 0, & \text{if } V = \varnothing \\ A, & \text{otherwise} \end{cases}$. This is not a sheaf, for many spaces X, e.g., $X = \mathbb{R}$ with classical topology. Take the real line $\mathbb{R}$, and two disjoint open subsets U_1 and U_2, then let $U = U_1 \cup U_2 \subseteq \mathbb{R}$. Now $7 \in S_{\mathbb{Z}}(U_1)$ and $8 \in S_{\mathbb{Z}}(U_2)$, then the sections agree on the intersection, but there is not $a \in S_{\mathbb{Z}}(U_1 \cup U_2) = \mathbb{Z}$ that restricts to both 7 and 8.

Example 9.16. For a topological space X and Abelian group A, the sheaf A_X of locally constant A-valued functions on X is $A_X(U)$, the set of functions $f : U \to A$ for $U \subseteq X$ open that are locally constant, i.e., for every $p \in U$, there exists $p \in V \subseteq U$ such that $f\ |_V$ is constant.

Definition 9.17 (Stalk)**.** Let A be a presheaf on a space X. The *stalk* of A at a point $p \in X$ is $A_p = \varinjlim_{p \in U \subseteq X} A(U)$ for any open U of X containing p. That is, an element A_p is a germ of section of A at p.

Example 9.18. For a quasi-projective algebraic set X over k, the stalk $\mathcal{O}_{X,p}$ is exactly the local ring of X at p.

Definition 9.19 (Homomorphism of Presheaves)**.** A *homomorphism of presheaves* of Abelian groups A and B on a space X is a natural transformation $A \to B$ (as contravariant functors on $\mathbf{Top}(X)$): for every open $U \subseteq X$ we are given a homomorphism $f_U : A(U) \to B(U)$ of Abelian groups such that for every open inclusion $U \subseteq V$, the diagram

$$\begin{array}{ccc} A(V) & \longrightarrow & B(V) \\ \downarrow & & \downarrow \\ A(U) & \longrightarrow & B(U) \end{array}$$

commutes.

10 Lecture 10

Algebraic geometry classifies closed subsets of $\mathbb{A}_k^n$, so it is the study of affine algebraic sets up to isomorphisms. As we mentioned, we have a correspondence between coordinate rings and the commutative k algebras (finitely-generated over k and reduced), but the latter is hard to classify. The main technique we use is to switch from affine algebraic geometry to projective algebraic geometry. In projective algebraic geometry, we get invariants as cohomology groups of sheaves, which is a measurement of difference between local and global behaviors.

Definition 10.1 (Homomorphism of Sheaves)**.** A *homomorphism* $f : A \to B$ *of sheaves of Abelian groups* over X is the same thing. That is, $\mathbf{Sh}(X)$ is a full subcategory of $\mathbf{PreSh}(X)$.

Remark 10.2. A map $f : A \to B$ of presheaves on a space X determines a homomorphism of Abelian groups $f_p : A_p \to B_p$ for every point $p \in X$. This well-defined mapping is given by $s \in A(U) \mapsto f(s) \in B(U)$, and thus is mapped to a germ $f(s)_p \in B_p$.

Proposition 10.3. Let $f : A \to B$ be a homomorphism of sheaves on X. Then $f_p : A_p \to B_p$ is an isomorphism for every $p \in X$ if and only if $f : A \to B$ is an isomorphism.

Remark 10.4. This is not true for presheaves.

Example 10.5. Let T be the constant presheaf on a space X associated to $\mathbb{Z}$. That is, $T(U) = \mathbb{Z}$ for every open $U \subseteq X$. Then there is a natural map $T \to \mathbb{Z}_X$ of presheaves where $\mathbb{Z}_X$ is the sheaf of local of locally constant $\mathbb{Z}$-valued functions.

For instance, if $X = \mathbb{R}$, f is not an isomorphism, but f induces an isomorphism on stalks. Both presheaves have stalk at every point as $\mathbb{Z}$.

Proof. It is clear that if f is an isomorphism, then $f_p : A_p \to B_p$ is an isomorphism for every $p \in X$.

Conversely, let $f : A \to B$ be a homomorphism of sheaves on X, with isomorphism of Abelian groups $f_p : A_p \to B_p$ at every point $p \in X$. We have to show that for every $U \subseteq X$, the homomorphism of Abelian groups $f_U : A(U) \to B(U)$ must be an isomorphism. First, we show that f_U is injective. Let $s \in A(U)$ be such that $f_U(s) = 0 \in B(U)$. We have a commutative diagram

$$\begin{array}{ccc} A(U) & \xrightarrow{f_U} & B(U) \\ \downarrow & & \downarrow \\ A_p & \xrightarrow[f_p]{} & B_p \end{array}$$

so s mapping from both ways goes to 0 since $f_U(s) = 0$, but f_p is an isomorphism here, so the germ of s at every point $p \in U$ is 0. Therefore, for every point $p \in U$ we can choose an open set $p \in U_p \subseteq U$ such that $s\ |_{U_p} = 0 \in A(U_p)$. By the definition of sheaves, it follows that $s = 0 \in A(U)$, hence injective. To show that f_U is surjective, let U be an open subset of X and let $t \in B(U)$. We want to find $s \in A(U)$ with $f(s) = t$. For each point $p \in U$, the germ $t_p \in B_p$ is the image of a unique element $s_p \in A_p$. That is, for each $p \in U$, there is an open set $p \in U_p \subseteq U$ and a section $w_p \in A(U_p)$ such that the germ of w_p at p is $s_p \in A_p$. It is not necessarily true that $w_p \mid_{U_p \cap U_q} = w_q \mid_{U_p \cap U_q} \in A(U_p \cap U_q)$. However, we know that $f(w_p) \in B(U_p)$ has germ at p equal to t_p, the germ of $t \in B(U)$ at p, which is the same thing as the germ of $t \mid_{U_p} \in B(U_p)$. Thus, there is an open neighborhood $p \in V_p \subseteq U_p \subseteq U$ such that $f(w_p) \mid_{V_p} = t \mid_{V_p} \in B(V_p)$. Clearly V_p's form an open cover of U since $p \in V_p$.

Claim 10.6. For every $p, q \in U$, $w_p \mid_{V_p}$ agrees with $w_q \mid_{V_q}$ in $A(V_p \cap V_q)$.

Subproof. We know that both $w_p \mid_{V_p \cap V_q}$ and $w_q \mid_{V_p \cap V_q}$ map to $t \mid_{V_p \cap V_q}$. In particular, $f(w_p) \mid_{V_p \cap V_q}$ has the same germ at every point of $V_p \cap V_q$ as $f(w_q) \mid_{V_p \cap V_q}$. By our assumption (that $f_x : A_x \to B_x$ is isomorphic for all $x \in X$), it follows that $w_p \mid_{V_p \cap V_q}$ and $w_q \mid_{V_p \cap V_q}$ have the same germ at every point in A_x for $x \in V_p \cap V_q$. By the proof of injectivity, we know that for a sheaf A on a space X, $A(U) \to \prod_{p \in U} A_p$ is injective. Therefore, $w_p \mid_{V_p \cap V_q} = w_q \mid_{V_p \cap V_q}$. ■

Since A is a sheaf, it follows that there is a unique section $s \in A(U)$ such that $s \mid_{V_p} = w_p \mid_{V_p}$ for every $p \in U$. We want to show that $f(s) = t \in B(U)$. Indeed, we know by construction that the sections in $B(U)$ have the same germ at every point in U, so since $A(U) \to \prod_{p \in U} A_p$ is injective, $f(s) = t \in B(U)$ as desired. □

Definition 10.7 (Kernel of Sheaves)**.** Let $f : A \to B$ be a homomorphism of sheaves (of Abelian groups) on a topological space X. The *kernel* of f, denoted $\ker(f)$, is the sheaf $(\ker(f))(U) = \ker(f : A(U) \to B(U))$ for $U \subseteq X$ open.

Definition 10.8 (Image of Sheaves)**.** Let $f : A \to B$ be a homomorphism of sheaves (of Abelian groups) on a topological space X. The *image* of f, denoted $\mathrm{im}(f)$, is defined by $(\mathrm{im}(f))(U) = \mathrm{im}(f : A(U) \to B(U))$ for $U \subseteq X$ open. Note that this only a presheaf in general.

11 Lecture 11

Example 11.1. Let $X = S^1$ for $U \subseteq X$ open. Let A be the sheaf of continuous $\mathbb{C}$-valued functions on S^1. Let B be the sheaf of $\mathbb{C}*$-valued continuous functions on S^1. The structure on $B(U)$ is $(fg)(x) = f(x)g(x) \in \mathbb{C}^*$ for $x \in S^1$.

We have a homomorphism of sheaves $\exp : A \to B$ given by $\exp(f)(x) = e^{f(x)} \in B(U)$ for $f \in A(U)$ and $x \in U$ where $U \subseteq S^1$ open. This is a homomorphism since $e^{f+g} = e^f \cdot e^g$.

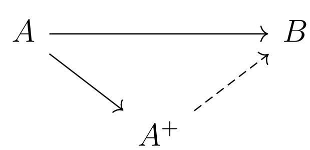
We claim that $\mathrm{im}(\exp)$ is a presheaf that is not a sheaf. Consider the section $z \in B(S^1)$, the obvious inclusion $S^1 \hookrightarrow \mathbb{C}^*$, then $z \notin (\mathrm{im}(\exp))(S^1)$: $z = e^{f(z)}$ for some continuous $\mathbb{C}$-valued function on S^1 corresponds to $f(z) = \log(z) + \mathbb{Z} \cdot 2\pi i$, which is not possible. But if

$U = S^1\backslash\{1\}$ and $V = S^1\backslash\{-1\}$, then $z\mid_U$ and $z\mid_V$ are in $(\mathrm{im}(f))(U)$ and $(\mathrm{im}(f))(V)$. These sections clearly agrees on $U \cap V$, but they cannot be glued to an element of $(\mathrm{im}(f))(S^1)$, so this is not a sheaf.

Definition 11.2 (Skyscraper Sheaf)**.** For a topological space X and a point $p \in X$ and an Abelian group A, the *skyscraper sheaf* A_p on X is defined by $A_p(U) = \begin{cases} A, & \text{if } p \in U \\ 0, & \text{if } p \notin U \end{cases}$ for $U \subseteq X$ open.

Example 11.3. Let k be an algebraically closed field $X = P_k^1$ (with Zariski topology), consider the sheaves $A = \mathcal{O}_X$ and $B = k_0 \oplus k_\infty$. The direct sum of two sheaves A and B is defined by $(A \oplus B)(U) = A(U) \oplus B(U)$. There is an inclusion homomorphism of sheaves $A \to B$ by evaluation at 0 and ∞, note that $0 = [1, 0] \in P_k^1$ and $\infty = [0, 1] \in P_k^1$. Define $f = \exp : A \to B$. We claim that $\mathrm{im}(f)$ is a presheaf but not a sheaf on $P_k^1 \cong \mathbb{A}_k^1$. Let $U = P_k^1\backslash\{0\}$ and $V = P_k^1\backslash\{\infty\}$. Consider the sections $0 \in (\mathrm{im}(f))(U) = k(U)$ and $1 \in (\mathrm{im}(f))(V) = k(V)$. Therefore, these sections agree on $(\mathrm{im})(U \cap V)$, but these sections cannot be glued to an element of $(\mathrm{im}(f))(P_k^1) = \mathrm{im}(A(P_k^1) \to B(P_k^1)) = \mathrm{im}(k \to k \oplus k) \cong k$.

Theorem 11.4. Let A be a presehaf of Abelian groups on a space. Then there is a sheaf A^+, the *sheafification* of A with a map $A \to A^+$ which is universal for maps of A to sheave. That is, for every sheaf B with a map $A \to B$, there is a unique map $A^+ \to B$ making the diagram commutes:



Proof Sketch. For an open $U \subseteq X$, define

$$A^+(U) = \left\{ \prod_{p \in U} S_p \text{ with } s_p \in A_p \right\} \subseteq \prod_{o \in U} A_p$$

for all $p \in U$, such that for all $p \in U$ there exists $p \in V_p \in U$ and $t \in A(V_p)$ such that s_p is the germ t_p for all $p \in V_p$. This is a sheaf and has the universal property. $\square$

Definition 11.5 (Image Sheaf)**.** For a map $f : A \to B$ of sheaves on X, let the *image sheaf* $\mathrm{im}(f)$ be the sheafification of the presheaf $U \to \mathrm{im}(A(U) \xrightarrow{f} B(U))$.

Definition 11.6 (Injective, Surjective)**.** A map $f : A \to B$ of sheaves is *injective* if $f : A(U) \to B(U)$ is injective for every open set $U \subseteq X$.

A map $f : A \to B$ of sheaves is *surjective* if the image sheaf $\mathrm{im}(f) \subseteq B$ is equal to B. Thus, we do not require that $f : A(U) \to B(U)$ to be surjective.

Equivalently, a map of sheaves $f : A \to B$ is surjective if and only if for every open $U \subseteq X$ and every $A \in B(U)$, there is a covering $\{U_\alpha\}_{\alpha \in I}$ of U such that $A \mid_{U_\alpha}$ is the image of f over some section in $A(U_\alpha)$.

Proposition 11.7. Let $f : A \to B$ be a map of sheaves on a space X, then f is injective (respectively, surjective, isomorphic) if and only if for every $p \in X$, $f_p : A_p \to B_p$ is injective (respectively, surjective, isomorphic).

Remark 11.8. The isomorphism of sheaves is just a bijective map on sheaves.

Definition 11.9 (Cokernel of Sheaves)**.** For a map of sheaves $f : A \to B$ of Abelian groups on a space X, the *cokernel sheaf* $\mathrm{coker}(f)$ is the sheafification of the presheaf $U \mapsto \mathrm{coker}(f : A(U) \to B(U))$, where the cokernel here is defined by $B(U)/f(A(U))$.

Remark 11.10. With these definitions, the category of sheaves of Abelian groups on a topological space X is an Abelian category.

Definition 11.11 (Direct Image)**.** Let $f : X \to Y$ be a continuous map of topological spaces. Let E be a sheaf on X. The *direct image sheaf* f_*E on Y is the sheaf $(f_*E)(U) = E(f^{-1}(U))$ for open $U \subseteq Y$.

Example 11.12. If $f : * \to X$ is a map, then an Abelian group A gives a sheaf on a point, and $(f_*)(A)$ is the skyscraper sheaf A_p.

Example 11.13. If Y is a closed subset of an algebraic set X over k, and $i : Y \to X$ is the inclusion, then $i_*(\mathcal{O}_Y)$ is a sheaf on X.

Definition 11.14 (Inverse Image)**.** For a continuous map $f : X \to Y$ of topological spaces, let E be a sheaf on Y. The *inverse image sheaf* $f^{-1}(E)$ on X is the sheafification of the presheaf $U \mapsto \varinjlim_{V} E(V)$ where V runs over all open subsets of Y that contains $f(U)$.

Example 11.15. Let $i : * \hookrightarrow X$ with image $p \in X$. For a sheaf E on X, the inverse image sheaf $f^{-1}(E)$ is the Abelian group E_p, the stalk at p.

12 Lecture 12

Definition 12.1 (Inverse Image)**.** Let $f : X \to Y$ be a continuous map and E a sheaf on Y. We define the inverse image sheaf $f^{-1}(E)$ on X as the sheafification of the presheaf $U \subseteq X \mapsto \varinjlim_{V \subseteq Y} E(V)$ for V open such that $f(U) \subseteq V$.

Remark 12.2. This is the left adjoint to f_*, that is,

$$\mathbf{Hom}_{\mathbf{Sh}(X)}(f^{-1}(E), F) \cong \mathbf{Hom}_{\mathbf{Sh}(Y)}(E, f_*F).$$

Remark 12.3. The sheafification in the definition of $f^{-1}(E)$ cannot be omitted. Take the map $f : X \to *$ for a topological space X. Take the sheaf $\mathbb{Z}_*$ on the point, the presheaf above is sent from $U \subseteq X$ open to $\mathbb{Z}$ if $U \neq \varnothing$, and to 0 if $U = \varnothing$.

