Homological Algebra Notes

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1 Abelian Category

1.1 Additive Category

Definition 1.1.1 (Additive Category)**.** A category $\mathcal{A}$ is called additive if:

- It admits a zero object (an object that is both initial and final), denoted 0.
- For any $A, B \in \mathcal{A}$, there exists coproduct $A \coprod B$ and product $A \times B$, with a cononical map $A \coprod B \to A \times B$ such that the following two diagrams commute and is an isomorphism:

$$\begin{tikzcd}
 & A \arrow[d, hook] \arrow[dr, "!"] \arrow[ddl, "\mathbf{id}"'] & \\
 & A \coprod B \arrow[d, "\cong"] & 0 \arrow[d, "!"] \\
A & A \times B \arrow[l, two heads] \arrow[r, two heads] & B
\end{tikzcd}$$

$$\begin{tikzcd}
 & B \arrow[d, hook] \arrow[dr, "!"] \arrow[ddl, "\mathbf{id}"'] & \\
 & A \coprod B \arrow[d, "\cong"] & 0 \arrow[d, "!"] \\
B & A \times B \arrow[l, two heads] \arrow[r, two heads] & A
\end{tikzcd}$$

  If this is the case, as the product and the coproduct coincide, we call it the biproduct or direct sum.

  As a result, $\mathbf{Hom}_{\mathcal{A}}(A, B)$ is an Abelian monoid via $f, g : A \to B$ with a defined operation on the set of homomorphism in this category $f + g$ defined by

$$A \xrightarrow{\triangle} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\triangledown} B$$

- Every morphism $f : A \to B$ has an opposite $-f$ such that $f + (-f) = 0 = (-f) + f$, i.e. $\mathbf{Hom}_{\mathcal{A}}(A, B)$ are Abelian groups.

Equivalently, we can define an additive category $\mathcal{A}$ in the following way: $\mathcal{A}$ is enriched over Abelian groups (every Hom set is an Abelian group and

$$\begin{aligned}
\mathbf{Hom}(A, B) \times \mathbf{Hom}(B, C) &\to \mathbf{Hom}(A, C) \\
(f, g) &\mapsto g \circ f
\end{aligned}$$

is bilinear), and has finite biproducts.

To be clear, we can define the biproduct/direct sum as $A \oplus B$ where

$$\begin{tikzcd} & B \arrow[d, "i_B"] & \\ A \arrow[r, "i_A"] & A \oplus B \arrow[r, "p_B"] \arrow[d, "p_A"] & B \\ & A & \end{tikzcd}$$

such that $p_A \circ i_A = \mathbf{id}_A, p_A \circ i_B = 0, p_B \circ i_A = 0, p_B \circ i_B = \mathbf{id}_B$, as well as $i_A \circ p_A + i_B \circ p_B = \mathbf{id}_{A \oplus B}$.

Example 1.1.2. Let R be a ring.

1. The category R-**mod** of left R-modules and R-linear homomorphisms is additive. So is **Mod**-R, that of right R-modules.

   Both categories above are Abelian.

   In particular, if $R = \mathbb{Z}$, the R-mod category is exactly $\mathbf{Ab}$.

2. The subcategory R-**proj** of (left) projective R-modules is additive. The category is usually not Abelian.

3. Same with R-**inj**, the category of (left) injective R-modules.

4. If $\mathcal{A}$ is additive, then so is $\mathcal{A}^{\text{op}}$.

5. If I is small and $\mathcal{A}$ is additive, then $\mathcal{A}^I = \mathbf{Func}(I, \mathcal{A})$, the category of functors $F : I \to \mathcal{A}$ with natural transformations as morphisms, remains additive: $(F \oplus G)(i) = F(i) \oplus G(i)$.

   An example of small category I is the following: suppose X is a topological space, then $I = \mathbf{Open}(X)$ is a small category, where objects are the open subsets $U \subseteq X$ and morphisms are $\mathbf{Mor}_I(U, V) = \begin{cases} \varnothing, & \text{if } U \nsubseteq V \\ * = \{i_{vmu} : U \to V\}, & \text{if } U \subseteq V \end{cases}$, with the composition given by the following: if $U \subseteq V \subseteq W$, then $i_{W,V} \circ i_{V,U}$ is defined to be $i_{W,U}$.

   Furthermore, let us look at the presheaves. We denote the presheaves on X with values in $\mathcal{A}$ to be the category of functors $\mathbf{Pre}_{\mathcal{A}}(X) = \mathcal{A}^{\mathbf{Op}(X)^{\text{op}}}$. A presheaf $P \in \mathbf{Pre}_{\mathcal{A}}(X)$ has $\mathcal{P}(U) \in \mathcal{A}$ for all open subsets $U \subseteq X$, and $p(i_{V,U}) : P(V) \to P(U)$ for all $U \subseteq V$, denoted as $\mathbf{Res}_U^V$ to be the restriction of V in U, such that $\mathbf{Res}_U^U = \mathbf{id}_{P(U)}$ and $\mathbf{Res}_U^W = \mathbf{Res}_U^V \circ \mathbf{Res}_V^W$ for all $U \subseteq V \subseteq W$.

Definition 1.1.3 (Additive Functor). A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is called additive if every $F : \mathbf{Hom}_{\mathcal{A}}(A_1, A_2) \to \mathbf{Hom}_{\mathcal{B}}(FA_1, FA_2)$ is a homomorphism of Abelian groups (preserves the sum of morphisms). Equivalently, F preserves the direct sum/biproduct of objects.

A convention we will use here is that unless specified, functors between additive categories are assumed to be additive.

Remark 1.1.4. A map $f : A_1 \oplus \cdots \oplus A_m \to B_1 \oplus \cdots \oplus B_n$ is described uniquely by $f_{ij} = \mathbf{proj}_i \circ f \circ \mathbf{inc}_j$:

$$\begin{tikzcd} A_j \arrow[r] \arrow[d, "\mathbf{inc}_j"] & B_i \\ A_1 \coprod \cdots \coprod A_m \arrow[r, "f"'] & B_1 \times \cdots \times B_n \arrow[u, "\mathbf{proj}_i"'] \end{tikzcd}$$

commutes. In particular, f corresponds to an $n \times m$ matrix (f_{ij}) for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, with $f = \sum_{i,j} \mathbf{inc}_i \circ f_{ij} \circ \mathbf{proj}_j$.

Furthermore, we have composition in the following sense: if $f = (f_{ij}) : A_1 \oplus \cdots \oplus A_m \to B_1 \oplus \cdots \oplus B_n$ and $g = (g_{kl}) : B_1 \oplus \cdots \oplus B_n \to C_1 \oplus \cdots \oplus C_p$, then $g \circ f : A_1 \oplus \cdots \oplus A_m \to C_1 \oplus \cdots \oplus C_p$ is given by $(g \circ f)_{lj} = \sum_{i=1}^{n} g_{lj} \circ f_{ij}$ for $1 \leqslant j \leqslant m$ and $1 \leqslant l \leqslant p$.

In short, we can compose morphisms between direct sums via matrix multiplication. (Of course, in $\mathbb{F}$-**mod**, i.e. $\mathbb{F}$-vector spaces, the vector spaces are essentially of the form $V = \mathbb{F}^m = \mathbb{F} \oplus \cdots \oplus \mathbb{F}$.

Example 1.1.5. The map

$$A \xrightarrow{\mathbf{inc}_A} A \oplus B \xrightarrow{\mathbf{proj}_B} B$$

can be considered as $\mathbf{inc}_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{proj}_B = \begin{pmatrix} 0 & 1 \end{pmatrix}$. In particular, the matrix multiplication ends up as the zero matrix as desired.