As we have seen, this is not a sheaf. For instance,take $X = \mathbb{R}$ with classical topology. With sheafification, $f^{-1}(\mathbb{Z}_*) = \mathbb{Z}_X$, the sheaf of locally constant $\mathbb{Z}$-valued functions.

Remark 12.4 (Motivating Scheme)**.** For an algebraically closed field k, there is an equivalence of categories (with reversed orderings) between affine algebraic sets over k and reduced commutative finitely-generated k-algebras.

Given an affine algebraic set X, we send it to $\mathcal{O}(X) = k[x_1, \dots, x_n]/I(X)$. This is contravariant, as it sends a morphism $X \to Y$ to the k-algebra homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$.

Given a k-algebra A, choose a presentation of A as $A \cong k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ for some elements f_i's, then we send it to $\{f_1 = 0, \dots, f_r = 0\} \subseteq \mathbb{A}^n_k$.

We now want to find a similar correspondence for all commutative rings. For example, the local ring of an algebraic set at a point is usually not a finitely-generated algebra, e.g., $\mathcal{O}_{\mathbb{A}^1_k,0} \cong k[x]_{(x)} = k[x, \frac{1}{x-a}$ for all $a \in k^*]$.

By the Nullstellensatz, in the case where k is algebraically closed, let $X = \mathbf{Max}(\mathcal{O}(X))$, the set of maximal ideals in $\mathcal{O}(X)$. For instance, the maximal ideals in $k[x_1, \dots, x_n]$ are given by elements $(a_1, \dots, a_n) \in k^n$, $I = (x_1 - a_1, \dots, x_n - a_n) \subseteq k[x_1, \dots, x_n]$. However, the right choice would be to send $A \mapsto \operatorname{Spec}(A)$, the set of prime ideals in A.

Remark 12.5. For a homomorphism $f : A \to B$ of commutative rings, there is a natural map $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ by sending a given prime $\mathfrak{p} \subseteq B$ to $f^{-1}(\mathfrak{p}) \subseteq A$, which is prime in A. Note that $A/f^{-1}(\mathfrak{p}) \subseteq B/\mathfrak{p}$.

If $\mathfrak{p}$ is maximal, then $f^{-1}(\mathfrak{p})$ need not be maximal. For example, take the ring homomorphism $\mathbb{Z} \to \mathbb{Q}$, then $(0) \subseteq \mathbb{Q}$ is maximal, but $f^{-1}(0) = 0 \subseteq \mathbb{Z}$ is prime but not maximal.

Definition 12.6 (Spectrum)**.** For a commutative ring A, its *spectrum* $\operatorname{Spec}(A)$ is the set of prime ideals in A. For an ideal $I \subseteq A$, define its zero set $Z(I) = \{\mathfrak{p} \in \operatorname{Spec}(A) : I \subseteq \mathfrak{p}\}$. For a commutative ring A, a closed subset of $\operatorname{Spec}(A)$ is a subset of the form $Z(I)$ for some ideal $I \subseteq A$.

Remark 12.7 (Why is this the right construction?)**.** Given an element $f \in A$, we can think of f as a function whose values "near a point $\mathfrak{p}p \in \operatorname{Spec}(A)''$ is $f \in A_{\mathfrak{p}}$ (with a ring homomorphism $A \to A_{\mathfrak{p}}$) and values "at the point $\mathfrak{p}$ is $f \in A/\mathfrak{p}$, which is a domain, or we can think of it as $f \in \mathrm{Frac}(A/\mathfrak{p})$, a field.

Therefore, $Z(I) = \bigcap_{f \in I} Z(f)$, where $Z(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) : f = 0 \in A/\mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Spec}(A) : f \in \mathfrak{p}\}$.

Proposition 12.8. For every commutative ring A, $\mathrm{Spec}(A)$ is a topological space.

Proof. We have to show:

- $\varnothing$ and $\mathrm{Spec}(A)$ are closed in $\mathrm{Spec}(A)$,
- the union of two closed subsets in $\mathrm{Spec}(A)$ is closed,
- the intersection of any collection of closed subsets is closed.

1. $Z((1)) = Z(A) = \{\mathfrak{p} \in \mathrm{Spec}(A)(1) \subseteq \mathfrak{p}\} = \varnothing$ because a prime ideal does not contain 1; and $Z((0)) = \{\mathfrak{p} : 0 \in \mathfrak{p}\} = \mathrm{Spec}(A)$, so those are closed.

2. Given closed subsets $Z(I)$ and $Z(J)$ in $\mathrm{Spec}(A)$ for ideals I and J, we want to show that $Z(I) \cup Z(J) = Z(K)$ for some ideal $K \subseteq A$. We could either take $K = IJ$ or $K = I \cap J$. Let us use $K = IJ$. That is, we want to show a prime ideal $\mathfrak{p} \subseteq A$ satisfies $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$ if and only if $IJ \subseteq \mathfrak{p}$.

   We have $IJ \subseteq I$ and $IJ \subseteq J$, so ($\Rightarrow$) is clear. Conversely, suppose that $IJ \subseteq \mathfrak{p}$ for some prime $\mathfrak{p} \subseteq A$, and suppose that $I, J \nsubseteq \mathfrak{p}$, then there are elements $f \in I \backslash \mathfrak{p}$ and $g \in J \backslash \mathfrak{p}$, then $fg \notin \mathfrak{p}$, but $fg \in IJ \subseteq \mathfrak{p}$, contradiction, so $Z(I) \cup Z(J) = Z(IJ)$.

3. $\bigcap_{\alpha \in S} Z(I_\alpha) = Z(\sum_{\alpha \in S} I_\alpha)$.

□

Definition 12.9 (Sheaf of Regular Functions of Spectrum)**.** Let A be a commutative ring. Then the *sheaf of regular functions* on the topological space $X = \mathrm{Spec}(A)$ is defined by: for an open subset $U \subseteq X$, $\mathcal{O}_X(U) = \{S = (s_p : p \in U)\}$ where $s_p \in A_p$ (the localization) such that U is covered by open subsets $V \subseteq U$, on which s can be written as $\frac{f}{g}$ for some $f, g \in A$ such that "$g \neq 0$ of every point of V", that is, $g \notin p$ for every point $p \in V$.

Remark 12.10. It is easy to see that $\mathcal{O}_X(U)$ is a commutative ring for each open $U \subseteq X = \mathrm{Spec}(A)$. It is also easy to verify that $\mathcal{O}_X$ is a sheaf of commutative rings on $X = \mathrm{Spec}(A)$.

Definition 12.11 (Standard Open Subset)**.** A *standard open subset* of $\mathrm{Spec}(A)$ for any commutative ring A is a subset of the form $\{f \neq 0\} = \mathrm{Spec}(A) \backslash Z((f)) \subseteq \mathrm{Spec}(A)$ for some $f \in A$.

Remark 12.12. It is easy to verify that the standard open sets form a basis for the topologies of Spec(A).

13 Lecture 13

Note that from now on a a ring will always be commutative, unless stated otherwise.

Definition 13.1 (Ringed Space)**.** A *ringed space* X is a topological space with a sheaf of commutative rings.

Example 13.2.
1. A quasi-projective algebraic set over algebraically closed field k is a ringed space.
2. For every ring A, Spec(A) is a ringed space.

Definition 13.3 (Affine Scheme)**.** An *affine scheme* is a ringed space that is isomorphic to Spec(A) as a ringed space, for some ring A.

Definition 13.4 (Quasi-compact)**.** A topological space X is *quasi-compact* if every open cover has a finite subcover.

Lemma 13.5. Let A be a ring. The topological space Spec(A) is quasi-compact.

Proof. That is, the topological space X is compact (but not necessarily Hausdorff). That is, for any open cover $\{U_\alpha\}_{\alpha\in I}$ of $X = \text{Spec}(A)$, here is a finite subset $J \subseteq I$ such that $X = \bigcup\limits_{\alpha\in J} U_\alpha$.

We can choose ideal $I_\alpha \subseteq A$ for $\alpha \in I$ such that $U_\alpha = X \backslash Z(I_\alpha)$, so $X = \bigcup\limits_{\alpha\in I} U_\alpha$, then $\bigcap\limits_{\alpha\in I} Z(I_\alpha) = \varnothing \subseteq X$, so $Z(\sum\limits_{\alpha\in I} I_\alpha = \varnothing$. Recall that $Z(I) \subseteq \text{Spec}(A)$ means $\{\mathfrak{p} \in \text{Spec}(A) : I \subseteq \mathfrak{p}\}$, then $Z(I) = \varnothing$ if and only if I is not contained in any prime ideal, but every ideal $I \subsetneq A$ is contained in some maximal ideal, so $1 \in \sum\limits_{\alpha\in I} I_\alpha$, and so there exists a finite subset $J \subseteq I$ with $1 \in \sum\limits_{\alpha\in J} I_\alpha$, thus $\bigcap\limits_{\alpha\in J} Z(I_\alpha) = \varnothing$, that is, $\bigcup\limits_{\alpha\in J} U_\alpha = X$. □

Theorem 13.6. Let A be a ring, and let $X = \text{Spec}(A)$ be a ringed space.

1. There is a natural isomorphism $A \cong \mathcal{O}(X)$.
2. For any element $g \in A$, there is a natural isomorphism $A\left[\frac{1}{g}\right] \cong \mathcal{O}(\{g \neq 0\})$, where $\{g \neq 0\}$ is called the standard open subset of $X = \text{Spec}(A)$.
3. For every $p \in \text{Spec}(A)$, the stalk $\mathcal{O}_{X,p} \cong A_p$, the localization of A at the prime ideal p.

Example 13.7. Let F be a field, e.g., $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for p a prime number, or $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$. Then $\operatorname{Spec}(F)$ is a point, corresponding to the prime ideal $0 \subseteq F$. We have $\mathcal{O}(\varnothing) = 0$ and $\mathcal{O}(*) = F$.

Lemma 13.8. For any ring A, there are a natural bijective correspondences between:

1. the closed points in $\operatorname{Spec}(A)$ and the maximal ideals in A.

2. the points in $\operatorname{Spec}(A)$ and the prime ideals in A.

3. the closed subsets of $\operatorname{Spec}(A)$ and the radical ideals in A.

Proof. (2) is clear. For (1), what is the closure of a prime ideal $p \in \operatorname{Spec}(A) = X$? It is the closed subset $Z(p) \subseteq X$. By definition, this is the set of primes q containing p. Clearly $Z(p)$ is a closed subset of X that contains the point p. If $Z(I)$ is some other closed subset containing p, then $I \subseteq p$, so $Z(p) \subseteq Z(I)$. Therefore, $Z(p)$ is the closure of the point p. So a point $p \in \operatorname{Spec}(A)$ is closed if and only if $Z(p) = \{p\}$ if and only if the only prime ideal containing p is p, i.e., p is a maximal ideal.

To prove (3), recall that for any commutative ring A, $\operatorname{nilrad}(A) = \{x \in A : x^n = 0 \text{ for some } n > 0\}$, also known as the intersection of all prime ideals in A. Therefore, for any ideal $I \subseteq A$, then $\operatorname{rad}(I)$ is the intersection of all prime ideals in A containing I. So for an ideal I, knowing $Z(I) = \{p \in \operatorname{Spec}(A) : I \subseteq p\}$ is equivalent to knowing the intersection of all primes containing I, i.e., knowing $\operatorname{rad}(I)$. □

Example 13.9 (What is $\operatorname{Spec}(\mathbb{Z})$?)**.** The prime ideals in $\mathbb{Z}$ are the maximal ideals (2), (3), (5), and so on, and the zero ideal $(0) \subseteq \mathbb{Z}$. Geometrically speaking, the points (2), (3), (5), and so on are closed, but the closure of the point (0) is $Z((0)) = \operatorname{Spec}(\mathbb{Z})$.

Definition 13.10 (Generic Point)**.** For a topological space X, a *generic point* of X is a point whose closure is X.

Remark 13.11. For every domain A, $\operatorname{Spec}(A)$ has a generic point, namely the prime ideal $(0) \subseteq A$.

The closed subsets of $\operatorname{Spec}(\mathbb{Z})$ are the subsets of the form $Z(I)$ for some ideal I. Since $\mathbb{Z}$ is a PID, $I = (a)$ for some $a \in \mathbb{Z}$. Therefore, every closed subset $\operatorname{Spec}(\mathbb{Z})$ is of the form $\{a = 0\}$ for some $a \in \mathbb{Z}$.

Example 13.12. $\{15 = 0\} \subseteq \operatorname{Spec}(\mathbb{Z})$ is $\{(3), (5)\}$, and $\{0 = 0\}$ is all of $\operatorname{Spec}(\mathbb{Z})$. Therefore, every closed subset of $\operatorname{Spec}(\mathbb{Z})$ is either $\operatorname{Spec}(\mathbb{Z})$ or a finite set of closed points. And we have, for example, $\mathcal{O}(\{15 \neq 0\} = \mathbb{Z}\left[\frac{1}{15}\right]$. So, for instance, $7, \frac{2}{3}, \frac{8}{15}$, are all regular functions on $\{15 \neq 0\}$.

Example 13.13 (What is Spec($\mathbb{Z}/6\mathbb{Z}$)?)**.** We can use that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} = \{(a, b) : a \in \mathbb{Z}/2\mathbb{Z}, b \in \mathbb{Z}/3\mathbb{Z}\}$. Therefore, $\text{Spec}(\mathbb{Z}/6\mathbb{Z}) \cong \text{Spec}(\mathbb{Z}/2\mathbb{Z}) \coprod \text{Spec}(\mathbb{Z}/3\mathbb{Z})$, therefore given by two closed points (2) and (3). Therefore, not every affine scheme has a generic point.

Example 13.14. Affine n-space over a Commutative Ring R, $\mathbb{A}^n_R$, means $\text{Spec}(R[x_1, \ldots, x_n])$.

Example 13.15. Let k be an algebraically closed field. What is $\mathbb{A}^1_k$ in this new sense? It is $\text{Spec}(k[x])$. The prime ideals are the maximal ideals $(x - a)$ for $a \in k$, and the prime (but not maximal) ideal (0). Therefore, $\mathbb{A}^1_k = k \cup \{*\}$, where $*$ denotes the generic point.

14 Lecture 14

Definition 14.1 (Discrete Valuation)**.** A *discrete valuation* v on a field F is a surjective function $v : F \to \mathbb{Z} \cup \{\infty\}$ such that

1. For $a \in F$, $v(a) = \infty$ if and only if $a = 0 \in F$.
2. $v(ab) = v(a) + v(b)$ for all $a, b \in F$.
3. $v(a + b) \geq \min(v(a), v(b))$ for all $a, b \in F$.

Definition 14.2 (Discrete Valuation Ring)**.** A *discrete valuation ring* (DVR) is the subring $\{a \in f : v(a) \geq 0\}$ of a discretely valued field (F, v).

Example 14.3. For a field k, get a valuation on the field $k(x)$ by $v(x^a \cdot \frac{p}{q}) = a$, if $a \in \mathbb{Z}$ and $p, q \in k[x]$ not multiples of x, so this valuation measure the order of vanishing of $f \in k(x)$ at $0 \in \mathbb{A}^1_k$. Therefore, it is negative if f has a pole at $0 \in \mathbb{A}^1_k$. The associated DVR is $\{f \in k(x) : v(f) \geq 0\} = k[x]_{(x)}$, the localization at this prime ideal.

Example 14.4. Get the p-adic valuation on $\mathbb{Q}$ for a prime number p by $v(p^n \frac{u}{v}) = a$ if $u, v \in \mathbb{Z}\backslash(p)$. The associated DVR is the local ring $\mathbb{Z}_{(p)}$.

Remark 14.5 (What is Spec(A) for a DVR A?)**.** The ideal in a DVR A are just $\{0\}$ and $J_a = \{f \in R : v(f) \geq a\}$ for $a \in \mathbb{N}$. The only prime ideals are (0) and $J_1 = \{f \in R : v(f) \geq 1\} = (q)$ for some $q \in R$ with $v(q) = 1$, which gives a maximal ideal. Therefore, Spec(A) is given a closed point J_1 and a generic point (0). The open subset of Spec(A) are $\varnothing$, the generic point $\{g \neq 0\}$, and Spec(A), so the ring of regular functions on these open subsets are 0, and $A[\frac{1}{g}] = \text{Frac}(A)$, and A.

Remark 14.6. Recall that for a commutative ring R, the affine n-space over R is the affine scheme $\mathrm{Spec}(R[x_1, \ldots, x_n])$. For k algebraically closed, the scheme $\mathbb{A}^1_k$ is $k \cup \{*\}$ where $*$ is the generic point. What about $\mathbb{A}^2_k$? The points of the scheme $\mathbb{A}^2_k$ are the prime ideals in $k[x, y]$. By the Nullstellensatz, the subset of closed points is k^2 (by $(a, b) \in k^2$, get the maximal ideal $(x-a, y-b) \subseteq k[x, y] = R$). The subspace topology on k^2 is the Zariski topology. The other point of the scheme $\mathbb{A}^2_k$ are in one-to-one correspondence with the irreducible closed subset of dimension at least 1 in k^2.

Remark 14.7. There is a one-to-one correspondence between open subset of the scheme $\mathbb{A}^2_k$ and open subsets of k^2, given by $U \mapsto U \cap k^2$. Moreover, the ring of regular functions on open $U \subseteq \mathbb{A}^2_k$ is the same as the regular functions on $U \cap k^2$.

Example 14.8. $\mathrm{Spec}(0) = \varnothing$ because $R \subseteq R$ is not prime ideal, because a domain is defined to have $1 \neq 0$. And if R is a non-zero ring, then $\mathrm{Spec}(R) \neq \varnothing$. Also, if R is a ring with $\mathrm{Spec}(R) = \varnothing$, then $R \cong \mathcal{O}(\mathrm{Spec}(R)) = 0$.

Example 14.9 (What is the scheme $\mathbb{A}^1_{\mathbb{R}}$?)**.** A point of $\mathbb{A}^1_{\mathbb{R}}$ is a prime ideal in $\mathbb{R}[x]$. For any field, this is a PID, the prime ideal are (0) and the maximal ideals, which are of the form (g) for an irreducible polynomial $g \in R[x]$. For an irreducible polynomial $g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ over $\mathbb{R}$, we have a complete factorization over $\mathbb{C}$ given by $g(x) = (x - c_1) \cdots (x - c_n)$ for some $c_1, \ldots, c_n \in \mathbb{C}$. Note that multiplying the complex conjugations maintain the coefficients as reals. Therefore, the irreducible real polynomials in one variable are $x - b$ for $b \in \mathbb{R}$, and $(x - c)(x - \bar{c})$ for $c \in \mathbb{C} \backslash \mathbb{R}$. Therefore, as a set $\mathbb{A}^1_{\mathbb{R}}$ is just the quotient of $\mathbb{C}$ over the action of complex conjugation, and the single generic point.

Note that $I = (x^2 + 1) \subseteq \mathbb{R}[x]$ has $Z(I) \subseteq R$ empty, which is the same as $Z((1))$, but $\mathrm{rad}(x^2 + 1) \neq \mathrm{rad}(1)$.

Definition 14.10 (Ringed Space)**.** A *ringed space* is a topological space X with a sheaf of commutative rings $\mathcal{O}_X$.

Definition 14.11 (Locally Ringed Space)**.** A *locally ringed space* is a ringed space such that for every $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring.

Example 14.12. For a commutative ring A, $X = \mathrm{Spec}(A)$ is a locally ringed space because $\mathcal{O}_{X,p} = A_p$, the localization of A at p.

Definition 14.13 (Affine Scheme)**.** An *affine scheme* X is a locally ringed space that is isomorphic (as a locally ringed space) to $\mathrm{Spec}(A)$ for some commutative ring A. A *scheme* is a locally ringed space that has an open cover by affine schemes (as locally ringed space).

Example 14.14. Every open subset of a scheme is a scheme.

Proof Sketch. Let X be a scheme, and $V \subseteq X$ open. We are given $X = \bigcup_{\alpha \in I} U_\alpha$, with $U_\alpha \cong \text{Spec}(R_\alpha)$ then U is a locally ringed space with U_α's. Then $V = \bigcup_{\alpha \in I} (V \cap U_\alpha)$, but $V \cap U_\alpha$ need not be an affine scheme, but it is an open subset of an affine scheme.

Note that every open subset of $U_\alpha = \text{Spec}(R_\alpha)$ is a union of some standard open subsets $\{g_\beta \neq 0\} \subseteq U_\alpha$ for $g_\beta \in R_\alpha$. Therefore, V is a union of affine scheme, namely the $\{g_\beta \neq 0\}$ are of the form $\text{Spec}(R_\alpha[\frac{1}{g}]_\beta)$, so it is a scheme. □

15 Lecture 15

Definition 15.1 (Morphism of Ringed Spaces)**.** A *morphism of ringed spaces* $f : X \to Y$ is a continuous map together with a homomorphism of sheave of rings $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ on Y. That is, for every open $U \subseteq Y$, we are given a ring homomorphism

$$f^\# : \mathcal{O}_Y(U) \to (f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U)),$$

which is compatible with restriction maps.

Example 15.2. For fields F_1, F_2, a morphism $\text{Spec}(F_1) \to \text{Spec}(F_2)$ of ringed spaces is equivalent to a ring homomorphism $F_2 \to F_1$ (which is necessarily injective).

Definition 15.3 (Local Homomorphism of Local Rings)**.** A *local homomorphism of local rings* is a ring homomorphism $f : A \to B$ with A, B as local rings such that $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Remark 15.4. A homomorphism of local rings $A \to B$ is a local homomorphism if and only if it induces a homomorphism of residue fields $A/\mathfrak{m}_A \hookrightarrow B/\mathfrak{m}_B$.

Example 15.5. The inclusion $\mathbb{Z}_{(2)} \hookrightarrow \mathbb{Q}$ is a homomorphism of local rings which is not a local homomorphism, given that $(2) \mapsto (0)$.

In other words, we don't have an inclusion of residue fields.

Definition 15.6 (Morphism of Locally Ringed Spaces)**.** A *morphism of locally ringed spaces* is a morphism $f : X \to Y$ of ringed spaces between locally ringed spaces X and Y such that for every $p \in X$, the associated homomorphism $\varphi : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism of local rings.

Example 15.7. For every homomorphism $\varphi : A \to B$ of commutative rings, the induced morphism $f : \text{Spec}(B) \to \text{Spec}(A)$ of ringed spaces is a morphism of locally ringed spaces.

Proof Sketch. Let $p \in \operatorname{Spec}(B)$, then $f(p) = \varphi^{-1}(p)$ as a prime ideal in A. The homomorphism $\varphi : \mathcal{O}_{\operatorname{Spec}(A),f(p)} \to \mathcal{O}_{\operatorname{Spec}(B),p}$ is the induced natural homomorphism $A_{\varphi^{-1}(p)} \to B_p$. We claim that φ_p is a local homomorphism of local rings $B \to B_p$. The maximal ideal in B_p is pB_p, and the residue field $B_p/pB_p \cong \operatorname{Frac}(B/p)$.

We need to show that if $\frac{u}{v} \in A_{\varphi^{-1}(p)}$ maps into pB_p, then $\frac{u}{v} \in \varphi^{-1}(p) \cdot A_{\varphi^{-1}(p)}$.

Lemma 15.8. For a prime ideal $\mathfrak{p}$ in a ring B and $u \in B$, $v \in B \backslash \mathfrak{p}$, then $\frac{u}{v} \in \mathfrak{p}B_{\mathfrak{p}}$ if and only if $u \in \mathfrak{p}$.

Subproof. Left as an exercise. ■

By lemma, if $\varphi\left(\frac{u}{v}\right) = \frac{\varphi(u)}{\varphi(v)} \in pB_p$, then $\varphi(u) \in p$, so $u \in \varphi^{-1}(p)$, so $\frac{u}{v}$ is contained in the maximal ideal of $A_{\varphi^{-1}(p)}$. □

Theorem 15.9. For any commutative rings A and B, there is a one-to-one correspondence between ring homomorphisms $A \to B$ and morphisms $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ of locally ringed spaces.

Remark 15.10. This is false for ringed spaces.

Proof Sketch. Given a ring homomorphism $\varphi : A \to B$, this corresponds to the natural homomorphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$. Conversely, given a morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ of locally ringed spaces, this gives a pullback ring homomorphism $\mathcal{O}(\operatorname{Spec}(A)) \to \mathcal{O}(\operatorname{Spec}(B)$, therefore gives the morphism from A to B. □

Definition 15.11 (Morphism of Schemes)**.** A *morphism of schemes* $f : X \to Y$ for X, Y schemes is a morphism of locally ringed spaces.

Remark 15.12. This makes schemes into a category.

Remark 15.13. Theorem 15.9 implies that the full subcategory of affine schemes is equivalent to (reversing arrows) to the category of commutative rings.

Definition 15.14 (Glued Schemes)**.** Let X_1 and X_2 be schemes. Let $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ be open subsets, and let $g : U_1 \to U_2$ be an isomorphism of schemes, then there is a scheme $X = X_1 \cup_g X_2$, called the *glued scheme*.

- As a set $X = (X_1 \coprod X_2)/\sim$, where $U_1 \ni x \sim g(x) \in U_2$.
- As a topological space, the glued scheme is the quotient space of the topological space $X_1 \coprod X_2$ by that equivalent relation. (We have a natural function $X_1 \coprod X_2 \to X$ and we say that a subset of X is closed (respectively, open) if and only if $f^{-1}(U)$ is closed (respectively, open).

To define $\mathcal{O}_X$, we just glue the sheaves together in the obvious way (that agrees on the intersection).

Example 15.15. For any commutative ring R, define the scheme P_R^1 as $P_R^1 = \mathbb{A}_R^1 \cup_g \mathbb{A}_R^1$, where we glue $g : \mathbb{A}_R^1 \supseteq \{x \neq 0\} \to \{x \neq 0\} \subseteq \mathbb{A}_R^1$ by $x \mapsto \frac{1}{x}$.

Remark 15.16. The open subset $\{x \neq 0\} \subseteq \mathbb{A}_R^1 = \operatorname{Spec}(R[x])$ is a standard open subset isomorphic to $\operatorname{Spec}(R[x]\left[\frac{1}{x}\right])$. Therefore, to define a morphism $g : \{x \neq 0\} \to \{y \neq 0\}$, it is equivalent to give a ring homomorphism $R[y]\left[\frac{1}{y}\right] \to R[x]\left[\frac{1}{x}\right]$ (by taking the identity map on R and sending y to $\frac{1}{x}$).

Definition 15.17 (Quasi-affine Scheme)**.** A *quasi-affine scheme* is a scheme which is isomorphic to some open subset of an affine scheme.

Example 15.18. For any field k, $U = \mathbb{A}_k^2 \backslash \{0\}$ is a quasi-affine scheme which is not affine.

Remark 15.19. For every scheme X, there is a natural morphism of schemes $X \to \operatorname{Spec}(\mathcal{O}(X))$.

Remark 15.20 (Gluing Morphisms)**.** For any scheme X with an open cover $\{U_\alpha\}_{\alpha \in I}$, a scheme morphism $X \to Y$ is equivalent to morphisms $f_\alpha : U_\alpha \to Y$ for all $\alpha \in I$ such that $f_\alpha |_{U_\alpha \cap U_\beta} = f_\beta |_{U_\alpha \cap U_\beta}$. In this way, we get to glue morphisms of schemes.

Proof. There is a natural morphism $\varphi : U \to \operatorname{Spec}(\mathcal{O}(U))$. Clearly the scheme U is affine if and only if φ_U is an isomorphism. Here U is the union of two standard open sets of $\mathbb{A}_k^2$, $U = \{x \neq 0\} \cup \{y \neq 0\} \subseteq \mathbb{A}_k^2 = \operatorname{Spec}(k[x,y])$. Therefore, $\mathcal{O}(U) = k[x,y]\left[\frac{1}{x}\right] \cap k[x,y]\left[\frac{1}{y}\right] \subseteq k[x,y]\left[\frac{1}{xy}\right] = \mathcal{O}(\{x \neq 0\} \cap \{y \neq 0\}) = \mathcal{O}(\{xy \neq 0\})$. Here $k[x,y]$ has a basis as a k-vector space $x^i y^i$ for all $i, j \geq 0$. Therefore, $k[x,y]\left[\frac{1}{xy}\right] = k[\{x^i y^j : i, j \in \mathbb{Z}\}]$. Note that we can find a basis when we only adjoint $\frac{1}{x}$ or $\frac{1}{y}$, and thus their intersection gives the ring $\mathcal{O}(U) = k[x,y]$. Hence, the morphism $U \to \operatorname{Spec}(\mathcal{O}(U))$ is the inclusion $\mathbb{A}_k^2 \backslash \{0\} \hookrightarrow \mathbb{A}_k^2$. But the morphism is not an isomorphism because it is not surjective, so the quasi-affine scheme $\mathbb{A}_k^2 \backslash \{0\}$ for any field k is not affine. □

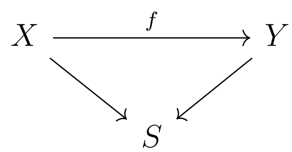
16 Lecture 16

Definition 16.1 (Residue Field)**.** For a scheme X and a point $p \in X$, the *residue field* $k(p)$ of X at p is the residue field $\mathcal{O}_{X,p}/\mathfrak{m}$. There is a natural morphism of schemes $\operatorname{Spec}(k(p)) = * \to X$ with the image as the point p.

For $p \in \operatorname{Spec}(R)$, we have a ring homomorphism $R \to R/p \to \operatorname{Frac}(R/p) \cong (R_p)/(pR_p) = k(p)$, and in turn a reversed map $\operatorname{Spec}(k(p)) \to \operatorname{Spec}(R) \hookrightarrow X$.

Example 16.2. We can think of points of $\mathrm{Spec}(\mathbb{Z})$ as images of $\mathrm{Spec}(\mathbb{F}_p)$ for p prime numbers and $\mathrm{Spec}(\mathbb{Q} \to \mathrm{Spec}(\mathbb{Z})$.

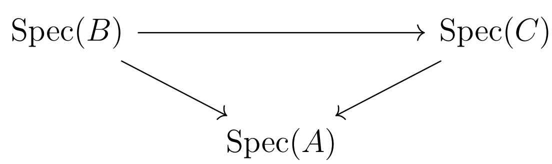
Definition 16.3 (Scheme Over Scheme)**.** Let S be a scheme. A *scheme over* S is a scheme X with a morphism $X \to S$. A morphism of scheme over S to a morphism $f : X \to Y$



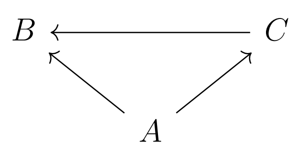
that makes the diagram commutes.

For a ring R, a scheme over R is a scheme over $\mathrm{Spec}(R)$.

Example 16.4. For rings A, B, and C, an affine scheme over A is equivalent to an A-algebra, i.e., a ring B with a morphism $A \to B$: the diagram



commutes given that



Example 16.5. A morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ over $\mathbb{C}$ is equivalent to a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[y] \to \mathbb{C}[x]$, i.e., to a polynomial in $\mathbb{C}[x]$. A morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ of schemes (not necessarily $\mathbb{C}$) is a ring homomorphism $\mathbb{C}[y] \to \mathbb{C}[x]$ could be given by an automorphism on the field $\mathbb{C}$, say $\pi \mapsto e$.

Lemma 16.6. For a scheme X and any commutative ring A, a morphism $X \to \mathrm{Spec}(A)$ of schemes is equivalent to a ring homomorphism $A \to \mathcal{O}(X)$.

Corollary 16.7. Every scheme is a scheme over $\mathbb{Z}$ in a unique way.