Example 1.1.6. Note that morphisms $f, g : A \to B$ induce $f \oplus g : A \oplus A \to B \oplus B$ described by the matrix $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$. In particular, we have

$$\begin{tikzcd}
A \arrow[r, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"] \arrow[rrr, bend left, "f+g"] \arrow[rr, bend right, "\begin{pmatrix} f \\ g \end{pmatrix}"'] & A \oplus B \arrow[r, "\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}"] & B \oplus B \arrow[r, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] & B
\end{tikzcd}$$

Exercise 1.1.7.
1. Every left adjoint $F : \mathcal{A} \to \mathcal{B}$ (or alternatively, right adjoint) is automatically additive.
2. What is the quotient of additive categories? The problem can be explained in the following sense:

   Suppose $A \subseteq B$ is a subcategory of additive category such that if $A_1, A_2 \in \mathcal{A}$, then $A_1 \oplus A_2 \in \mathcal{A}$ and $0 \in \mathcal{A}$, so $\mathcal{A}$ is additive itself. We want

$$\begin{tikzcd}
A \arrow[r, hook, "\mathbf{inc}"] \arrow[rr, bend left, "0"] & B \arrow[r, "F"] & B/A
\end{tikzcd}$$

   to be universal, i.e. for all $G : B \to C$ additive functor such that $G(A) = 0$ for all $A \in \mathcal{A}$, there exists a unique (up to isomorphism of categories) morphism $\bar{G} : B/A \to C$ additive functor such that

$$\begin{tikzcd}
B \arrow[r, "G"] \arrow[d, "F"'] & C \\
B/A \arrow[ur, dashed, "\exists ! \bar{G}"'] &
\end{tikzcd}$$

   commutes.

Proof.
1. Recall that left adjoints preserve colimits. Therefore, since products and coproducts coincide as biproduct in an additive category, then left adjoint functors between additive categories preserve biproducts, which means it is an additive functor. In a dual argument, one can show that right adjoint functors are also additive functors.

2. We define the quotient category $\mathcal{B}/\mathcal{A}$ in an analogous sense as the usual $\mathcal{B}/\sim$ construction: the objects in the category is exactly the objects in $\mathcal{B}$. The morphisms in $\mathcal{B}/\sim$ are the equivalence classes of morphisms in $\mathcal{A}$, such that for a pair of morphisms $f, g : X \to Y$ in $\mathcal{B}$, we say $f \sim g$ when $f - g$ factors through some object in $\mathcal{A}$. This construction has the universal property that the quotinet functor $Q : \mathcal{B} \to \mathcal{B}/\sim$ is the universal additive functor from $\mathcal{B}$ to an additive category such that $Q(\mathcal{A}) = 0$ (i.e., like the universal property stated in the edit to the question, but with "additive" inserted everywhere).

   This construction is used quite a bit in the representation theory of finite dimensional algebras. For example, the "stable module category" is the quotient of the module category by the subcategory consisting of projective modules. □

1.2 Kernels and Cokernels

We first recall the concept of pushout and pullback. A pushout of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

is an object P along with morphisms $h : B \to P$ and $k : C \to P$ such that the square commutes and satisfies the following universal property: for any object T and morphisms $l : B \to T$ and $m : C \to T$ that satisfies $m \circ g = l \circ f$, there is a unique morphism $n : n \to T$ that makes all related diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[ddr, bend left, "l"] & \\ C \arrow[r, "k"] \arrow[drr, bend right, "m"'] & P \arrow[dr, dashed, "\exists! n"] & \\ & & T \end{tikzcd}$$

Similarly one can define a pullback of the diagram

$$\begin{tikzcd} & B \arrow[d, "f"'] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

to be the an object Q along with two morphisms such that the following diagram commutes:

$$\begin{tikzcd} T \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Q \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ & C \arrow[r, "g"'] & A \end{tikzcd}$$

Definition 1.2.1 (Kernel, Cokernel)**.** Let $\mathcal{A}$ be an additive category and let $f : A \to B$ be a morphism in $\mathcal{A}$. We want to consider the pullback of

$$\begin{tikzcd} & 0 \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

and the pushout of

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ 0 & \end{tikzcd}$$

The kernel of f in $\mathcal{A}$ is the limit of the first diagram (if it exists). The cokernel of f in $\mathcal{A}$ is the colimit of the second diagram (if it exists). If the kernel exists, we have a pullback square (also called a Cartesian square)

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d, "i"'] & 0 \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

i.e. $f \circ i = 0$, and the pair $(\ker(f), i)$ is universal in the following sense: for all $t : T \to A$ such that $f \circ t = 0$, there exists a unique map $\tilde{t} : T \to \ker(f)$ such that $t = i \circ \tilde{t}$:

$$\begin{tikzcd}
T \arrow[d, dashed, "\exists!\tilde{t}"'] \arrow[dr, "t"] \arrow[drr, bend left, "0"] & & \\
\ker(f) \arrow[r, "i"'] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

Respectively, we have a pushout square (also called a Cocartesian square)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d, "p"] \\
0 \arrow[r, "f"'] & \mathbf{coker}(f)
\end{tikzcd}$$

i.e. $p \circ f = 0$ and $(\mathbf{coker}(f), p)$ is universal:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left, "0"] \arrow[drr, bend right, "0"'] & B \arrow[r, "p"] \arrow[dr, "t"'] & \mathbf{coker}(p) \arrow[d, dashed, "\exists!\bar{t}"] \\
& & T
\end{tikzcd}$$

Remark 1.2.2. When kernel/cokernel exists, it is unique up to isomorphism (of pairs).

Recall that a morphism f is a monomorphism if $f \circ g = f \circ h \Rightarrow g = h$, and is an epimorphism if $g \circ f = h \circ f \Rightarrow g = h$.

Example 1.2.3.

1. In $\mathbf{Ring}$, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism, but not an isomorphism.
2. If $\mathcal{A}$ is additive, then
   - f is a monomorphism if and only if $\ker(f) = 0$;
   - f is an epimorphism if and only if $\mathbf{coker}(f) = 0$;
   - If f has a kernel $(\ker(f), i)$, then i is a monomorphism.
   - If f has a cokernel $(\mathbf{coker}(f), p)$, then p is an epimorphism.

Notation: We usually denote a monomorphism by $\rightarrowtail$ and an epimorphism by $\twoheadrightarrow$.

Example 1.2.4. In R-**Mod**, any morphism $f : A \to B$ has a kernel and a cokernel. In particular, $\ker(f) = \{a \in A \mid f(a) = 0 \in B\} \subseteq A$ is a submodule, where the associated morphism $i : \ker(f) \to A$ is the inclusion map. The cokernel is $\mathbf{coker}(f) = B/\mathbf{im}(f)$ where $\mathbf{im}(f) = \{f(a) \mid a \in A\}$. There is an associated map $p : B \to B/\sim$ where $b \sim b'$ indicates $b - b' \in \mathbf{im}(f)$, sending $b \mapsto [b]_\sim = b + \mathbf{im}(f)$. This induces the following diagram, where $\bar{t}([b]_\sim) := t(b)$:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "p"] \arrow[d, "t"] & B/\sim \arrow[dl, dashed, "\exists!\bar{t}"] \\
& T &
\end{tikzcd}$$

Remark 1.2.5. The notions, like all limits and colimits, really depend on the ambient category $\mathcal{A}$. This is illustrated by the following example.

Example 1.2.6. Consider the category $\mathbb{Z}$-$\mathbf{Mod}$ and $f : \mathbb{Z} \to \mathbb{Z}$ that sends $x \mapsto 2x$. Note that $\ker(f) = 0$ and $\mathbf{coker}(f) = \mathbb{Z}/2\mathbb{Z}$ given by $p : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$.