Definition 16.8 (Faithful, Full)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* if for all objects $A, B \in \mathcal{C}$, the function $\mathbf{Hom}_{\mathcal{C}}(A, B) \to \mathbf{Hom}_{\mathcal{D}}(FA, FB)$ is injective.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* if for all objects $A, B \in \mathcal{C}$, the function $\mathbf{Hom}_{\mathcal{C}}(A, B) \to \mathbf{Hom}_{\mathcal{D}}(FA, FB)$ is surjective.

We say a functor is *fully faithful* if it is full and faithful.

Remark 16.9. If a fully faithful functor $F : \mathcal{C} \to \mathcal{D}$ is surjective on objects, i.e., for every object Z in $\mathcal{D}$, there is $A \in \mathcal{C}$ such that $F(A) = Z$, then F is an equivalence of categories.

Theorem 16.10. For an algebraically closed field k, there is a fully faithful functor from the quasi-projective algebraic sets over k to the schemes over k.

Proof Sketch. For an affine algebraic set over algebraically closed k, there is closed subset $X \subseteq k^n$ for some $n \geq 0$, and associates the affine scheme $\mathrm{Spec}(\mathcal{O}(X)) = \mathrm{Spec}(k[x_1, \ldots, x_n])/I(X)$. For X quasi-projective, we can glue together those affine schemes over k. □

Definition 16.11 (Projective Scheme)**.** For a commutative ring R and $n \geq 0$, the projective n-space over R, denoted P^n_R, defined a *projective scheme* by gluing $n+1$ copies of the affine scheme $\mathbb{A}^n_R$.

Remark 16.12. Let $g \in [x_0, \ldots, x_n]$ be homogeneous of positive degree, then any rational function of the form $\frac{f}{g^r}$ with f homogeneous of degree $r \cdot \deg(g)$ is a regular function on $\{g \neq 0\} \subseteq P^n_k$. In fact, this open subset $\{g \neq 0\} \subseteq P^n_k$ is an affine scheme, namely $\mathrm{Spec}(S\left[\frac{1}{g}\right])_{\deg 0}$, given $S = k[x_0, \ldots, x_n]$ graded by $|x_i| = 1$.

17 Lecture 17

Let S be a graded ring, i.e., $S = S_0 \oplus S_1 \oplus \cdots$.

Definition 17.1 (Irrelevant Ideal)**.** We call $S_+ = \bigoplus_{a \geq 1} S_1$ the *irrelevant ideal*.

Example 17.2. Think of $S = k[x_0, \ldots, x_n]$ for a field k graded by $|x_i| = 1$.

Definition 17.3 (Proj)**.** We denote $\mathrm{Proj}(S)$ to be the set of homogeneous prime ideals that do not contain S_+.

Definition 17.4 (Projective Scheme)**.** For any commutative ring A and $n \geq 0$, the *projective n-space* over A is $P^n_A = \mathrm{Proj}(A[x_0, \ldots, x_n])$ with grading $|x_i| = 1$.

Remark 17.5.
- One can define a topology on $\mathrm{Proj}(S)$ with closed subsets $Z(I)$ for $I \subseteq S$ a homogeneous ideal.
- We previously defined a sheaf of rings $\mathcal{O}_X$ on $X = \mathrm{Proj}(S)$.

Proposition 17.6. Let S be a graded ring and let $X = \mathrm{Proj}(S)$ be the ringed space. Then

1. for each $p \in X$, the stalk $\mathcal{O}_{X,p}$ is the local ring $(S_p)_{\deg 0}$, with elements of the form $\frac{f}{g}$ with $f, g \in S$ homogeneous of same degree where $g \notin p$.

2. for each homogeneous element $f \in S_+$, let $\{f \neq 0\} \subseteq X$ be an open subset, then $\{f \neq 0\}$ is isomorphic to $\text{Spec}(S\left[\frac{1}{f}\right])_{\deg 0}$ (with elements of the form $\frac{h}{f^r}$ for $h \in S$ homogeneous, $r \geq 0$, and $\deg(h) = r\deg(f)$) as a locally ringed space.

3. $\text{Proj}(S)$ is a scheme.

Exercise 17.7. Let A be a ring. Show that the scheme $P^n_A = \text{Proj}(A[x_0, \ldots, x_n])$ is covered by $n+1$ affine open subsets $\{x_i \neq 0\}$ for $0 \leq i \leq n$, and that $\{x_i \neq 0\} \cong \mathbb{A}^n_A$.

Definition 17.8 (Connected Scheme)**.** A scheme X is *connected* if it is connected as a topological space.

Definition 17.9 (Irreducible Scheme)**.** A scheme X is *irreducible* if it is irreducible as a topological space.

Exercise 17.10. Let A be a ring, then $\text{Spec}(A)$ is connected if and only if $A \neq 0$ and we cannot write $A = B \times C = \{(b, c) : b : B, c \in C\}$ with B and C as non-zero rings.

Proof. Indeed, $\text{Spec}(B \times C) = \text{Spec}(B) \coprod \text{Spec}(C)$ as a scheme. Note that if $p \subseteq B$ is a prime ideal, then $p \times C$ is a prime ideal of $B \times C$. This determines the structure of the spectrum. Correspondingly, $\mathcal{O}(X) \cong \mathcal{O}(Y) \times \mathcal{O}(Z)$. □

Definition 17.11 (Reduced Scheme)**.** A scheme X is *reduced* if for every open subset $U \subseteq X$, the ring $\mathcal{O}(U)$ is reduced, i.e., every nilpotent element of $\mathcal{O}(U)$ is zero.

Exercise 17.12. 1. $\text{Spec}(A)$ is reduced if and only if $A = \mathcal{O}(\text{Spec}(A))$ is reduced.

2. A scheme X is reduced if and only if for every point $p \in X$, the local ring $\mathcal{O}_{X,p}$ is reduced. Therefore, being reduced is a local property of a scheme.

Example 17.13. $X = \text{Spec}(\mathbb{Q}[x]/(x^2))$ is a non-reduced scheme. For instance, by (1). Indeed, $0 \neq x \in \mathbb{Q}[x]/(x^2)$, but $x^2 = 0 \in \mathbb{Q}[x]/(x^2)$. Here X is just a single point. However, this point has fuzz (*fat point*) with it, in the sense that there is a natural embedding $\text{Spec}(\mathbb{Q}) \hookrightarrow X$, with $\text{Spec}(\mathbb{Q})$ as a single point as well.

Remark 17.14 (Geometry of non-reduced Scheme)**.** Consider $\mathbb{A}^2_{\mathbb{C}}$. Suppose $X = Z(I)$ and $Y = Z(J)$ for ideals $I, J \subseteq R = \mathbb{C}[x, y]$, then $X \cap Y = Z(I + J)$, so the scheme-theoretic intersection $X \cap Y$ is just $\text{Spec}(\mathbb{C}[x, y])/(I + J)$.

- Consider X and Y as a line and a circle intersecting at two points, then in this case $X \cap Y = \text{Spec}(\mathbb{C}) \coprod \text{Spec}(\mathbb{C}) = \text{Spec}(\mathbb{C} \times \mathbb{C})$.

- Consider X and Y as a line and a circle intersecting at a single point, i.e., as a tangent, then $X \cap Y = \cong \text{Spec}(\mathbb{C}[x]/(x^2))$.
- In the case of $x = 0$ with $y = x^3$, $X \cap Y = \text{Spec}(\mathbb{C}[x]/(x^3))$ since $\mathbb{C}[x, y]/(y, y - x^3) = \mathbb{C}[x]/(x^3)$.

Definition 17.15 (Integral Scheme)**.** A scheme X is *integral* if $X \neq \varnothing$ and for every non-empty open subset $U \subseteq X$, the ring $\mathcal{O}(U)$ is a domain.

Lemma 17.16. For a ring A, the scheme $\text{Spec}(A)$ is integral if and only if A is a domain.

Proposition 17.17. A scheme X is integral if and only if it is reduced and irreducible.

Remark 17.18. Although the reduced property is local, the irreducible property only depends on X as a topological space, and in turn the integral property is a global property.

Proof. ($\Rightarrow$): Let X be an integral scheme, then for every open subset $U \subseteq X$, the ring $|mathcalO(U)$ is the zero ring (if $U = \varnothing$) or a domain (if $U \neq \varnothing$). In both cases, $\mathcal{O}(U)$ is reduced, so the scheme X is reduced. Also, $X \neq \varnothing$. Suppose $X = S_1 \cup S_2$ with S_1, $S_2 \subsetneq X$ as closed subsets. Then $X \backslash S_1$ and $X \backslash X_2$ are disjoint non-empty open subsets of X. Therefore, $\mathcal{O}(X \backslash S_1 \cup X \backslash S_2) = \mathcal{O}(X \backslash S_1) \times \mathcal{O}(X \backslash S_2)$ as the product of two non-zero rings, which is not a domain since $(1, 0) \times (0, 1) = (0, 0)$. This contradicts the fact that X is integral.

($\Leftarrow$): Let X be a reduced irreducible scheme. Since X is irreducible, $X \neq \varnothing$. Let $U \subseteq X$ be a non-empty open subset, then $\mathcal{O}(U) \neq 0$. Suppose $f, g \in \mathcal{O}(U)$ have $fg = 0 \in \mathcal{O}(U)$, then $U = \{f = 0\} \cup \{g = 0\}$. (Recall that in an affine scheme $\text{Spec}(R)$, for $f \in R$, $\{f = 0\} := \{p \in \text{Spec}(R) : f = 0 \in R_p\}$.)

Since X is irreducible, U is irreducible (by homework 2), then either U is $\{f = 0\}$ or $\{g = 0\}$. Suppose $U = \{f = 0\}$. (Note that this does not say $f = 0$ directly.) Let $V = \text{Spec}(R) \subseteq U$ be a non-empty open subset, then $f \in \bigcap_{p \in \text{Spec}(V)} = \text{nilrad}(\mathcal{O}(V)) = \text{nilrad}(R)$. Since X is reduced, so $f = 0$. Since this works for all affine $V \subseteq U$, we have $f = 0 \in \mathcal{O}(U)$, therefore $\mathcal{O}(U)$ is a domain, and so X is an integral scheme. □

18 Lecture 18

Lemma 18.1. A scheme X is quasi-compact if and only if it can be written as a finite union of affine open subschemes.

Proof. We have shown that every affine scheme $\text{Spec}(R)$ is quasi-compact. If a scheme X is quasi-compact, then it can be covered by some affine open subscheme $U_\alpha \subseteq X$ with $\alpha \in I$, hence can be covered by finitely many of these. The converse is easy. □

Example 18.2. For any set of schemes Y_α with $\alpha \in I$, there is a scheme $Y = \coprod_{\alpha \in I} Y_\alpha$. If Y_α is non-empty for every $\alpha \in I$, and I is infinite, then Y is not quasi-compact.

Note that $\operatorname{Spec}(R_1) \coprod \operatorname{Spec}(R_2) \cong \operatorname{Spec}(R_1 \times R_2)$ and it is affine in particular. Therefore, we have $\mathcal{O}(Y) \cong \prod_{\alpha \in I} \mathcal{O}(Y_\alpha)$, but Y is not quasi-compact, and hence not affine.

Example 18.3. $\coprod_{i=1}^{\infty} \operatorname{Spec}(\mathbb{C})$ is not isomorphic to $\operatorname{Spec}(\prod_{i=1}^{\infty} \mathbb{C})$.

Exercise 18.4. If you know about ultrafilters, try to describe $\operatorname{Spec}(\prod_{i=1}^{\infty} \mathbb{C})$ as a set at least.

Definition 18.5 (Locally Noetherian)**.** A scheme X is *locally Noetherian* if it can be written as a union of affine open subschemes $\operatorname{Spec}(R_\alpha)$ with R_α Noetherian.

Definition 18.6 (Noetherian)**.** A scheme X is Noetherian if it can be written as a finite union of affine open subschemes isomorphic to $\operatorname{Spec}(R_\alpha)$ with R_α's Noetherian.

Definition 18.7 (Noetherian)**.** A scheme is *Noetherian* if and only if it is locally Noetherian and quasi-compact.

Remark 18.8. If a scheme X is Noetherian, then its underlying topological space is Noetherian. That is, for any sequence $X \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$, there exists some N such that $Y_N = Y_{N+1} = \cdots$.

Example 18.9. $X = \operatorname{Spec}(R)$.

Exercise 18.10. Given an example of a ring A which is not Noetherian with $\operatorname{Spec}(A)$ Noetherian as a topological space. (Hint: make $\operatorname{Spec}(A)$ a point.)

Lemma 18.11. Every Noetherian topological space is a finite union of irreducible closed subsets $X = Y_1 \cup \cdots Y_n$ for some $n \geq 0$. If we assume no Y_i contained in Y_j for all $i \neq j$, then the $Y_i's$ are unique up to reordering. In that case, they are called the irreducible components.

Proposition 18.12. A scheme X is locally Noetherian if and only if for every affine open subscheme $U \cong \operatorname{Spec}(R) \subseteq X$, the ring R is Noetherian.

Corollary 18.13. An affine scheme $X = \operatorname{Spec}(A)$ is Noetherian if and only if A is Noetherian.

Definition 18.14 (Locally of Finite Type)**.** A morphism $f : X \to Y$ of schemes is *locally of finite type* if Y can be covered by some affine open subschemes $V_i = \operatorname{Spec}(B_i) \subseteq Y$, such that $f^{-1}(V_i)$ can be written as a union of affine open subschemes $\operatorname{Spec}(A_{ij})$ with the homomorphisms $B_i \to A_{ij}$ make A_{ij} a finitely-generated B_i-algebra.

Definition 18.15 (Finite Type)**.** A morphism $f : X \to Y$ of schemes is of *finite type* if Y and X have open covers as Definition 18.14 with $f^{-1}(V_i)$ a finite union of the open subschemes $\operatorname{Spec}(A_{ij})$.

Example 18.16. For every ring R and $n \geq 0$, the affine n-space $\mathbb{A}_R^n$ is of finite type over R because $R[x_1, \ldots, x_n]$ is finitely-generated as an R-algebra.

Example 18.17. For any ring R and any $n \geq 0$, the projective n-space P_R^n is of finite type over R.

Example 18.18. The morphism $\coprod_{i=1}^{\infty} \operatorname{Spec}(\mathbb{C}) \to \operatorname{Spec}(\mathbb{C})$ is locally of finite type but not of finite type.

Definition 18.19 (Finite)**.** A morphism $f : X \to Y$ of schemes is *finite* if Y has a covering by some affine open subscheme $V_i \cong \operatorname{Spec}(B_i) \subseteq Y$, and A_i is finitely-generated as an B_i-module for all i.

Example 18.20. For a field k, the morphism $\mathbb{A}_k^1 \to \operatorname{Spec}(k)$ is of finite type but not finite. Indeed, $k[x]$ is finitely-generated as a k-algebra, but not as a k-vector space since $k[x] = k\{1, x, x^2, \ldots\}$.

Example 18.21. Let k be a field, then the morphism $f : \mathbb{A}_k^1 \to \mathbb{A}_k^1$ over k given by $f(x) = x^2$ is finite. Indeed, think of the morphism as $\operatorname{Spec}(k[x]) \to \operatorname{Spec}(k[y])$, then it is defined as $y = x^2$. Now this is true because the k-algebra homomorphism $k[y] \to k[x]$ defined by $y \mapsto x^2$ takes $k[x]$ into a $k[y]$-module generated by 1 and x. Indeed, $k[y] \cdot 1 = k \cdot \{1, x^2, x^4, \ldots\}$, and $k[y] \cdot x = k \cdot \{x, x^3, x^5, \ldots\}$

19 Lecture 19

Remark 19.1.

- If $f : X \to Y$ is a finite morphism of schemes, then f is quasi-finite, that is, for all $y \in Y$, $f^{-1}(y)$ is a finite set.
- Also, f is closed: the image in Y of every closed subset of X is closed in Y.

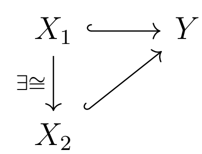
Example 19.2. For a field k, the inclusion $\mathbb{A}_k^1 \backslash \{0\} \hookrightarrow \mathbb{A}_k^1$ is not a finite morphism, even though it is quasi-finite, because $f(\mathbb{A}_k^1 \backslash \{0\}) = \mathbb{A}_k^1 \backslash \{0\}$ is not closed in $\mathbb{A}_k^1$. Indeed, $k[x, x^{-1}]$ is not finitely-generated as a module over its subring $k[x]$. Indeed, as a k-vector space, $k[x, x^{-1}] = k[\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots]$. Indeed, the $k[x]$-submodule is included in $x^{-n} \subseteq k[x]$, for some $n \geq 0$, and that is not $k[x, x^{-1}]$.

Definition 19.3 (Open Subscheme, Immersion)**.** An *open subscheme* of a scheme X is an open subset $U \subseteq X$ with the subspace topology and with the sheaf $\mathcal{O}_U := i^{-1}(\mathcal{O}_X)$ for $i : U \to X$ as the obvious inclusion.

A morphism $f : X \to Y$ is an *open immersion* if it is an isomorphism from X to an open subscheme of Y.

A *closed immersion* $f : X \to Y$ is a morphism that is a homeomorphism from X to a closed subset of Y such that the associated map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective (as a map of sheaves).

Definition 19.4 (Closed Subscheme)**.** A closed subscheme of Y is an equivalence class of closed immersions with the equivalence relation



such that the diagram commutes.

Example 19.5 (Hartshorne II.3.11(b))**.** For a ring A, the clsoed subschemes of $\mathrm{Spec}(A)$ are in one-to-one correspondence with the ideals in A, $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ (which is a closed immersion). The image as a set is just $\mathrm{Spec}(A/\mathrm{rad}(I))$.

Remark 19.6. Note that the closed subset of $\mathrm{Spec}(A)$ are in one-to-one correspondence with the radical ideals in A.

For elements $a_1, a_2, \ldots$ in a ring A, the closed subscheme $\{a_1 = 0, a_2 = 0, \ldots\} \subseteq \mathrm{Spec}(A)$ means $\mathrm{Spec}(A/(a_1, a_2, \ldots)) \hookrightarrow \mathrm{Spec}(A)$ given by a closed immersion.

Example 19.7. Let k be a field. In $\mathbb{A}^2_k$, we have the closed subscheme $Z_1 = \{x = 0\}$, $Z_2 = \{x^2 = 0\}$, $Z^3 = \{x^2 = 0, xy = 0\} \subseteq \mathbb{A}^2_k$. They correspond to $\mathcal{O}(Z_1) = k[x, y]/(x) \cong k[y]$, $\mathcal{O}(Z_2) = k[x, y]/(x^2)$, $\mathcal{O}(Z_3) = k[x, y]/(x^2, xy)$.

What information about a function $f \in \mathcal{O}(\mathbb{A}^2_k)$ is determined by its restriction to Z_1, Z_2, or Z_3? For Z_1: $f \mapsto f(0, y) \in k[y]$; for Z_2: $f \mapsto f(0, y)$ and $\frac{\partial f}{\partial x}|_{(0,y)}$; for Z_3: $f \mapsto f(0, y)$ and $\frac{\partial f}{\partial x}|_{(0,0)}$.

Remark 19.8. Every closed subset Z of a scheme X as a unique structure as a reduced closed subscheme.

Remark 19.9. Note that closed immersion implies finite implies affine.

Correspondingly, in ring theory, an affine commutative ring homomorphism $A \to B$ is surjective implies $A \to B$ with B finitely-generated as an A-module.

Definition 19.10 (Scheme Dimension)**.** The *dimension of a scheme* X is its dimension as a topological space, i.e.,

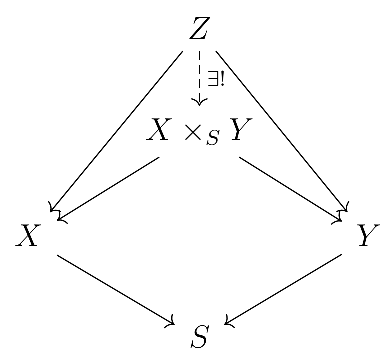
$$\sup\{r \geq 0: \text{ there exists chain } Z_0 \subsetneq \cdots \subsetneq Z_r \subseteq X \text{ of irreducible closed subsets}\}$$

Example 19.11. For a ring A, $\dim(\mathrm{Spec}(A)) = \dim(A)$ as the Krull dimension.

Example 19.12. For a field k, $\dim(\mathbb{A}_k^n) = n$, and so $\dim(P_k^n) = n$.

For a quasi-projective algebraic set over k algebraically closed, the dimension of scheme $\dim(\mathcal{A}(X)) = \dim(X)$.

Definition 19.13 (Fiber Product)**.** Let X, Y be schemes over a scheme S. The *fiber product* $X \times_S Y$ is a scheme with morphism to X and to Y making the square commute and which is universal for that property. That is, for any scheme Z with morphism $Z \to X$ and $Z \to Y$ over S that makes the diagram commute.



Remark 19.14. The fiber product of set $X \xrightarrow{f} S \xleftarrow{g} Y$ is just $X \times_S Y = \{(x, y) : x \in X, y \in Y, f(x) = g(y)\}$.

Theorem 19.15. Fiber product always exists in the category of schemes. It is unique up to a unique isomorphism.

Proof Sketch. For affine scheme $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, with $S = \mathrm{Spec}(C)$, then A and B are C-algebras. The fiber product is also affine: $X \times_S Y = \mathrm{Spec}(A \otimes_C B)$. In general, construct $X \times_S Y$ by gluing affine schemes as above. □

Example 19.16. For a field k, $\mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \cong \mathbb{A}_k^2$. Here $\times_k$ means $\times_{\mathrm{Spec}(k)}$.

Proof. $k[x] \otimes_k k[y] \cong k[x, y]$. □

Definition 19.17. Let $f : X \to Y$ be a morphism of schemes and $y \in Y$. Let $k(y)$ be the residue field of the scheme Y at y. We have a natural morphism $\mathrm{Spec}(k(y)) \to Y$. The fiber X_y of X over y means $X \times_Y \mathrm{Spec}(k(y))$.

Remark 19.18. As a set, the fiber X_y can be identified with $f^{-1}(y)$. Notice that the fiber X_y is a scheme over the field $k(y)$.

20 Lecture 20

Remark 20.1. Even if k is algebraically closed, the product scheme $\mathbb{A}^1_k \times_k \mathbb{A}^1_k \cong \mathbb{A}^2_k$ is not (as a topological space) the product of the topological spaces $\mathbb{A}^1_k$ and $\mathbb{A}^1_k$. (Not even the product as a set.)

Example 20.2. For X, Y open subschemes of a scheme S, the fiber product $X \times_S Y$ is simply $X \cap Y$, also an open subscheme of S.

Example 20.3. For X, Y closed subschemes of a scheme, we define the scheme-theoretic intersection $X \cap Y = X \times_S Y$.

In the affine case, for a ring A, the scheme-theoretic intersection is

$$\begin{aligned}\operatorname{Spec}(A/I) \cap \operatorname{Spec}(A/J) &= \operatorname{Spec}(A/I) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A/J) \\ &= \operatorname{Spec}((A/I) \otimes_A (A/J)) \\ &= \operatorname{Spec}(A/(I+J)).\end{aligned}$$

Example 20.4. The scheme-theoretic intersection of the (reduced) curves $\{y = 0\} \subseteq \mathbb{A}^2_{\mathbb{C}}$ and $\{y = \pm x^2\} \subseteq \mathbb{A}^2_{\mathbb{C}}$ is the closed subscheme $\{y = 0, y = x^2\} \subseteq \mathbb{A}^2_{\mathbb{C}}$, which is not reduced, since $\mathbb{C}[x, y]/(y, y - x^2) \cong \mathbb{C}[x]/(0 - x^2) \cong \mathbb{C}[x]/(x^2)$, which is not reduced.

Example 20.5 (What are the fibers of the morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ by $x \mapsto x^2$?)**.** The fiber over $a \in \mathbb{C}$ is simply the closed subscheme $\{x^2 = a\} \subseteq \mathbb{A}^1_{\mathbb{C}}$, with $\mathbb{C}[x] \otimes_{\mathbb{C}[y]} \mathbb{C}[y]/(y - a) = \mathbb{C}[x]/(x^2 - a)$. If $a \neq 0$, this is two points, isomorphic to $\operatorname{Spec}(\mathbb{C}) \coprod \operatorname{Spec}(\mathbb{C})$. If $a = 0$, this is a non-reduced scheme, with underlying set a single point.

What is the fiber of f over the generic point? The local ring of $\mathbb{A}^1_{\mathbb{C}}$ at its generic point is exact the field $\mathbb{C}(y)$, or $\mathbb{C}[y]_{(0)} \cong \mathbb{C}(y)$. So the fiber of f over the generic point of $\mathbb{A}^1_{\mathbb{C}}$ is $\operatorname{Spec}(\mathbb{C}[x] \otimes_{\mathbb{C}[y]} \mathbb{C}(y))$, here $\mathbb{C}[x] \cong \mathbb{C}[y][x]/(x^2 - y) \cong \mathbb{C}[y]\{1, x\}$, so the tensor product is just $\mathbb{C}(y)[x]/(x^2 - y)$, which is a degree 2 extension field of $\mathbb{C}(y)$, namely $\mathbb{C}(x)$.

Definition 20.6 (Base Change)**.** For a scheme X over a scheme T and any morphism $S \to T$, the *base change* of $X \to T$ with respect to S is the morphism $X \times_T S \to S$. We may call $X \times_T S = X_S$.

Example 20.7. Let X be a scheme over a field k. Let E be any extension field of k. The base change $X_E := X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(E)$ is a scheme over E.

Example 20.8. Let $X = \{f_1 = 0, f_2 = 0, \ldots\} \subseteq \mathbb{A}^n_k$ for a field k, then for any field extension E/k, the scheme X_E over E is $X_E = \{f_1 = 0, f_2 = 0, \ldots\} \subseteq \mathbb{A}^n_E$.

Definition 20.9 (Rational Point)**.** Let X be a scheme over a field k, then a *k-rational point* s of X is a morphism $s : \mathrm{Spec}(k) \to X$ such that the composition $\mathrm{Spec}(k) \to X \to \mathrm{Spec}(k)$ is the identity. We write $X(k)$ for the set of k-rational points of X.

Example 20.10. If $X = \{f_1 = 0, f_2 = 0, \ldots\} \subseteq \mathbb{A}_k^n$, then $X(k) = \{(a_1, \ldots, a_n) \in k^n : f_1(a_1, \ldots, a_n) = 0, f_2(a_1, \ldots, a_n) = 0, \ldots\}$.

Proof Sketch. In terms of rings, a k-rational point is a ring homomorphism

$$k \to k[x_1, \ldots, x_n]/(f_1, f_2, \ldots) \to k$$

with a composition as the identity map. □

Example 20.11. Let X be a scheme of finite type over $\mathbb{C}$, then the set $X(\mathbb{C})$ has a natural topology, the classical (Euclidean) topology. So we get an invariant of schemes X over $\mathbb{C}$ by the fundamental group $\pi_1(X(\mathbb{C}))$, or $H^j(X(\mathbb{C}), \mathbb{Z})$, etc.

Example 20.12. The line with two origins over a field k is $\mathbb{A}_k^1 \cup_{\mathbb{A}_k^1 \setminus \{0\}} \mathbb{A}_k^1$ with $x \in \mathbb{A}_k^1 \setminus \{0\}$ mapped to $x \in \mathbb{A}_k^1 \setminus \{0\}$. Let $k = \mathbb{C}$, then the space $X(\mathbb{C})$ with classical topology is not Hausdorff.

Exercise 20.13. A topological space X is Hausdorff if and only if the diagonal $\Delta_x := \{(x, x) : x \in X\} \subseteq X^2$ is a closed subset of X^2, if and only if the map $\Delta_x : X \to X^2$ defined by $x \mapsto (x, x)$ is a homeomorphism from X to a closed subset of X^2.

The scheme $\mathbb{A}_{\mathbb{C}}^1$ as a topological space is not Hausdorff, so $\mathbb{A}_{\mathbb{C}}^1 \subseteq \mathbb{A}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1$ (as a product topological space) is not closed.

Definition 20.14 (Separated Morphism of Schemes)**.** A morphism $f : X \to Y$ of schemes is *separated* if the diagonal morphism $\Delta_X : X \to X \times_Y X$ is a closed immersion.

Example 20.15. For a field k, $\mathbb{A}_k^1$ is separated over k (since the morphism $\mathbb{A}_k^1 \to \mathbb{A}_k^2$ defined by $x \mapsto (x, x)$ is a closed immersion.

Example 20.16. The line X with two origins is not separated over k.

Proof. We want to see whether $X \to X \times_k X$ is a closed immersion, or show the image is not a closed subset. Here $X = U_1 \cup U_2$ as the union of two affine opens, both isomorphic to $\mathbb{A}_k^1$, then $X \to X \times_k X = U_1 \times_k U_1 \cup U_1 \times_k U_2 \cup U_2 \times_k U_1 \cup U_2 \times_l U_2$ as open affine subschemes.

Case 1: In $U_1 \times_k U_1 \cong \mathbb{A}_k^2$, the diagonal Δ_X is $U_1 \times_K U_1 \cong \mathbb{A}_k^2$, then $\{x = y\} \subseteq \mathbb{A}_k^2$, which is closed.

Case 2: In $U_1 \times_k U_2 \cong \mathbb{A}_k^2$, the diagonal Δ_X gives $\{x = y\} \setminus \{0\}$, which is not closed, so X is not separated over k. □

21 Lecture 21

Example 21.1. For a field k, P_k^1 is separated over k.

Proof. We have to show that the diagonal morphism $\Delta_p : P_k^1 \to P_k^1 \times_k P_k^1$ is a closed immersion (or, by Hartshorne, the image is a closed subset). We have $P_k^1 = U_1 \cup U_2$ open subschemes, and $U_1 \cong \mathbb{A}_k^1$ and $U_2 \cong \mathbb{A}_k^1$. Here we glue by $A_k^1 \backslash \{0\} \xrightarrow{\cong} \mathbb{A}_k^1 \backslash \{0\}$ via $x \mapsto \frac{1}{x}$. So $P_k^1 \times_k P_k^1$ is the union of four affine open subschemes, isomorphic to $U_1 \times_k U_1$, $U_1 \times_k U_2$, $U_2 \times_k U_1$, and $U_2 \times_k U_2$. All four of them are isomorphic to $\mathbb{A}_k^2$. We have to show that for all $i, j \in \{1, 2\}$, $\Delta_P \cong (U_i \times_k U_j)$ closed in $U_i \times_k U_j \cong \mathbb{A}_k^2$.

Case 1: For $i = j = 1$, $\Delta_P \cap (U_1 \times_k U_1) = \Delta_{U_1} \subseteq U_1 \times_k U_1 = \{x = y\} \subseteq \mathbb{A}_k^2$, which is just $\mathbb{A}_k^2$.

Case 2: For $i = 1$ and $j = 2$, we can show that $\Delta_P \cap (U_1 \times_k U_2)$ closed in $U_1 \times U_2 \cong \mathbb{A}_k^2$. Indeed, this is just $\{xy = 1\} \subseteq U_1 \times U_2 = \mathbb{A}_k^2$, which is a closed subset of $\mathbb{A}_k^2$, then P_k^1 is separated over k. □

Definition 21.2 (Affine Morphism)**.** A morphism of schemes $f : X \to Y$ is *affine* if Y has an open covering by affine schemes U_α such that $f^{-1}(U_\alpha)$ is affine.

Lemma 21.3. Every affine morphism is separated.