However, consider the category of $\mathbb{Z}$-$\mathbf{proj}$ (which happens to be free), or only the finitely-generated ones (which are torsion free since projective), and the same map f. This time, both the kernel and the cokernel are 0.

Lemma 1.2.7. Consider a commutative square in an additive category $\mathcal{A}$:

$$\begin{tikzcd}
A' \arrow[r, "\alpha"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B' \arrow[r, "\beta"'] & B
\end{tikzcd}$$

(a) Suppose the square is Cartesian, and that f has a kernel. Then f' has the same kernel, i.e. $\exists! i' : \ker(f) \to A'$ such that $(\ker(f), i')$ is a kernel of f'.

$$\begin{tikzcd}
 & \ker(f) \arrow[dl, dashed, tail, "\exists! i'"'] \arrow[d, tail, "i"] \\
A' \arrow[r, "\alpha"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B' \arrow[r, "\beta"'] & B
\end{tikzcd}$$

(b) Suppose the square is Cocartesian, and that f' has a cokernel:

$$\begin{tikzcd}
A' \arrow[r, "\alpha"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B' \arrow[r, "\beta"'] \arrow[d, two heads, "p'"'] & B \arrow[dl, dashed, two heads, "\exists! p"] \\
\mathbf{coker}(f) &
\end{tikzcd}$$

Then f has the same cokernel as f' does.

Proof. It suffices to prove the first statement since the second statement follows from a dual argument.

Consider the pullback property for the square on $\ker(f)$, we have a commutative diagram

$$\begin{tikzcd}
\ker(f) \arrow[r, "i"] \arrow[d, "0"'] & A \arrow[d, "f"] \\
B' \arrow[r, "\beta"'] & B
\end{tikzcd}$$

Note that $f \circ i = 0 = \beta \circ 0$. This induces a unique morphism $i' : \ker(f) \to A'$ such that $f' \circ i' = 0$ and $\alpha \circ i' = i$.

Now suppose there is an object T with morphism $t : T \to A'$ such that $f' \circ t = 0$. Note that $\alpha \circ t : T \to A$ satisfies $f \circ (\alpha \circ t) = f\alpha t = \beta(f t) = 0$. By the universal property of $\ker(f)$ on f, there exists a unique map $s : T \to \ker(f)$ such that $i \circ s = \alpha \circ t$.

We claim that $\tilde{t} = s : T \to \ker(f)$ is the unique map we want that satisfies the universal property for $\ker(f)$ to be the kernel of f'. Notice that the morphism $i' \circ \tilde{t}$ satisfies $f'(i\tilde{t}) = 0 = f't$ and $\alpha(i'\tilde{t}) = \alpha i's = is = \alpha t$. Therefore, both t and $i' \circ t$ satisfies the pullback property. By the uniqueness of the pullback, $i' \circ \tilde{t} = t$. Therefore, $\tilde{t} = s$ is a morphism we want. Finally, $\tilde{t} = s$ is the unique morphism that satisfies the universal property because i' is a monomorphism (as $\alpha \circ i' = i$ is monomorphism).

$$\begin{tikzcd}
T \arrow[rr, dashed, "\tilde{t}"] \arrow[dr, "t"] \arrow[ddr, bend right, "0"'] & & \ker(f) \arrow[dl, dashed, tail, "\exists! i'"'] \arrow[d, tail, "i"] \\
 & A' \arrow[r, "\alpha"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
 & B' \arrow[r, "\beta"'] & B
\end{tikzcd}$$

□

Corollary 1.2.8.
- Pullback of a monomorphism is a monomorphism.
- Pullback of an epimorphism is an epimorphism.

Remark 1.2.9. Given a corner

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd},$$

the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd},$$

if exists, is the cokernel of

$$A \xrightarrow{\begin{pmatrix} f \\ -g \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} h & k \end{pmatrix}} D \ .$$

Note that $\begin{pmatrix} h & k \end{pmatrix} \circ \begin{pmatrix} f \\ -g \end{pmatrix} = 0$ if and only if $hf - kg = 0$ if and only if $hf = kg$.

Similarly, the pushback is a suitable kernel.

1.3 Definition of Abelian Category

Let $\mathcal{A}$ be an additive category. We have the following diagram.

$$\begin{tikzcd} \ker(f) \arrow[r, tail, "i"] & A \arrow[r, "f"] \arrow[d, "q"] & B \arrow[r, two heads, "p"] & \textbf{coker}(f) \\ & \textbf{coker}(i) \arrow[r, "\exists! \bar{f}"] & \ker(p) \arrow[u, tail, "j"] & \end{tikzcd}$$

Note that $\bar{f}$ is induced by the universal properties of the diagram.

Definition 1.3.1 (Abelian Category)**.** An Abelian category $\mathcal{A}$ is an additive category in which every morphism admits a kernel and cokernel and such that $\forall f : A \to B$ the cokernel of the kernel of f is canonically isomorphic to the kernel of the cokernel of f. i.e. $\bar{f}$ is an isomorphism.

Remark 1.3.2. It follows that f factors as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & \textbf{im}(f) \arrow[ur, tail] & \end{tikzcd}$$

i.e. the composition of an epimorphism followed by a monomorphism.

Since the kernel of epimorphism has to be $\ker(f)$, the epimorphism is the cokernel of $\ker(f)$, and similarly the monomorphism is the kernel of $\textbf{coker}(f)$. Therefore, our construction is unique (up to unique isomorphism); the intermediate object is called $\textbf{im}(f)$.

Revisting the first diagram, if we define (as an alternative) the image of f as $\ker(p)$, and define the coimage of f as $\textbf{coker}(i)$, then the definition says that the image and coimage coincide.

Exercise 1.3.3. The category $\mathcal{A} = \mathbf{R}\text{-}\mathbf{Mod}$ for a ring R is an Abelian category.

Example 1.3.4. Take $\mathcal{A} = \mathbb{Z}\text{-}\mathbf{proj}$ and $f : \mathbb{Z} \to \mathbb{Z}$ as $x \mapsto 2x$, then we have

$$\begin{tikzcd} 0 \arrow[r, tail] & A \arrow[r, "f", "\times 2"'] \arrow[d, "\mathbf{id}"] & B \arrow[r, two heads] & 0 \\ & \mathbf{coim}(f) \arrow[r, "\exists ! \bar{f}: \times 2", "\not\cong"'] & \mathbf{im}(f) \arrow[u, hook] & \end{tikzcd}$$

Therefore, the category is not Abelian.

Example 1.3.5. Let $\mathcal{A}$ be the additive category of Hausdorff topological Abelian groups. It has (usual) kernels (preimage of 0) and cokernels $(B/\overline{\mathbf{im}(f)})$. However, $\mathcal{A}$ is not Abelian.

For example, a dense subgroup $A \subseteq B$ yields a homomorphism. The kernel and cokernel are both zero, so the induced map is just the original map itself.

Proposition 1.3.6. In an Abelian category $\mathcal{A}$,

1. i is a kernel if and only if $\ker(i) = 0$ if and only if i is a monomorphism.
2. Dually speaking, i is a cokernel if and only if $\mathbf{coker}(i) = 0$ if and only if i is an epimorphism.
3. f is an isomorphism if and only if it is a monomorphism and an epimorphism if and only if $\ker(f) = 0 = \mathbf{coker}(f)$.

Proof. We would only prove the last part. Suppose $\ker(f) = 0 = \mathbf{coker}(f)$. By definition, we know f is a monomorphism and an epimorphism.