Proof. Let $f : X \to Y$ be an affine morphism. The problem is local on$Y$, so we can assume that Y is affine, say $Y = \mathrm{Spec}(A)$. Therefore, X is also affine, and $X = \mathrm{Spec}(B)$. Here $X \times_Y X = \mathrm{Spec}(B \otimes_A B)$. The morphism $\Delta_X : X \to X \times_Y X$ (with projections π_1 and π_2 both onto X)corresponds to the ring homomorphism $B \rightrightarrows B \otimes_A B \to B$ where the last map is given by $b_1 \otimes b_2 \mapsto b_1 b_2$. Clearly this ring homomorphism is surjective. So $\Delta_X : X \to X \times_Y X$ is a closed immersion, so X is separated over Y. □

Definition 21.4 (Quasi-projective Scheme)**.** A *quasi-projective scheme* X over a field k is an open subscheme of a closed subscheme of P_k^n for some $n \geq 0$.

Remark 21.5.
1. Every open immersion is separated.
2. Every quasi-projective scheme X over a field k is separated over k.

Proof. Use that any composite of separated morphisms is separated. Here X is quasi-projective and is open in $\bar{X}$, which is a closed immersion of $P_k^n \to \mathrm{Spec}(k)$.

Note that closed immersion implies finite implies affine implies separated. And, for any $n \geq 0$, P_k^n is separated over k, by the same argument as for P_k^1. (In fact, this argument works for P_A^n for any ring A.) □

Remark 21.6. For a scheme X of finite type over $\mathbb{C}$, X is separated over $\mathbb{C}$ if and only if the space $X(\mathbb{C})$ in the classical topology is Hausdorff.

Definition 21.7 (Algebraic Variety)**.** An *algebraic variety* over a field k is an integral separated scheme of finite type over k, i.e., as a scheme morphism $X \to \text{Spec}(k)$.

Remark 21.8. Several big properties in algebraic geometry are preserved by arbitrary base change. That is, if a property is true on $f : X \to Y$, then for any morphism $Z \to Y$, the map $f_Z : X_Z \to Z$ given by the fiber product should have the same property, i.e., invariant under base change.

Such properties include: closed immersions, open immersions, affine morphisms, locally of finite type, finite type, finite, separated.

Exercise 21.9. A topological space X is compact if and only if the continuous map $X \to *$ is *universally closed*, that is, for any topological space Y, the induced map $X \times_k Y \to Y$ is closed as well.

Example 21.10. Use this to show that $\mathbb{R}$ is not compact, since $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x$ is not closed.

Definition 21.11 (Closed, Universally Closed, Proper)**.** A morphism $f : X \to Y$ of schemes is *closed* if the image of every closed subset of X via f is closed in Y.

A morphism $f : X \to Y$ is *universally closed* if every base change of $X \to Y$ is closed.

A morphism $f : X \to Y$ is *proper* if it is separated, of finite type, and universally closed.

Example 21.12. $\mathbb{A}^1_k$ is not proper over k, for a field k.

Proof. $\mathbb{A}^1_k \to \text{Spec}(k)$ is not universally closed since $\{xy = 1\} \subseteq \mathbb{A}^1_k$ is a closed subset and its image in $\mathbb{A}^1_k$ is $\mathbb{A}^1_k \backslash \{0\} \subseteq \mathbb{A}^1_k$, which is not closed in $\mathbb{A}^1_k$. □

Theorem 21.13. For every $n \geq 0$, $P^n_{\mathbb{Z}}$ is proper over $\mathbb{Z}$.

Remark 21.14.

1. Properness is preserved by arbitrary pullback. Therefore, for every commutative ring R, P^n_R is proper over R.

22 Lecture 22

Remark 22.1. Let $f : X \to Y$ be a morphism of separated schemes of finite type over $\mathbb{C}$. Then f is proper if and only if $f : X(\mathbb{C}) \to Y(\mathbb{C})$ is proper (for the classical topology), meaning that the preimage of compact subset is compact.

Example 22.2. The inclusion $\mathbb{A}^1_{\mathbb{C}}\backslash\{0\} \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$ is not proper.

Lemma 22.3.
1. A finite morphism is proper (separated, of finite type, and universally closed), hence a closed immersion is proper.
2. A composite of proper morphisms is proper.
3. Properness is preserved by arbitrary base change.

Theorem 22.4. For every $n \geq 0$, $P^n_{\mathbb{Z}}$ is proper over $\mathbb{Z}$.

Corollary 22.5. For every commutative ring R, and $n \geq 0$, P^n_R is proper over R. More generally, for any scheme X and $n \geq 0$, $P^n_X = P^n_{\mathbb{Z}} \times_{\mathbb{Z}} X$ is proper over X.

Corollary 22.6. Every projective scheme over a commutative ring R is proper over R.

Proof. Let $X \subseteq P^n_R$ be a closed subscheme, then there is a closed immersion $X \hookrightarrow P^n_R$ and proper morphism $P^n_R \to \operatorname{Spec}(R)$, then $X \to \operatorname{Spec}(R)$ is proper. □

Proof of Theorem. The slogan of this argument is called "elimination theory".

We have mentioned that $P^n_{\mathbb{Z}}$ is separated over $\mathbb{Z}$. Clearly it is of finite type over $\mathbb{Z}$. We now want to show that for every scheme X, $P^n_X \to X$ is closed. That is, for every closed subset $Z \subseteq P^n_X$, its image in X is closed.

The problem is local on X, so we can assume that X is affine, and so $X = \operatorname{Spec}(A)$ for some ring A. Given a closed subset $Z \subseteq P^n_A$, here Z is the zero set $\{f_1 = 0, f_2 = 0, \ldots\}$, with $f_1, f_2, \ldots \in A[x_0, \ldots, x_n]$ as homogeneous of positive degree. Let $\pi : P^n_A \to \operatorname{Spec}(A)$ be the projection, and let $p \in \operatorname{Spec}(A)$, then $p \in \pi(Z)$ if and only if the fiber of Z over $\operatorname{Spec}(k(p))$ is not empty, if and only if $\{f_1, f_2, \ldots\} \subseteq P^n_{k(p)}$ is not empty, but these polynomials are mapped to polynomials in $k(p)[x_0, \ldots, x_n]$, so this is true if and only if $\{f_1 = 0, f_2 = 0, \ldots\} \subseteq \mathbb{A}^{n+1}_k$ is more than just the origin, if and only if $\operatorname{rad}(f_1, f_2, \ldots)$ does not contain $I = (x_0, \ldots, x_n)$, if and only if for every $r \geq 0$, $(f_1, f_2, \ldots)$ does not contain I^r, if and only if for all $r \geq 1$, the $k(p)$-linear map $\bigoplus\limits_i k(p)[x_0, \ldots, x_n]_{r-a_i} \to k(p)[x_0, \ldots, x_n]_r$ is not surjective, where $a_i = \deg(f_i)$, and this map is defined by $(g_1, g_2, \ldots) \mapsto \sum\limits_i f_i g_i$. Note that this map comes (by $\otimes_A k(p)$) from an A-linear map

$$\bigoplus_i A[x_0, \ldots, x_n]_{r-a_i} \to A[x_0, \ldots, x_n]_r$$

$$(g_1, g_2, \ldots) \mapsto \sum f_i g_i$$

This A-linear map is given by a matrix over A of size $\sum\limits_i \binom{n+r-a_i}{r-a_i} \times \binom{n+r}{r}$. Therefore, $p \in \pi(Z)$ if and only if for all $r \geq 1$, all minors of size $\binom{n+r}{r} \times \binom{n+r}{r}$ map to zero in $k(p)$. So

$\pi(p) \subseteq \operatorname{Spec}(A)$ is the zero set of all minors of size $\binom{n+r}{r} \times \binom{n+r}{r}$ in the matrix above, for all $r \geq 1$. This is a closed subset of $\operatorname{Spec}(A)$, and is the zero set of an infinite set of elements of A, so $P^n_{\mathbb{Z}} \to \operatorname{Spec}(\mathbb{Z})$ is proper. $\square$

Lemma 22.7. For a proper scheme X over scheme S and a separated scheme Y over S, and a morphism $X \to Y$ over S:

$$\begin{array}{ccc} X & \xrightarrow{\qquad\qquad} & Y \\ & {\scriptstyle\text{proper}}\searrow \qquad \swarrow{\scriptstyle\text{separated}} & \\ & S & \end{array}$$

Then $f(X)$ is a closed subset of Y.

Remark 22.8. We can compare it to a theorem in topology: For X a compact topological space and Y a Hausdorff space, and $f : X \to Y$ is continuous, then $f(X)$ is closed in Y.

Proof. Factor f as $X \xrightarrow{(1_X,f)} X \times_S Y \xrightarrow{\pi_Y} Y$. (The image of the first map is called the graph of f. The second map is the projection onto Y.) Since $X \to S$ is proper over S, then $X \times_S Y \to Y$ is proper, hence closed. So it suffices to show that the graph of f is closed in $X \times_S Y$. Here the graph of f is the inverse image of $\Delta_Y \subseteq Y \times_S Y$ by the morphism $X \xrightarrow{(1_X,f)} X \times_S Y$. Since Y is separated over S, Δ_Y is closed in $Y \times_S Y$. So the inverse image is contained in $X \times_S Y$ and is closed. $\square$

Corollary 22.9. Let X be a quasi-projective scheme over a field k. So X is an open subscheme of a closed subscheme of P^n_k for some $n \geq 0$. Then X is proper over k if and only if X is closed in P^n_k (as a subset).

Proof. ($\Leftarrow$): We have shown.

($\Rightarrow$): Let X be quasi-projective and proper over k. Apply Lemma 22.7 to the given map $X \to P^n_k$. Since X is proper over k and P^n_k is separated over k, then X is closed in P^n_k by Lemma 22.7. $\square$

23 Lecture 23

Theorem 23.1. Let X be a proper variety over an algebraically closed field, then $\mathcal{O}(X) = k$.

Remark 23.2. This is totally false for $\mathbb{A}^1_k$, since $\mathcal{O}(\mathbb{A}^1_k) = k[x]$.

Proof. Let $f \in \mathcal{O}(X)$. (Here $\mathcal{O}$ is integral, hence $X \neq \varnothing$, so the homomorphism $k \to \mathcal{O}(X)$ is injective.) We want to show that $k \to \mathcal{O}(X)$ is surjective. Think of f as a morphism of schemes over k, as $f : X \to \mathbb{A}^1_k$, then this corresponds to $k[x] \to \mathcal{O}(X)$ defined by $x \mapsto f$.

Since f is proper over k and $\mathbb{A}^1_k$ is separated over k, the image $f(X) \subseteq \mathbb{A}^1_k$ is a closed subset. So $f(X)$ is either all of $\mathbb{A}^1_k$ or a finite set of closed points. Consider the composition $X \xrightarrow{f} \mathbb{A}^1_k \hookrightarrow P^1_k$. Since X is proper over k and P^1_k is separated over k, then $f(X)$ is closed in P^1_k. So $f(X)$ is not all of $\mathbb{A}^1_k$, so in fact $f(X) \subseteq \mathbb{A}^1_k$ is a finite set of closed points. Here X is integral so X is irreducible as a topological space, then $f(X)$ is just a one-point space. As a set, $\mathbb{A}^1_k = k \cup \{*\}$ where $*$ is the generic point. So there is an element $a \in k$ such that $Z(f-a) = X$. That means $f - a \in \mathcal{O}(X)$ restricted to every affine open subscheme of X is nilpotent.

Example 23.3. Let $Y = \{x^2 = 0, xy = 0\} \subseteq \mathbb{A}^2_{\mathbb{C}}$, then $Z(x) = Y$, but $x \neq 0 \in \mathcal{O}(Y)$, but $x^2 = 0 \in \mathcal{O}(Y)$.

Here X is integral, so X is reduced, and therefore $f - a \mid_U = 0 \in \mathcal{O}(U)$ for every affine open subset U, hence $f - a = 0 \in \mathcal{O}(X)$. □

Remark 23.4. If X is a proper variety over a field k (perhaps not algebraically closed), then $\mathcal{O}(X)$ is a field, a finite extension of k.

Definition 23.5 (Rational Function)**.** Let X be an integral scheme. Then a *rational function* on X is an element of $\mathcal{O}_{X,p}$, where p is the generic point of X.

Remark 23.6. Let X be an integral scheme, and let $U \subseteq X$ be a non-empty, open affine subscheme, i.e., $U = \mathrm{Spec}(R)$. Here R is a domain. Then the prime ideal $(0) \in \mathrm{Spec}(R)$ is a generic point, i.e., its closure in U is all of U. By irreducibility of X, the closure in X of this point in U is all of X. (Recall the closure of $p \in \mathrm{Spec}(R)$ is just $Z(p)$. Note that the generic point of X is now contained in every non-empty open subset.)

Definition 23.7 (Function Field)**.** The *function field* of an integral scheme means the local ring $\mathcal{O}_{X,p}$, where p is the generic point.

Recall that an element of $\mathcal{O}_{X,p}$ is represented by a regular function f on some open neighborhood of p in X, i.e., on a non-empty open subset of X. Let $U = \mathrm{Spec}(R) \subseteq X$ be a non-empty affine open subscheme, so R is a domain. Then $\mathcal{O}_{X,p} = \mathcal{O}_{U,p} = R_{(0)}$, i.e., localized at prime ideal, which is just the fraction field of R, as a field.

Example 23.8. The function field of $\mathbb{A}^n_k$ (for field k and $n \geq 0$) is $\mathrm{Frac}(k[x_1, \ldots, x_n]) = k(x_1, \ldots, x_n)$, the field of rational functions.

The function field of P^n_k is also isomorphic to $k(x_1, \ldots, x_n)$, since P^n_k also contains $\mathbb{A}^n_k$ as a non-empty open subset.

Example 23.9. The function field of the hypersurface $\{x_{n+1}^2 = g(x_1, \ldots, x_n)\} \subseteq \mathbb{A}_k^{n+1}$ for a field k, $g(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ is not a square. This is a variety over k, in particular integral. Therefore the function field of X is $k(x_1, \ldots, x_n)(\sqrt{g})$.

Definition 23.10 (Rational Map)**.** Let X and Y be varieties over a field k. A *rational map* $f : X \dashrightarrow Y$ (not necessarily a morphism) is given by a morphism over k from some non-empty open subset U of X to Y, i.e., $f : U \to Y$. (Precisely, a rational map is an equivalence class of pairs (U, f), with equivalent relation: (U, f), with equivalence relation: $(U_1, f_1) \sim (U_2, f_2)$ if there is a non-empty open $V \subseteq U_1 \cap U_2$ with $f_1 |_V = f_2 |_V$.

Definition 23.11 (Dominant)**.** A rational map $f : X \dashrightarrow Y$ of varieties (over a field k) is *dominant* if, for some open subset $U \subseteq X$ on which f is defined, $f(U)$ is dense in Y. (Note that this is independent of the choice of U.)

Remark 23.12. Equivalently, a rational map is dominant if and only if maps generic point of X to generic point of Y.

Definition 23.13 (Birational Map)**.** A *birational map* $f : X \dashrightarrow Y$ of varieties over k is a dominant rational map over k that has a rational inverse. That is, there is a rational map $g : Y \dashrightarrow X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$.

Remark 23.14. Dominant rational maps and varieties form a category, but rational maps do not.

Definition 23.15 (Birational Variety)**.** Two varieties X and Y over k are *birational* if there is a birational map $f : X \dashrightarrow Y$. (Clearly this is an equivalence relation on varieties.)

Remark 23.16. Two varieties X and Y over k are birational if and only if some non-empty open $U \subseteq X$ is isomorphic (over k) to some non-empty open subset of Y.

Example 23.17. $\mathbb{A}_k^n$ is birational to P_k^n because $\mathbb{A}_k^n$ is isomorphic to an open subset of P_k^n.

24 Lecture 24

Definition 24.1 (Birational Variety)**.** We say varieties X and Y over a field k are *birational* if there exists non-empty open subsets $U \subseteq X$ and $V \subseteq Y$, with $U \cong V$ over K. Here $X \backslash U$ and $Y \backslash V$ have dimension less than $\dim(X) = \dim(Y)$.

Definition 24.2 (Rational Variety)**.** A variety X over a field k is *rational* over k if it is birational to P_k^n (where $n = \dim(X)$), or equivalently birational to $\mathbb{A}_k^n$.

Example 24.3. Let k be a field of characteristic not 2. Then the affine curve $X = \{x^2+y^2 = 1\} \subseteq \mathbb{A}^2_k$ is a rational curve.

Remark 24.4.
- A curve is a variety of dimension 1 over a field.
- A surface has dimension 2.
- A 3-fold has dimension 3, etc.

Proof. To show that X is rational over k, note that the line through $(1,0)$ with slope $t \in k$ is given by $y = t(x-1)$. Intuitively, we should expect this line to meet X in 2 points. These points should include $(1,0)$ and another point, so we need to find the other one. To determine the intersection, we have $x^2 + t^2(x-1)^2 = 1$, so $x^2(t^2+1) - 2t^2x + (t^2-1) = 0$. Here $x-1$ is a solution to this equation, so we must be able to factor out $x-1$. We get $(x-1)[(t^2+1)x - (t^2-1)] = 0$. So the other point of the intersection is given by $x = \frac{t^2-1}{t^2+1}$, a rational function in t. This gives a rational map $\mathbb{A}^1_k \dashrightarrow X \subseteq \mathbb{A}^2_k$ explicitly. Namely, it is sending $t \in \mathbb{A}^1_k$ to

$$\left(\frac{t^2-1}{t^2+1}, t\left(\frac{t^2-1}{t^2+1} - 1\right)\right) = \left(\frac{t^2-1}{t^2+1}, \frac{-2t}{t^2+1}\right).$$

Note that this is not defined in the closed subset $\{t^2+1=0\} \subseteq \mathbb{A}^1_k$. The rational inverse map $X \dashrightarrow \mathbb{A}^1_k$ over k is $(x,y) \mapsto \frac{-y}{1-x} = \frac{y}{x-1}$. Correspondingly, this is not defined at the point $(1,0) \in X(k)$. ☐

Remark 24.5. This shows that for any field k of characteristic not 2, and $X(k) = \{x^2+y^2 = 1\}$, we have $X(k) \backslash S \cong \mathbb{A}^1_k(k) \backslash T$ for some finite sets S and T.

This is interesting for $k = \mathbb{Q}$: we have described all $\mathbb{Q}$-solutions of the equation $x^2+y^2 = 1$. That gives the $\mathbb{Z}$-solution of $x^2+y^2 = z^2$, i.e., $\left(\frac{x}{z}\right) + \left(\frac{y}{z}\right)^2 = 1$.

Remark 24.6. For a variety X over a field k, the field of rational functions $k(X)$ is a finitely-generated extension field of k.

Proof. That is, there are finitely many elements $f_1, \ldots, f_N \in k(X)$ such that $k(X)$ is the smallest subfield containing k, and $f_1, \ldots, f_N$. Let $\mathrm{Spec}(A) \subseteq X$ be a non-empty affine open subscheme, then $k(X) = \mathrm{Frac}(A)$ for a domain A. Then A is a finitely-generated k-algebra. Let $f_1, \ldots, f_n$ be generators for A as a k-algebra, then these elements generate $k(X)$ as afield over k. ☐

Remark 24.7. Every finitely-generated field extension of a field k is the function field of some variety X/k. We can choose X to be either affine or projective.

Proof. Let $f_1, \ldots, f_N \in E$ generate E as an extension of k. Let A be the k-subalgebra of E generated by $f_1, \ldots, f_N$. Clearly A is a finitely-generated k-algebra and a domain. So $X = \operatorname{Spec}(A)$ is an affine variety over k, then $\operatorname{Frac}(A) \subseteq E$, and this is an equality because $f_1, \ldots, f_N$ generate E over k. This shows the affine case.

We have a closed immersion $X \hookrightarrow \mathbb{A}_k^N$ with the backwards map $k[x_1, \ldots, x_n]/I \hookleftarrow k[x_1, \ldots, x_N]$. Think of open subset $\mathbb{A}_k^N P_k^N$, and let $\bar{X}$ be the closure of X in P_k^n. This is a projective variety over k. We have $k(\bar{X}) \cong k(X) = E$. This shows the projective case. □

Theorem 24.8. Let k be a field, then there is an arrow-reversing equivalence of categories between

- category of varieties over k with dominant rational maps over k, and
- category of finitely-generated field extensions of k, with morphisms being injective homomorphisms of fields over k.

Remark 24.9. Two varieties over k are isomorphic in the first category if and only if they are birational over k.

Proof Sketch. Go from a variety X to the field $k(X)$, which contains k. Given a dominant rational map $f : X \dashrightarrow Y$, we get an inclusion of fields $k \subseteq k(Y) \subseteq k(X)$. □

Theorem 24.10. Every variety of dimension n over a field k of characteristic 0 is birational to a projective hypersurface in P_k^{n+1}.

Remark 24.11. Note that Y usually has to be singular.

Geometric Argument. X is birational to an affine variety over k, hence to a projective variety called Y. Consider a general linear projection $X \to Y$ (which is not a morphisms since there is somewhere not defined). One can show that this is birational though usually not an isomorphism. □

Algebraic Argument. The field $k(X)$ is a finitely-generated extension field of k, so we can factor $k \subseteq k(t_1, \ldots, t_n) = \operatorname{Frac}(k[t_1, \ldots, t_n]) \subseteq k(X)$ as a finite extension. Here the transcendental degree of $k(X)/k$ is n, which is also the dimension of X. Since k has characteristic 0, $k(t_1, \ldots, t_n)$ is a perfect field. Then $k(X)$ is a finite separable extension of $k(t_1, \ldots, t_n)$. That is, $\operatorname{Spec}(k(X))$ is a smooth 0-dimensional variety over $k(t_1, \ldots, t_n)$. Then by the Primitive Element Theorem, $k(X)$ is generated by a single element $u \in k(X)$ as an extension of $k(t_1, \ldots, t_n)$. So $k(X)$ is generated as an extension of k by $n+1$ elements $t_1, \ldots, t_n, u \in k(X)$.

Now let A be the k-algebra generated by $t_1, \ldots, t_n, n$, then $Y = \text{Spec}(A)$ is an affine variety in $\mathbb{A}_k^{n+1}$, and $\text{Frac}(A) = k(Y) \cong k(X)$, so $\dim(Y) = n$, so Y is a hypersurface in $\mathbb{A}_k^{n+1}$. Now take the closure of Y in P_k^{n+1}, we get a projective hypersurface. $\square$

25 Lecture 25

Lemma 25.1 (Noether Normalization Lemma)**.** Let k be a field and A is a non-zero finitely-generated k-algebra. Then there is $n \geq 0$ and inclusion $k[x_1, \ldots, x_n] \hookrightarrow A$ of k-algebras such that A is finitely-generated as a module over $k[x_1, \ldots, x_n]$. Moreover, $n = \dim(A)$.

Remark 25.2. Geometrically, that says: let k be a field, X is a non-empty affine scheme of finite type over k. Then there is an $n \geq 0$ (namely, $n = \dim(X)$) and a finite surjective morphism $X \to \mathbb{A}_k^n$ over k.

Remark 25.3 (Smooth Scheme over field k)**.** Let k be a field, $X \subseteq \mathbb{A}_k^n$ be a closed subscheme. So $X = \{f_1 = 0, f_2 = 0, \ldots\} \subseteq \mathbb{A}_k^n$ for some $f_1, f_2, \ldots \in k[x_1, \ldots, x_n]$. Let p be a point in X (not necessarily a k-point or even a closed point). The matrix of derivatives $\left(\frac{\partial f_i}{\partial x_j}\right) |_p \in M_{r \times n}(k(p))$ can be given by $dF : k[x_1, \ldots, x_n] \to k(p) = \text{Frac}(k[x_1, \ldots, x_n]/p)$ viewed as a linear map $k(p)^n \to k(p)^r$.

Definition 25.4 (Zariski Tangenet Space)**.** The *Zariski tangent space* to X at p is $\ker(dF) \subseteq k(p)^n$.

Suppose that X has dimension m near p. Then we always have $m \subseteq \dim(T_p(X))$, then we always have $m \leq \dim_{k(p)} T_p(X)$.

Definition 25.5 (Smooth)**.** X is *smooth* over k at p if equality holds.

Example 25.6. Let $X \subseteq \mathbb{A}_k^1$ be a closed 0-dimensional subvariety. When is smooth over k? For example, let $X = \{x^2 = a\} \subseteq \mathbb{A}_k^1 = \text{Spec}(k[x])$ for some given $a \in k$. This subscheme is a subvariety if and only if the polynomial $x^2 - a$ is irreducible over k, if and only if a is not a square in k. Suppose $a \in k$ is not a square, so X is an affine variety of dimension 1 over k, and $X = \text{Spec}(k(\sqrt{a})) = \text{Spec}(k[x]/(x^2 - a))$. Then X is smooth in k if and only if $\partial x = \frac{\partial}{\partial x}(x^2 - a) |_p \neq 0$, if and only if $\partial x \neq 0 \in k(p) = k[x]/(x^2 - a)$.

Case 1: $\text{char}(k) \neq 2$. Then $2 \neq 0 \in k(p)$, and so $x \neq 0$ since $a \neq 0 \in k$. So, in $\text{char}(k) \neq 2$, X is smooth over k.

Case 2: $\text{char}(k) = 2$, then X is not smooth over k. This is equivalent to saying $k(\sqrt{a})$ is an inseparable extension of k in characteristic 2.

Remark 25.7. Smoothness is a relative notion on schemes, i.e., a property of the morphism $X \to \text{Spec}(k)$.

Theorem 25.8 (Segre Embedding)**.** Let k be a field. For any poisitive integers a, b there is a closed immersion $P_k^{a-1} \times_k P_k^{b-1} \hookrightarrow P_k^{ab-1}$ called the *Segre embedding*.

Remark 25.9. Therefore, the product of any two projective schemes over k is projective over k. (Since if closed $X \subseteq P^{a-1}$ and closed $Y \subseteq P^{b-1}$ then $X \times Y \subseteq P^{a-1} \times P^{b-1} \subseteq P^{ab-1}$.)

Remark 25.10. It is easy to see that the product of two proper schemes over k is also proper over k, because properness is preserved by base change.

Also, it follows from Segre embedding that the product of two quasi-projective schemes is also quasi-projective over k.

Proof. Let V and W be vector spaces over k with $\dim_k(V) = a$ and $\dim_k(W) = b$. Think of $P(V) = P^{a-1}$ and $P(W) = P^{b-1}$. (Here $P(V)$ means the set of 1-dimensional k-linear subspaces of V.) Here $\dim_k(V \otimes_k W) = ab$, so we want a closed immersion $P(V) \times_k P(W) \hookrightarrow P(V \otimes_k W)$. Namely the Segre morphism is $[L] : [M] \mapsto [L \otimes_k M]$ where $L \subseteq V$ and $M \subseteq W$ are 1-dimensional linear subspaces. Equivalently, for non-zero $v \in V$, write $[v]$ to be the line $k \cdot v \subseteq V$, then $[v] \times [w] \mapsto [v \otimes w]$. Note that $(av) \otimes w = a(v \otimes w)$. In terms of bases $V \cong k^a$ and $W \cong k^b$, the morphism is $\left(\sum_{i=1}^{a} a_i e_i\right) \otimes \left(\sum_{i=1}^{b} b_i f_i\right) = \sum_{i,j} a_i b_j (e_i \otimes f_j)$ where $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are bases for V and W, respectively. So the Segre embedding is $[x_1, \ldots, x_a] \times [y_1, \ldots, y_b] \mapsto [x_1y_1, x_1y_2, \ldots, x_ay_b]$ in lexicographical order.

Why is $f : P_k^{ab-1} \times)_k P_k^{b-1} \hookrightarrow P_k^{ab-1}$ a closed immersion? We can work over open subsets of P^{ab-1}, say the open subset $\{Z_{ij} \neq 0\} \cong \mathbb{A}_k^{ab-1}$. Look at the open set $\{Z_{11} \neq 0\} \cong \mathbb{A}_k^{ab-1}$. The restriction $f^{-1}(U) \to U$ is $\mathbb{A}_k^{a-1} \times_k \mathbb{A}_k^{b-1} \xrightarrow{f} \mathbb{A}^{ab-1}$. The morphism is mapping $([1, x_2, \ldots, x_a],$ $[1, y_2, \ldots, y_b])$ to $[1, x_2, \ldots, x_1, y_2, \ldots, y_b, \ldots, x_i y_j]$ with $2 \leq i \leq a$ and $2 \leq j \leq b$. An inverse map is given by $Z_{ij} \mapsto (x_2 = Z_{21}, \ldots, y_2 = Z_{12}, \ldots, x_a = Z_{a1}, \ldots, y_b = Z_{1b})$, where $Z_{ij} = (Z_{i1})(Z_{1j})$ for $2 \leq i \leq a$ and $2 \leq j \leq b$, so the image of f is a closed subscheme given by the form above. $\square$

26 Lecture 26

Example 26.1. Recall that Segre embedding $P_k^{a-1} \times_k P_k^{b-1} \to P_k^{ab-1}$. When $a = b = 2$, the Segre embedding is $([u_{,}0, u_1], [v_0, v_1]) \mapsto [u_0v_0, u_0v_1, u_1v_0, u_1v_1] \in P^3$. The image is the surface $xw = yz \subseteq P_k^3$.

Definition 26.2 (Quadratic Hypersurface)**.** A *quadratic hypersurface* means a hypersurface $X = \{f = 0\} \subsetneq P_k^{n+1}$ with f homogeneous of degree 2 over k. where $f \neq 0$. Similarly we can define hyperplane (of degree 1), cubic hypersurface (of degree 3), quartic hypersurface (of degree 4), etc.

Remark 26.3. Over an algebraically closed field k, all smooth quadratics of dimension n are isomorphic.

Remark 26.4 (Plane Conics over $\mathbb{R}$)**.** In $\mathbb{A}^2_{\mathbb{R}}$, there are three nondegenerate types of conics: ellipses, parabolas, and hyperbolas. In $P^2_{\mathbb{R}}$, they all are isomorphic. Ellipses do not intersect the line at infinity. Parabolas are tangent to it. Hyperbolas intersect it at two points.

Theorem 26.5 (Krull's Projective Ideal Theorem)**.** Let A be a Noetherian ring and $f \in A$ is not a zero divisor. Then every minimal prime ideal containing f has codimension 1.

Definition 26.6 (Codimension)**.** The *codimension* of a prime ideal $\mathfrak{p}$ in a ring A is the supremum of the length of chains if prime ideal contained in $\mathfrak{p}$, i.e., $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}$.

Remark 26.7. This is given by the irreducible closed subset of $\operatorname{Spec}(A)$ containing $Z(\mathfrak{p}) \subseteq \operatorname{Spec}(A)$.

Remark 26.8. In other words, the theorem says: if X is a Noetherian affine scheme and $f \in \mathcal{O}(X)$ is not a zero divisor in $\mathcal{O}(X)$, then every irreducible component of $\{f = 0\} \subseteq \operatorname{Spec}(A)$ has codimension 1 in $\operatorname{Spec}(A)$.

Theorem 26.9 (Affine Dimension Theorem)**.** Let k be a field and let Y, Z be subvarieties of $\mathbb{A}^n_k$. Let $r = \dim(Y)$ and $s = \dim(Z)$, then every irreducible component of $Y \cap Z$ has dimension at least $r + s - n$.

Proof. First, suppose that Z is a hypersurface (a subvariety with codimension 1) in $\mathbb{A}^n_k$. Because $k[x_1, \ldots, x_n]$ is a UFD, every codimension 1 prime ideal in $k[x_1, \ldots, x_n]$ is principal. So $Z = \{f = 0\} \subsetneq \mathbb{A}^n_k$ with $f \in k[x_0, \ldots, x_n]$ is irreducible over k. Here $Y \cap Z = \{f \mid_Y = 0\}$ inside Y. If $f \mid_Y = 0 \in \mathcal{O}(Y)$, then $Y \cap Z = Y$, so $Y \cap Z$ has the same dimension as the dimension of Y, which is $r \geq r + s - n = r - 1$ (since $s = n - 1$ here). Otherwise, $f \in \mathcal{O}(Y)$ is not a zero divisor since $\mathcal{O}(Y)$ is a domain. By Krull's theorem, $Y \cap Z = \{f = 0\} \subseteq Y$ has every irreducible component of dimension equals to $\dim(Y) - 1 = r - 1 = r + s - n$.