In particular, we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r, tail] & A \arrow[r, "f", "\times 2"'] \arrow[d, "\mathbf{id}"] & B \arrow[r, two heads] & 0 \\ & A \arrow[r, "\bar{f}=f", "\cong"'] & B \arrow[u, "\mathbf{id}"] & \end{tikzcd}$$

Therefore f is an isomorphism. $\square$

Remark 1.3.7. Let $\mathcal{A}$ be Abelian. Then $\ker(-)$ and $\mathbf{coker}(-)$ are functorial when considered as $\mathbf{Ar}(\mathcal{A}) \to \mathcal{A}$:

$$\begin{tikzcd} \ker(f) \arrow[r, tail] \arrow[d, dashed, "\exists ! \ker(\alpha,\beta)"'] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, two heads] \arrow[d, "\beta"] & \mathbf{coker}(f) \arrow[d, dashed, "\exists ! \mathbf{ker}(\alpha,\beta)"] \\ \ker(f') \arrow[r, tail] & A' \arrow[r, "f'"] & B' \arrow[r, two heads] & \mathbf{coker}(f') \end{tikzcd}$$

In particular, they preserve isomorphisms.

Example 1.3.8 (Presheaves)**.** Let $\mathcal{A}$ be Abelian and I be small. Consider $\mathcal{A}^I$ be the category of functors $F : I \to \mathcal{A}$. Then $\mathcal{A}^I$ is Abelian with objectwise limits and colimits:

Let $f : F \to G$ be a natural transformation with $f_i : Fi \to Gi$ for all $i \in I$. Then $\ker(f) \in \mathcal{A}^I$ together with $\alpha : \ker(f) \to F$ is given by $(\ker(f))(i) = \ker(f_i) \in \mathcal{A}$. For $\theta : i \to j$ in I,

$$\begin{tikzcd} \ker(f_i) \arrow[r] \arrow[d, "(\ker(f))(\theta)"', "\exists !"] & Fi \arrow[r, "f_i"] \arrow[d, "F\theta"] & Gi \arrow[d, "G\theta"] \\ \ker(f_j) \arrow[r] & Fj \arrow[r, "f_i"] & Gj \end{tikzcd}$$

The same recipe works for any limits, and similarly for colimits.

Example 1.3.9. If X is a topological space and $I = \mathbf{Open}(X)^{\mathrm{op}}$, then $\mathbf{Pre}_{\mathcal{A}}(X) = \mathcal{A}^{\mathbf{Open}(X)^{\mathrm{op}}}$ is Abelian with open-wise kernel and cokernel.

Remark 1.3.10. In Abelian category $\mathcal{A}$, we have all finite limits and colimits.

Note that if we have direct sum, then there are finite product and coproduct, as well as pushouts and pullbacks.

Also note the coequalizer of $f, g : A \to B$ is just the cokernel of $f - g : A \to B$. Therefore, equalizers, kernels, and pullbacks/pushouts are internally related.

Example 1.3.11 (Sheaves)**.** Let X be a topological space. A sheaf on X with values in a category $\mathcal{A}$ is a presheaf $P \in \mathcal{A}^{\mathrm{Open}(X)^{\mathrm{op}}}$ (given $P(U) \in \mathcal{A}$ for every open subset $U \subseteq X$ and restriction $\mathbf{Res}_V^U : P(U) \to P(V)$ for $V \subseteq U$ such that $\mathbf{Res}_U^U = \mathbf{id}$ and $\mathbf{Res}_W^V \circ \mathbf{Res}_V^U = \mathbf{Res}_W^U$ for $W \subseteq V \subseteq U$), such that for every open cover $U = \bigcup_{j \in J} V_j$ and any family $s_j \in P(V_j)$ for all $j \in J$ such that $\mathbf{Res}_{V_i \cap V_j}^{V_i}(s_i) = \mathbf{Res}_{V_i \cap V_j}^{V_j}(s_j)$ for all $i, j \in J$, there exists a unique $s \in P(U)$ such that $\mathbf{Res}_{V_j}^U(s) = s_j$ for all $j \in J$.

If $\mathcal{A}$ has products, the above can be phrased as saying the following is an equalizer:

$$P(U) \xrightarrow{(\mathbf{Res}_{V_i}^U)_i} \prod_{i \in J} P(V_i) \rightrightarrows \prod_{j,k \in J} P(V_j \cap V_k)$$

Here the two maps in the equalizer are induced by $\prod_{i \in J} P(V_i) \to P(V_j) \xrightarrow{\mathbf{Res}_{V_j \cap V_k}^{V_j}} P(V_j \cap V_k)$ componentwise and $\prod_{i \in J} P(V_i) \to P(V_k) \xrightarrow{\mathbf{Res}_{V_j \cap V_k}^{V_k}} P(V_j \cap V_k)$ componentwise.

We define a category of sheaves as a full subcategory of presheaves: $\mathbf{Shv}_{\mathcal{A}}(X) \subseteq \mathbf{Pre}_{\mathcal{A}}(X)$.

Problem 1 (Exam Problem 1)**.** Suppose $\mathcal{A} = \mathbf{R}$**-Mod** for a ring R. Show that $\mathbf{Shv}_{\mathcal{A}}(X)$ is Abelian, with same kernels as in $\mathbf{Pre}_{\mathcal{A}}(X)$. Give an example of a morphism of sheaves whose cokernel in $\mathbf{Shv}_{\mathcal{A}}(X)$ differs from that in $\mathbf{Pre}_{\mathcal{A}}(X)$.

More precisely, as a fact, if $f : P \to Q$ is a morphism of sheaves, then the presheaf $\ker(f)$ is actually a sheaf. But $\mathbf{coker}(f)$ in $\mathbf{Shv}(X)$ is obtained by sheafification a. Note that $a \dashv i$ is an adjunction, where i is the inclusion/forgetful functor (fully faithful embedding)from the category of sheaves to the category of presheaves.

$$\begin{tikzcd}
\mathbf{Pre}_{\mathcal{A}}(X) \arrow[d, shift right, "a"'] \\
\mathbf{Shv}_{\mathcal{A}}(X) \arrow[u, hook, shift right, "i"']
\end{tikzcd}$$

We now briefly explain the idea of sheafification. The trick is just "doing it twice", referring to the diagram below:

$$\begin{tikzcd}
\mathbf{Pre}_{\mathcal{A}}(X) \arrow[d, shift right, "b"'] \arrow[dd, bend right=60, "a"'] \\
\mathbf{SepPre}_{\mathcal{A}}(X) \arrow[u, hook, shift right] \arrow[d, shift right, "c"'] \\
\mathbf{Shv}_{\mathcal{A}}(X) \arrow[u, hook, shift right]
\end{tikzcd}$$

Here $\mathbf{SepPre}_{\mathcal{A}}(X) = \{P \in \mathbf{Pre}_{\mathcal{A}}(X) \text{ separated}\}$ is the set of separated presheaves P, i.e. $P(U) \to \prod_{j \in J} P(V_j)$ is injective for all open cover $U = \bigcup_{j \in J} V_j$ of U open in X. Moreover, $a = c \circ b$, where b and c are obtained from the equalizer e of

$$\prod_{i \in J} P(V_i) \rightrightarrows \prod_{j,k \in J} P(V_j \cap V_k)$$

In particular, $bP(V)$ and $cP(V)$ are defined in the same way, which is the equalizer's colimit under refinement of $\{V_j\}_{j \in J}$ forming cover of U. Here we say the family $\{W_k\}_{k \in K}$ is a refinement if there exists an open cover $\{V_j\}$ of U such that for all $k \in K$, $W_k \subseteq V_{j_k}$ for some j_k.

In this sense, by performing the same operation twice, we get the sheafification.

1.4 Exact Sequences

For this whole section, we assume $\mathcal{A}$ to be an Abelian category.

Definition 1.4.1 (Exact)**.** A sequence of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if $g \circ f = 0$ and $\operatorname{im}(f) = \ker(g)$ via the canonical map:

$$\begin{tikzcd}
\ker(f) \arrow[r] & A \arrow[r, "f"] \arrow[d, two heads] & B \arrow[r, "g"] & C \\
& \operatorname{im}(f) \arrow[ur, tail] \arrow[r, "\exists!\bar{f}"'] & \ker(g) \arrow[u, tail] &
\end{tikzcd}$$

Note that the cokernel of $\bar{f}$ is the homology by the complex $A \to B \to C$.

Moreover, a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

or

$$\begin{tikzcd}
A \arrow[r, tail, "f"] & B \arrow[r, two heads, "g"] & C
\end{tikzcd}$$

is one exact at A, B and C. This is equivalent to saying f is a monomorphism and g is an epimorphism (and, in fact, g is the cokernel of f and f is the kernel of g).

Exercise 1.4.2. The sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

is exact if and only if $f = \ker(g)$.

The sequence

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

is exact if and only if $g = \operatorname{coker}(f)$.

Exercise 1.4.3. For $A \xrightarrow{f} B \xrightarrow{g} C$ such that $g \circ f = 0$, this induces $\tilde{f} : A \to \ker(g)$ and $\tilde{g} : \operatorname{coker}(f) \to C$, then the sequence is exact at B if and only if $\tilde{f}$ is an epimorphism and $\tilde{g}$ is a monomorphism if and only if the epi-mono factorization

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads] & & B \arrow[rr, "g"] \arrow[dr, two heads] & & C \\
& I \arrow[ur, tail] & & J \arrow[ur, tail] &
\end{tikzcd}$$

has the short sequence $0 \to I \to B \to J \to 0$ is exact.

Exercise 1.4.4. In $\mathbf{Shv}_{\mathcal{A}}(X)$, a sequence

$$P' \longrightarrow P \longrightarrow P''$$

is exact if and only if the sequence of stalks

$$P'_x \longrightarrow P_x \longrightarrow P''_x$$

is exact in $\mathcal{A}$ for every $x \in X$. (Here $\mathcal{A}$ should be at least a Grothendieck category, or just think about R-module categories.) The notion of stalk is given by $P_x = \operatorname{colim}_{\text{open } U \ni x} P(U)$ in $\mathcal{A}$.

Theorem 1.4.5 (Five Lemma)**.** Consider a commutative diagram in $\mathcal{A}$ with exact rows:

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\
B_1 \arrow[r, "\beta_1"'] & B_2 \arrow[r, "\beta_2"'] & B_3 \arrow[r, "\beta_3"'] & B_4 \arrow[r, "\beta_4"'] & B_5
\end{tikzcd}$$

Figure 1: Five Lemma

where f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is also an isomorphism.

Proof. We first consider a special case where $A_1 = A_5 = B_1 = B_5 = 0$, and use it to prove the general statement. In this case, we have

$$\begin{tikzcd}
A_2 \arrow[r, tail, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, two heads, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[d, "f_4"'] \\
B_2 \arrow[r, tail, "\beta_2"'] & B_3 \arrow[r, two heads, "\beta_3"'] & B_4
\end{tikzcd}$$

We want to show that $\ker(f_3) = 0 = \operatorname{coker}(f_3)$. Consider $i : \ker(f_3) \to A_3$. As $f_3 i = 0$, then $f_4\alpha_3 i = \beta_3 f_3 i = 0$. Because f_4 is an isomorphism, $\alpha_3 i = 0$. Because α_2 is a kernel of α_3, there exists $\tilde{i}$ such that $i = \alpha_2\tilde{i}$. Then $\beta_2 f_2 \tilde{i} = f_3\alpha_2\tilde{i} = f_3 i = 0$. Since β_2 and f_2 are monomorphisms, then $\tilde{i} = 0$. Therefore, $i = 0$ and so $\ker(f_3) = 0$. Dually, we have $\operatorname{coker}(f_3) = 0$.

Note that the proof does not work in the general case because in general we cannot get a lift from the kernel. We now prove the general case.

Consider the epi-mono factorization. By exactness of the diagram, we have

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] & A_2 \arrow[rr, "\alpha_2"] \arrow[dr, two heads] & & A_3 \\
& & \operatorname{coker}(\alpha_1) \arrow[ur, tail] \arrow[d, "\exists \overline{f_2}"] & \\
& & \operatorname{coker}(\beta_1) \arrow[dr, tail] & \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[rr, "\beta_2"] \arrow[ur, two heads] & & B_3
\end{tikzcd}$$

Here $\overline{f_2} = \operatorname{coker}(f_1, f_2)$ is an isomorphism by the functoriality of the cokernel. Similarly, there is

$$\begin{tikzcd}
A_3 \arrow[rr, "\alpha_3"] \arrow[dr, two heads] & & A_4 \\
& \ker(\alpha_4) \arrow[ur, tail] \arrow[d, "\exists \overline{f_4}"] & \\
& \ker(\beta_4) \arrow[dr, tail] & \\
B_3 \arrow[rr, "\beta_3"] \arrow[ur, two heads] & & B_4
\end{tikzcd}$$

where $\overline{f_4} = \ker(f_4, f_5)$ is an isomorphism by the functoriality of the kernel. We then obtain

$$\begin{tikzcd}
\operatorname{coker}(\alpha_1) \arrow[r, tail] \arrow[d, "\cong"] & A_4 \arrow[r, two heads] \arrow[d, "f_3"] & \ker(\alpha_4) \arrow[d, "\cong"] \\
\operatorname{coker}(\beta_1) \arrow[r, tail] & B_4 \arrow[r, two heads] & \ker(\beta_4)
\end{tikzcd}$$

and by the special case, we know f_3 is an isomorphism. □

Proposition 1.4.6. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. Then the following are equivalent:

1. f is a split monomorphism, i.e. it admits a retraction $r : B \to A$ such that $r \circ f = \mathbf{id}$.
2. g is a split epimorphism, i.e. it admits a section $s : C \to B$ such that $g \circ s = \mathbf{id}$.
3. The sequence is split exact, i.e. there exists $r : B \to A$ and $s : C \to B$ such that $r \circ f = g \circ s = \mathbf{id}$ and $\mathbf{id}_B = fr + sg$.
4. There exists an isomorphism $h : B \xrightarrow{\cong} A \oplus C$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "h"', "\cong"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

(where the bottom row is given by the usual embedding and projection) such that $h \circ f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \end{pmatrix} \circ h = g$.

Remark 1.4.7. Note that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

is Cartesian (pullback) if and only if $0 \to A \xrightarrow{\begin{pmatrix} f \\ -g \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} h & k \end{pmatrix}} D$ is exact, and it is Cocartesian (pushout) if and only if $A \xrightarrow{\begin{pmatrix} f \\ -g \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} h & k \end{pmatrix}} D \to 0$ is exact. It is Bicartesian (both Cartesian and Cocartesian) if and only if $0 \to A \xrightarrow{\begin{pmatrix} f \\ -g \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} h & k \end{pmatrix}} D \to 0$ is exact.

Lemma 1.4.8. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a commutative square in an Abelian category. The following are equivalent:

1. The square is Cartesian.
2. The canonical map $\ker(g) \to \ker(h)$ is an isomorphism and the canonical map $\operatorname{coker}(g) \to \operatorname{coker}(h)$ is a monomorphism.
3. The canonical map $\ker(f) \to \ker(k)$ is an isomorphism and the canonical map $\operatorname{coker}(f) \to \operatorname{coker}(k)$ is a monomorphism.