That argument extends to the case where Z is a complete intersection, i.e., $Z \subseteq \mathbb{A}^n_r$ can be defined by only $n - s$ equations, which is also the codimension of $Y \subseteq \mathbb{A}_n$. In that case, if $Z = W_1 \cap \cdots \cap W_{n-s}$ hypersurfaces, then $Y \cap Z = Y \cap W_1 \cap \cdots \cap W_{n-s}$. So every irreducible component of $Y \cap Z$ has dimension at least $r - (n - s) = r + s - n$.

Now consider $Y \times Z \subseteq \mathbb{A}_k^n \times_k \mathbb{A}_k^n$, then $Y \cap Z = (Y \times_k Z) \cap \Delta_{A_k^n} \subseteq \mathbb{A}_k^n \times \mathbb{A}_k^n \cong \mathbb{A}_k^{2n}$, where $Y \times_k Z$ has dimension $r + s$. Here $\delta_{\mathbb{A}_k^n}$ is a complete interior of $\mathbb{A}_k^{2n}$, since $\Delta_{A^n} = \{x_1 = y_1, \ldots, x_n = y_n\} \subseteq \mathbb{A}_k^n \times \mathbb{A}_k^n$. Then by our previous argument, every irreducible component $\{(x_1, \ldots, x_n, y_1, \ldots, y_n)\}$ of $Y \cap Z = (Y \times Z) \cap \Delta_{\mathbb{A}_k^n}$ has dimension at least $r + s - n$. □

Theorem 26.10 (Projective Dimension Theorem)**.** Let Y, Z be subvarities of P_k^n where k is a field. Let $r = \dim(Y)$ and $s = \dim(Z)$, then every irreducible component of $Y \cap Z$ has dimension at least $r + s - n$. Moreover, if $r + s - n \geq 0$, then $Y \cap Z$ is not empty.

Proof. Consider the affines $CY, CZ \subseteq \mathbb{A}_k^{n+1}$. here $\dim(CY) = r + 1$ and $\dim(CZ) = s + 1$. By the affine dimension theorem, every irreducible component of $C(Y \cap Z) = CY \cap CZ$ has dimension at least $(r+1) + (s+1) - (n+1) = r + s - n + 1$. So every irreducible component of $Y \cap Z$ has dimension at least $r + s - n$.

Suppose now that $r + s - n \geq 0$. As above, every irreducible component of $C(Y \cap Z)$ has dimension at least $r + s - n + 1 \geq 1$, but this interior contains $0 \in \mathbb{A}_k^{n+1}$. So $C(Y \cap Z)$ has an irreducible component of dimension at least 1, so $Y \cap Z \subseteq P_k^n$ is not empty. □

27 Lecture 27

Definition 27.1 (Degree of Projective Variety)**.** Let $X \subseteq \mathbf{P}_k^n$ be a closed subscheme for k a field and $n \geq 0$. Let $r = \dim(X)$. The *degree* of X is, for a linear subspace $L \subseteq \mathbf{P}_k^n$ of codimension r in $\mathbf{P}_k^n$ such that $X \cap L$ is finite, the dimension of $\mathcal{O}(X \cap L)$ as a k-vector space.

Remark 27.2. This is well-defined (independent of choice of L). It is unchanged under field extensions. In particular, $\deg_k(X) = \deg_{\bar{k}}(X_{\bar{k}})$.

Example 27.3. If k is algebraically closed and if X intersects L transcendental, then $\deg(X) = \#(X \cap L)$. Indeed, this is given by $T_pX \oplus T_pL \twoheadrightarrow T_pP_k^n$ for $p \in (X \cap L)(k)$.

Example 27.4. The degree of every linear subspace of $P^n = 1$. Degree of hypersurface of degree d is d.

Remark 27.5. Note that a subvariety has degree 1 if and only if $X \subseteq P^n$ is a linear subspace.

Remark 27.6 (Bezout's Theorem)**.** For two subvarieties $X, Y \subseteq P^n$ that intersect in the expected dimension (i.e., equals $\dim(X) + \dim(Y) - n$), then (as a scheme) $\deg(X \cap Y) = \deg(X)\deg(Y)$.

Example 27.7. The twisted cubic curve X in P^3 is not a complete intersection, i.e., there are not hypersurfaces $Y, Z \subseteq P^3$ such that $X = Y \cap Z$ as schemes.

Proof. X is the image of the 3rd Veronese map $P^1 \hookrightarrow P^3$ given by $[u, v] \mapsto [u^3, u^2v, uv^2, v^3]$. Here $\deg(X) = 3$, since the intersection of X and a hyperplane is the zero set of degree-3 homogeneous polynomial on p. Therefore, $\dim_k(\mathcal{O}(X \cap L)) = 3$, and so $\deg(X) = 3$, given by $\{au^3 + bu^2v + cuv^2 + dv^3 = 0\} \subseteq P^1$. Therefore, if the twisted cubic curve in P^3 were a complete intersection in P^3l, then $X = Y \cap Z$, then $3 - \deg(Y)\deg(Z)$ by Bezout's theorem. Since 3 is prime, $\deg(Y) = 1$ and $\deg(Z) = 3$. Hence, $\deg(Y)$ is a hyperplane. But it is easy to check that X is not contained in any hyperplane, so it is not a complete intersection in P^3. □

Remark 27.8. The twisted cubic $X \cong P^1 \subseteq P^3$ can be defined (as a scheme) by the equation $x_0x_2 = x_1^2$, $x_1x_3 = x_2^2$, and $x_1x_2 = x_0x_3$.

Theorem 27.9. Let R be a Noetherian ring and M be a finitely-generated R-module. Then the following are equivalent:

1. M is projective,
2. M is flat,
3. M is locally free, i.e, there are $g_1, \ldots, g_r \in R$ such that $(g_1, \ldots, g_r) = R$, and such that $M\left[\frac{1}{g_i}\right]$ is a free module over $R\left[\frac{1}{g_i}\right]$.

Definition 27.10 (Sheaf of Modules)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* is a sheaf of Abelian groups M on X such that for each open subset $U \subseteq X$, $M(U)$ is given the structure of an $\mathcal{O}(U)$-module, such that for every open subset $V \subseteq U \subseteq X$, the diagram commutes:

$$\begin{array}{ccc} \mathcal{O}(U) \times M(U) & \longrightarrow & M(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(V) \times M(V) & \longrightarrow & M(V) \end{array}$$

We then say M is an $\mathcal{O}_X$-module.

Remark 27.11. The category of $\mathcal{O}_X$-module is Abelian. Given a map $f : A \to B$ of $\mathcal{O}_X$-modules, then the sheaves $\ker(f)$, $\operatorname{im}(f)$, and $\operatorname{coker}(f)$ are $\mathcal{O}_X$-modules. Recall that if $C = \operatorname{im}(f : A \to B)$, then we may have $C(U) \neq \operatorname{im}(f : A(U) \to B(U))$.

We also have the notions of direct sum ($(A \oplus B)(U) = A(U) \oplus B(U)$), product, and direct limit and inverse limit of $\mathcal{O}_X$-modules.

Definition 27.12. Let X be an affine scheme, $X = \text{Spec}(R)$. Given an R-module M, we define an *associated* $\mathcal{O}_X$*-module* $\tilde{M}$ on X such that for any open set $U \subseteq X$, $\tilde{M}(U)$ is defined as the set of $(m_p : p \in U)$ such that m_p is contained in the localization M_p, and there is an open covering of U by open set V such that there is an element $a \in M$ and $s \in R$ such that $s \notin q$ for every $q \in V$ (i.e., $V \subseteq \{s \neq 0\} \subseteq X$) and $m_p = \frac{a}{s}$ for all $p \in V$.

Theorem 27.13. Let X be an affine scheme, $X = \text{Spec}(R)$, M is an R-module, then

1. $\tilde{M}$ is an $\mathcal{O}_X$-module,
2. the stalk of $\tilde{M}$ at each point $p \in X$ is the localization M_p,
3. for $f \in R$, $\tilde{M}(\{f \neq 0\}) = M\left[\frac{1}{f}\right]$,
4. in particular (taking $f = 1$), $\tilde{M}(\text{Spec}(R)) = M$.

Definition 27.14. Let X be a scheme. A *free* $\mathcal{O}_X$*-module* is an $\mathcal{O}_X$-module of the form $\mathcal{O}_X^{\oplus I}$ for some set I. A *free* $\mathcal{O}_X$*-module of finite rank* is $\mathcal{O}_X^{\oplus r}$ for some $r \geq 0$. A *vector bundle* M on a scheme X is an $\mathcal{O}_X$-module that is locally free of finite rank.

28 Lecture 28

Remark 28.1. Given a ring R and a R-module M, we can associate a bunch of vector spaces over fields to M, denoted $M \otimes_R \text{Frac}(R/p)$ where p prime ideals of R. Equivalently, we can associate modules over local rings to M, so we have $M_p = M \otimes_R R_p$ for $p \in \text{Spec}(R)$. Therefore, given a R-module M, we obtain an $\mathcal{O}_X$-module $\tilde{M}$ over $X = \text{Spec}(R)$. Recall that $M = \tilde{M}(X)$ for $X = \text{Spec}(R)$.

Definition 28.2 (Vector Bundle)**.** A *vector bundle* on a scheme X is a locally free $\mathcal{O}_X$-module of finite rank.

Remark 28.3. Let M be a vector bundle, then $M(U)$ is the set of vector spaces of section of M on V. Here $M(U)$ is a module over $\mathcal{O}(U)$, the ring of continuous functions $U \to \mathbb{R}$. This sheaf of $\mathcal{O}_X$-modules carries the same information as the geometric notion.

Remark 28.4. The rank of a vector bundle on a scheme is locally constant. If X is connected, then the rank is constant.

Definition 28.5 (Line Bundle)**.** A *line bundle* is a vector bundle of rank 1.

Example 28.6. Let k be a field and $n \geq 0$. Consider P^n_k. For an integer j, the line bundle $\mathcal{O}(j)$ on P^n_k is: for an open subset $U \subseteq P^n_k$, $\mathcal{O}(j)(U) = \{f \in \mathcal{O}(pi^{-1}(U)) :$ f is homogeneous of degree $j\}$. That is $f(tx) = t^j f(x)$ for $t \in \mathbb{A}^1_k \backslash \{0\}$ and $x \in \pi^{-1}(U)$. Formally, we interpret this equation as the equality of functions on $\mathbb{A}^1_k \backslash \{0\} \times_k \pi^{-1}(U)$.

Remark 28.7. $\mathcal{O}(j)$ is clearly an $\mathcal{O}_{P^n_k}$-module. To show that it is a line bundle, we will show that for each $0 \leq i \leq n$, let $U_i = \{x_i \neq 0\} \subseteq P^n_k$ and $U_i \simeq \mathbb{A}^n_k$. Then $\mathcal{O}(j) \mid_{U_i} \simeq \mathcal{O}_{U_i}$ as $\mathcal{O}_{U_i}$-modules. Namely, map $\mathcal{O}_{U_i} \hookrightarrow \mathcal{O}(j) \mid_{U_i}$ by sending regular function f on open subset $V \subseteq U_i$ to $fx_i^j \in \mathcal{O}(j)(U)$. This makes sense for all $j \in \mathbb{Z}$ because $x_i \neq 0$ on $\pi^{-1}(V)$. The inverse map is defined by $g \in \mathcal{O}(j) \mid_V \mapsto g/(x_i)^j$.

Remark 28.8. If $n = 0$, then $\mathcal{O}(j) simeq \mathcal{O}$ on $P^0_k = \operatorname{Spec}(k)$ because every vector bundle on point and is trivial.

Remark 28.9 (What is $\mathcal{O}(j)(P^n_k)$ for $n \geq 1$?)**.** Such structure is a module over $\mathcal{O}(P^n_k) = k$, i.e., a k-vector space. We have

$$\begin{aligned}\mathcal{O}(j)(P^n_k) &= \{f \in \mathcal{O}(\mathbb{A}^{n+1}_k \backslash \{0\}) : f \text{ is homogeneous of degree } j\} \\ &= \{f \in k[x_0, \ldots, x_n] : f \text{ is homogeneous of degree } j\}\end{aligned}$$

So

$$\dim_k(\mathcal{O}(j)(P^n_k)) = \begin{cases} 0, & \text{if } j < 0 \\ \binom{n+j}{j}, & \text{if } j \geq 0 \end{cases}$$

Example 28.10. For P^1_k, $\dim(\mathcal{O}(j)(P^1_k)) = \begin{cases} 0, & \text{if } j < 0 \\ j+1, & \text{if } j \geq 0 \end{cases}$. Therefore, for $a \neq b$, we have $\mathcal{O}(a) \ncong \mathcal{O}(b)$.

Remark 28.11. For any two line bundles L and M on a scheme X, their tensor product $L \otimes M := L \otimes_{\mathcal{O}_X} M$ by sheafification of $U \mapsto L(U) \otimes_{\mathcal{O}_X(U)} M(U)$ is a line bundle. We always have $L \otimes M \cong M \otimes L$. Also, let the dual line bundle L^* to be $L^* = \mathbf{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X)$. If L is a line bundle, we have $L \otimes L^* \simeq \mathcal{O}_X$.

Definition 28.12 (Picard Group)**.** The *Picard group* L of scheme X is the Abelian group of all isomorphism classes of line bundles. The group operation is $\otimes$ and the inverse is L^*.

Remark 28.13. On P^n_k, $\mathcal{O}(a) \otimes \mathcal{O}(b) \simeq \mathcal{O}(a+b)$, so we have a group homomorphism $\mathbb{Z} \to \operatorname{Pic}(P^n_k)$ by $j \mapsto [\mathcal{O}(j)] \in \operatorname{Pic}(P^n_k)$. For $n \geq 1$, this is injective. If not, there would be some $j > 0$ such that $\mathcal{O}(j) \simeq \mathcal{O}_{P^n_k}$, but that is false by the dimension of the global section.

Remark 28.14. In fact, $\mathbb{Z} \xrightarrow{\simeq} \text{Pic}(P^n_k)$ for $n \geq 1$.

Remark 28.15. A closed subscheme of P^n_k is defined by the zero set of a collection of homogeneous polynomials in $k[x_0, \ldots, x_n]$. These polynomials are not regular functions on P^n_k, but they are sections of the line bundles $\mathcal{O}(j)$ for $j > 0$.

Definition 28.16 (Rank)**.** Let M be a finitely-generated R-module. Let $p \in \text{Spec}(R)$. Then the *rank* of M at p is $\dim_{k(p)}(M \otimes_R k(p)) \in \mathbb{N}$.

Example 28.17. Let $R - \mathbb{Z}$, then the rank of $M = \mathbb{Z} \oplus \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^2$ at the generic point is $\dim_{\mathbb{Q}}(M \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{Q}}(\mathbb{Q}) = 1$, at (2) is 2, at (3) is 3, and at $p \neq 2, 3$ is 1.

Definition 28.18 (Quasi-coherent)**.** An $\mathcal{O}_X$-module on a scheme X is *quasicoherent* if there is an open covering of X by affine schemes $U_i = \text{Spec}(R_i)$ such that $F\mid_{U_i} \simeq \tilde{M}_i$ for some R_i-module M.

Definition 28.19 (Coherent)**.** An $\mathcal{O}_X$-module is *coherent* if F is as above with M_i being a finitely-generated R_i-module, for each i (in the case where X is locally Noetherian).

Remark 28.20. Every vector bundle on a scheme X is coherent sheaf.

Theorem 28.21. Let R be a reduced Noetherian ring, and let M be a finitely-generated R-module. Then M is projective if and only if M is flat if and only if M is locally free if and only if its rank on $\text{Spec}(R)$ is locally constant.

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