Remark 1.4.9. From the above, it follows that a pullback along an epimorphism yields a Bicartesian square.

In short, the pullback of an epimorphism is an epimorphism. Similarly, the pushout of a monomorphism is a monomorphism.

Proof. Note that statement 2 and 3 are dual, so we only have to show that statement 1 is equivalent to statement 2. We only show that statement 1 implies statement 2 here. We have already proved the first part of the statement. Now, consider the diagram below:

$$\begin{tikzcd}
F \arrow[rrr, bend left, "l"] \arrow[r, dashed, "\exists n"] \arrow[d, two heads, "n"] \arrow[dr, phantom, "\lrcorner"] & A \arrow[rr, bend left, "f"] \arrow[d, "g"] \arrow[rr, phantom, "\lrcorner"] & & B \arrow[dl, two heads] \arrow[d, "h"] \\
 & & \mathrm{im}(h) \arrow[dr, tail] & \\
E \arrow[urr, dashed] \arrow[r, tail, "i'"] \arrow[d, two heads, "p'"] \arrow[dr, phantom, "\lrcorner"] & C \arrow[rr, "k"] \arrow[d, two heads, "p"] & & D \arrow[d, two heads, "q"] \\
\ker(\bar{k}) \arrow[r, tail, "i"] & \mathrm{coker}(g) \arrow[rr, "\bar{k}"] & & \mathrm{coker}(h)
\end{tikzcd}$$

Consider the preliminary construction, where we have the Cartesian square on the top right and the kernels and cokernels at the bottom. We want to show that $\ker(\bar{k}) = 0$, i.e. $i = 0$, then by definition we know the map $\bar{k}$ is a monomorphism as desired. We first construct a pullback E of the mappings p and i, and get mappings p' and i'. In particular, observe that p' is an epimorphism and i' is a monomorphism.[1] Moreover, by commutativity we have $qki' = \bar{k}ip' = 0$ since $\bar{k}i = 0$. Because the image of h acts as the kernel of q, and the notion of kernel is just an equalizer (of q and the zero morphism), then the universal property says that we have a unique map from E to $\mathrm{im}(h)$.

Moreover, we can construct another pullback F with respect to $k \circ i$ and h.[2] By the universal property of pullback at A, we construct a map $n : F \to A$ such that $g \circ n = i' \circ m$ and $f \circ n = l$. Therefore, $i \circ p' \circ m = p \circ i' \circ m = p \circ g \circ n = 0$. However, p' and m are epimorphisms, then by right cancellation, we conclude that $i = 0$. This concludes the proof. □

Exercise 1.4.10. Statement 2 implies Statement 1.

Proof. In brief terms, we consider the diagram below:

$$\begin{tikzcd}
\ker(g) \arrow[r, "\cong"] \arrow[d, tail] & \ker(h) \arrow[d, tail] \\
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\
C \arrow[r, "k"] \arrow[d, "p"] & D \arrow[d, "q"] \\
\mathrm{coker}(g) \arrow[r, tail, "\bar{k}"] & \mathrm{coker}(h)
\end{tikzcd}$$

First, we construct the pullback P over $k : C \to D$ and $h : B \to D$, and by the pullback property we know there exists a map $\varphi : A \to P$. We then construct the diagram below:

[1] Recall that in a pullback diagram, the pullback of an epimorphism is an epimorphism (for Abelian categories) and the pullback of a monomorphism is a monomorphism (in any category). When we discuss the pullback of a single map, it is referring to map on the opposite side of the square.

[2] It could be more suitable to build this pullback with respect to the image of h.

$$\begin{tikzcd}
\ker(g) \arrow[rr, "\cong"] \arrow[d, tail] & & \ker(h) \arrow[d, tail] \\
A \arrow[rr, "f"] \arrow[dr, "\varphi"] \arrow[ddd, "g"] & & B \arrow[ddd, "h"] \\
& P \arrow[ur] \arrow[d, "h'"] & \\
& C \arrow[dl, equal] \arrow[dr, "k"] & \\
C \arrow[rr, "k"] \arrow[d, "p"] & & D \arrow[d, "q"] \\
\operatorname{coker}(g) \arrow[rr, tail, "\bar{k}"] & & \operatorname{coker}(h)
\end{tikzcd}$$

From the pullback square, we note that $\ker(h') \cong \ker(h)$, and so $\ker(g) \cong \ker(h') \cong \ker(h)$. Also, because $\bar{k}$: $\operatorname{coker}(g) \to \operatorname{coker}(h)$ is a monomorphism, then so is $m : \operatorname{coker}(g) \to \operatorname{coker}(h')$. But $m \circ p = r$ according to the diagram below, then m is also an epimorphism, then because we are working in an Abelian category, m is an isomorphism.

$$\begin{tikzcd}
\ker(g) \arrow[rr, "\cong"] \arrow[dr, "l", "\cong"'] \arrow[dd, tail] & & \ker(h) \arrow[dd, tail] \\
& \ker(h') \arrow[ur, "\cong"] \arrow[dd, tail] & \\
A \arrow[rr, "f"] \arrow[dr, "\varphi"] \arrow[ddd, "g"] & & B \arrow[ddd, "h"] \\
& P \arrow[ur] \arrow[d, "h'"] & \\
& C \arrow[dl, equal] \arrow[dr, "k"] \arrow[dd, two heads, "r"] & \\
C \arrow[rr, "k"] \arrow[dd, "p"] & & D \arrow[dd, "q"] \\
& \operatorname{coker}(h') \arrow[dr] & \\
\operatorname{coker}(g) \arrow[ur, "m", "\cong"'] \arrow[rr, tail, "\bar{k}"] & & \operatorname{coker}(h)
\end{tikzcd}$$

Now from the above construction we consider the sequences

$$\begin{tikzcd}
0 \arrow[r] & \ker(g) \arrow[r] \arrow[d, "l"', "\cong"] & A \arrow[r, "g"] \arrow[d, "\varphi"] & C \arrow[r] \arrow[d, equal] & \operatorname{coker}(g) \arrow[r] \arrow[d, "m"', "\cong"] & 0 \\
0 \arrow[r] & \ker(h') \arrow[r] & P \arrow[r, "h'"] & C \arrow[r] & \operatorname{coker}(h') \arrow[r] & 0
\end{tikzcd}$$

By Five Lemma, $\varphi : A \to P$ is an isomorphism. □

Corollary 1.4.11. The square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

is Bicartesian if and only if $\ker(g) = \ker(h)$ and $\operatorname{coker}(g) = \operatorname{coker}(h)$ if and only if $\ker(f) = \ker(h)$ and $\operatorname{coker}(f) = \operatorname{coker}(h)$.

Lemma 1.4.12. Every morphism of short exact sequences

$$\begin{tikzcd} A_1 \arrow[r, tail, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, two heads, "\beta_1"] \arrow[d, "g"] & C_1 \arrow[d, "h"] \\ A_2 \arrow[r, tail, "\alpha_2"] & B_2 \arrow[r, two heads, "\beta_2"] & C_2 \end{tikzcd}$$

factors uniquely as

$$\begin{tikzcd} A_1 \arrow[r, tail, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, two heads, "\beta_1"] \arrow[d, "g'"] & C_1 \arrow[d, equal] \\ A_1 \arrow[r, tail, "\alpha_3"] \arrow[d, equal] & B_3 \arrow[r, two heads, "\beta_3"] \arrow[d, "g''"] & C_1 \arrow[d, "h"] \\ A_2 \arrow[r, tail, "\alpha_2"] & B_2 \arrow[r, two heads, "\beta_2"] & C_2 \end{tikzcd}$$

where the top-left and bottom-right squares are Bicartesian. Note that such B_3 is unique.

Proof. Let us define as the pushout of the upper-left square, then we have the diagram

$$\begin{tikzcd} & A_1 \arrow[r, tail, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, two heads, "\beta_1"] \arrow[d, "g'"'] \arrow[dd, bend left, "g"] & C_1 \arrow[d, equal] & \ker(g'') = \ker(h) \arrow[l, "k"'] \arrow[dll, "l"] \\ & A_1 \arrow[r, tail, "\alpha_3"] \arrow[d, equal] & B_3 \arrow[r, two heads, "\beta_3"'] \arrow[d, dashed, "\exists g''"'] \arrow[dll, "m"'] & C_1 \arrow[d, "h"] & \\ \operatorname{coker}(f) = \operatorname{coker}(g') & A_2 \arrow[l, two heads, "p"] \arrow[r, tail, "\alpha_2"] & B_2 \arrow[r, two heads, "\beta_2"] & C_2 & \end{tikzcd}$$

By the remark before, the pushout of a monomorphism is still a monomorphism, so α_3 is a monomorphism. Also note that C_1 along with β_1 produces a cokernel of α_1. Then by lemma, $\beta_3 : B_3 \to C_1$ gives a cokernel of α_3. In particular, the square is Bicartesian.

By the pushout property for α_2 and g, there exists $g'' : B_3 \to B_2$ such that $g'' \circ g' = g$ and $g''\alpha_3 = \alpha_2$.

Claim 1.4.13. $\beta_2 \circ g'' = h \circ \beta_3$.

Subproof. Because B_3 is given as a pushout, it suffices to check by precomposition with α_3 and g'. We have

$$\begin{cases} \beta_2 g'' \alpha_3 = \beta_2 \alpha_2 = 0 \\ h\beta_3\alpha_3 = 0 \end{cases}$$

and so $\beta_2 g'' g' = \beta_2 g_2 = h\beta_1 = h\beta_3 g'$. ■

Finally, we check the bottom-right square. It has two epimorphisms β_2 and β_3 with isomorphic kernels, then it is bicartesian as well because the cokernels are both 0. □

Remark 1.4.14 (Connecting Homomorphism)**.** Suppose given a construction of short exact sequences

$$\begin{tikzcd} A_1 \arrow[r, tail, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, two heads, "\beta_1"] \arrow[d, "g"] & C_1 \arrow[d, "h"] \\ A_2 \arrow[r, tail, "\alpha_2"] & B_2 \arrow[r, two heads, "\beta_2"] & C_2 \end{tikzcd}$$

We construct $\delta : \ker(h) \to \operatorname{coker}(f)$ as follows: consider the unique factorization by the previous lemma. Because the top left square gives a pushout, then f and g' shares the same cokernel. Similarly, g'' and h shares the same kernel. Define δ as the composition $\ker(h) \cong \ker(g'') \rightarrowtail B_3 \twoheadrightarrow \operatorname{coker}(g') \cong \operatorname{coker}(f)$ from the figure above, i.e. given by $m \circ l$.

Theorem 1.4.15 (Snake Lemma)**.** Given a morphism of short exact sequences

$$\begin{tikzcd}
A_1 \arrow[r, tail, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, two heads, "\beta_1"] \arrow[d, "g"] & C_1 \arrow[d, "h"] \\
A_2 \arrow[r, tail, "\alpha_2"] & B_2 \arrow[r, two heads, "\beta_2"] & C_2
\end{tikzcd}$$

we can derive the diagram

$$\begin{tikzcd}
\ker(f) \arrow[r, "\tilde{\alpha}_1"] \arrow[d, tail, "i"] & \ker(g) \arrow[r, "\tilde{\beta}_1"] \arrow[d, tail, "j"] & \ker(h) \arrow[d, tail, "k"] \arrow[dddll, rounded corners, "\delta"] \\
A_1 \arrow[r, tail, "\alpha_1"] \arrow[d, "f"] & B_1 \arrow[r, two heads, "\beta_1"] \arrow[d, "g"] & C_1 \arrow[d, "h"] \\
A_2 \arrow[r, tail, "\alpha_2"] \arrow[d, two heads, "p"] & B_2 \arrow[r, two heads, "\beta_2"] \arrow[d, two heads, "q"] & C_2 \arrow[d, two heads, "r"] \\
\operatorname{coker}(f) \arrow[r, "\tilde{\alpha}_2"] & \operatorname{coker}(g) \arrow[r, "\tilde{\beta}_2"] & \operatorname{coker}(h)
\end{tikzcd}$$

Then the sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r, "\tilde{\alpha}_1"] & \ker(g) \arrow[r, "\tilde{\beta}_1"] & \ker(h) \arrow[dll, "\delta"] & \\
& \operatorname{coker}(f) \arrow[r, "\tilde{\alpha}_2"] & \operatorname{coker}(g) \arrow[r, "\tilde{\beta}_2"] & \operatorname{coker}(h) \arrow[r] & 0
\end{tikzcd}$$

is exact.

Proof. We will focus on the exactness around the connecting homomorphism δ. In particular, we will prove exactness at $\ker(h)$, and the other side would follow by a dual argument. For a element-wise argument (i.e. via diagram chasing), see 210A Homework 9. It suffices to show that $\operatorname{im}(\tilde{\beta}_1) = \ker(\delta)$. Observe that we have the diagram

$$\begin{tikzcd}
\ker(f) \arrow[r, "\tilde{\alpha}_1"] \arrow[dd, tail, "i"] & \ker(g) \arrow[rr, "\tilde{\beta}_1"] \arrow[dd, tail, "j"] \arrow[dr, two heads, "s"] & & \ker(h) = \ker(g'') \arrow[dd, tail, "k"] \arrow[dddll, bend right, tail, "l"'] \\
& & \operatorname{im}(\tilde{\beta}_1) \arrow[ur, tail, "b"] \arrow[dd, "j"] & \\
A_1 \arrow[r, tail, "\alpha_1"] \arrow[dd, "f"] & B_1 \arrow[rr, two heads, "\beta_1"] \arrow[dd, "g'"] \arrow[dr, two heads, "t"] & & C_1 \arrow[dd, equal] \\
& & \operatorname{im}(g') \arrow[dl, tail, "c"] & \\
A_2 \arrow[r, tail, "\alpha_3"] \arrow[d, equal] & B_3 \arrow[rr, two heads, "\beta_3"] \arrow[d, "g''"] \arrow[ddl, bend left, two heads, "m"] & & C_1 \arrow[d, "h"] \\
A_2 \arrow[r, tail, "\alpha_2"] \arrow[d, two heads, "p"] & B_2 \arrow[rr, two heads, "\beta_2"] \arrow[d, two heads, "q"] & & C_2 \arrow[d, two heads, "r"] \\
\operatorname{coker}(g') = \operatorname{coker}(f) \arrow[r, "\tilde{\alpha}_2"] & \operatorname{coker}(g) \arrow[rr, "\tilde{\beta}_2"] & & \operatorname{coker}(h)
\end{tikzcd}$$

First we consider the epi-mono factorization of $\tilde{\beta}_1$ and g', obtaining s and b as well as t and c. Note that $t \circ j$ vansihes on $\tilde{\alpha}_1$, and because we have exactness at $\ker(g)$, it induces $\bar{j} : \mathrm{im}(\tilde{\beta}_1) \to \mathrm{im}(g')$, making the square $\ker(g)$-$\mathrm{im}(\tilde{\beta}_1)$-$\mathrm{im}(g')$-$B_1$ commute. Note that $\ker(s) = \ker(\tilde{\beta}_1) = \ker(f)$, but $\ker(f) = \ker(g')$ since the square is Cartesian. Now $\ker(t) = \ker(g')$ again because c is a monomorphism. In particular, $\ker(s) = \ker(f) = \ker(t)$. Also, $\mathrm{coker}(s) = \mathrm{coker}(t) = 0$ because they are epimorphisms. Therefore, the square $\ker(g)$-$\mathrm{im}(\tilde{\beta}_1)$-$\mathrm{im}(g')$-$B_1$ is bicartesian.

We also have the square $\mathrm{im}(\tilde{\beta}_1)$-$\ker(h)$-$B_3$-$\mathrm{im}(g')$, along with the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathrm{im}(\tilde{\beta}_1) \arrow[r, tail, "b"] \arrow[d, tail, "\bar{j}"] & \ker(h) \arrow[d, tail, "l"] & \\
B_1 \arrow[r] \arrow[dr, two heads] & \mathrm{im}(g') \arrow[r, tail, "c"] \arrow[d, two heads] & B_3 \arrow[d, two heads] \arrow[dr, "g''"] & \\
 & \mathrm{im}(g) \arrow[r, tail] \arrow[rr, tail, bend right] & \mathrm{im}(g'') \arrow[r, tail] & B_2
\end{tikzcd}$$

We describe the cokernels vertically from construction: because the square $\ker(g)$-$\mathrm{im}(\tilde{\beta}_1)$-$\mathrm{im}(g')$-$B_1$ is cocartesian, then $\bar{j}$ and j have the same cokernel. The construction above induces $\mathrm{im}(g) \rightarrowtail \mathrm{im}(g'')$. By lemma, the square $\mathrm{im}(\tilde{\beta}_1)$-$\ker(h)$-$B_3$-$\mathrm{im}(g')$ is Cartesian. Similarly, g' and f have the same cokernel via m,

$$\begin{tikzcd}
\ker(m) \arrow[r, tail] \arrow[d, equal] & B_3 \arrow[r, two heads, "m"] & \mathrm{coker}(g') \\
\mathrm{im}(g') \arrow[ur, tail, "c"] & &
\end{tikzcd}$$

then we have

$$\begin{tikzcd}
 & \ker(\delta) \arrow[r] & \ker(h) \arrow[d, tail, "l"] \arrow[dr, "\delta"] & \\
\mathrm{im}(g') \arrow[r, equal] & \ker(m) \arrow[r, tail, "c"] & B_3 \arrow[r, two heads, "m"] & \mathrm{coker}(f)
\end{tikzcd}$$

with $\ker(\delta)$ is the pullback of the square. In other words, by definition of $\delta = m \circ l$, we have a Cartesian square. However, because the square $\mathrm{im}(\tilde{\beta}_1)$-$\ker(h)$-$B_3$-$\mathrm{im}(g')$ is Cartesian, $\ker(\delta) = \mathrm{im}(\tilde{\beta}_1)$ as desired. $\square$

Corollary 1.4.16. If α_1 is not a monomorphism and β_2 is not an epimorphism, then the long sequence may not be exact at the two ends, i.e. we may have to delete the two 0's. In particular, $\tilde{\alpha}_2$ is not a monomorphism and $\tilde{\beta}_2$ is not an epimorphism.

Exercise 1.4.17. For two composable morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

there exists a long exact sequence

$$\begin{tikzcd}
 & & A & & \\
0 \arrow[r] & \ker(f) \arrow[r] \arrow[ur, tail] & \ker(gf) \arrow[r, "\tilde{f}"] \arrow[u, tail] & \ker(g) \arrow[dll, "\delta"] & \\
 & \mathrm{coker}(f) \arrow[r, "\tilde{g}"'] & \mathrm{coker}(gf) \arrow[r] & \mathrm{coker}(g) \arrow[r] & 0 \\
 & & C \arrow[u, two heads] \arrow[ur, two heads] & &
\end{tikzcd}$$

Problem 2 (Exam Problem 2)**.** Let $F : A \to B$ be an exact functor (i.e. additive functor that preserves exact sequences) between two Abelian categories. State formally and prove that "F preserves connecting homomorphisms." Do appreciate how this would be difficult to prove "with elements".

1.5 Grothendieck Category

The rationale is that we want to find a setting for Abelian categories which would be precisely generalizing the R-modules, in which we can do some tricks that are the same as in module theory.

Definition 1.5.1 (Grothendieck Category)**.** An Abelian category $\mathcal{A}$ is called a Grothendieck category if

1. $\mathcal{A}$ admits arbitrary (small) coproducts, i.e. all small colimits.
2. Filtered colimits are exact. Equivalently, filtered colimits of monomorphisms are monomorphisms.
3. $\mathcal{A}$ has a generator G, i.e. an object such that $\mathbf{Hom}_{\mathcal{A}}(G, -) : \mathcal{A} \to \mathbf{Ab}$ is faithful: for any $f : X \to Y$, for all $g : G \to X$ we know $f \circ g = 0$ implies $f = 0$.

Example 1.5.2. The category of R-modules is Grothendieck.

Remark 1.5.3. The idea is first introduced by Grothendieck in **the Tôhoku paper** in 1957, that we want to replace elements of X by morphisms $G \to X$, of which is only a set's size.

Given $X \in \mathcal{A}$ in a Grothendieck category, one can prove that there is only a set of isomorphism classes of monomorphisms $Y \rightarrowtail X$, i.e. only a set of subobjects.

Definition 1.5.4 (Subobject)**.** A subobject of an object $c \in \mathcal{C}$ is a monomorphism $c' \rightarrowtail c$ with codomain c. Isomorphic subobjects, that is, subobjects $c' \rightarrowtail c \leftarrowtail c''$ with a commuting isomorphism $c' \cong c''$, are typically identified.

Example 1.5.5.

1. For a ring R, $\mathbf{R}$-**Mod** and $\mathbf{Ab}$.
2. If I is small and $\mathcal{A}$ is Grothendieck, then $\mathcal{A}^I$ is Grothendieck. In particular, if X is a topological space, then $\mathbf{Pre}_{\mathcal{A}}(X)$ remains Grothendieck.
3. Via localization/sheafification, $\mathbf{Shv}_{\mathcal{A}}(X)$ is Grothendieck.
4. Let X be a quasi-compact and quasi-separated scheme, then $\mathbf{QCoh}(X)$ is Grothendieck.

Remark 1.5.6. No assumption about existence of projectives is needed. In fact, if $\mathcal{A}$ has a projective generator P, then $\mathcal{A}$ has enough projectives.

$\mathcal{A}$ being Grothendieck does not imply $\mathcal{A}^{\mathrm{op}}$ is also Grothendieck.

Lemma 1.5.7. An object $I \in \mathcal{A}$ in a Grothendieck category is injective (lifting property of monomorphism), i.e. for any object M along with $M \to I$ and a monomorphsim $M \xrightarrow{Y}$, there exists a lift $Y \to I$:

$$\begin{tikzcd} M \arrow[r] \arrow[d, tail] & I \\ Y \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

if and only if I has the extension property with respect to monomorphisms $M \xrightarrow{G}$ where G is the generator.

Theorem 1.5.8. Every Grothendieck category $\mathcal{A}$ has enough injectives: every object $X \in \mathcal{A}$ admits a monomorphism $X \rightarrowtail I$ where I is injective.

This can be proven by an argument called "small object argument" (1957), sketched as the following.

Proof. Take $X \in \mathcal{A}$, then consider $T(X)$ which is the collection tuples $(M \rightarrowtail G, M \xrightarrow{m} X)$, up to isomorphism. This induces a pushout square

$$\begin{tikzcd}
\coprod\limits_{(\alpha,m)\in T(X)} M \arrow[r, "(m)_{(\alpha,m)}"] \arrow[d, "\coprod\limits_{(\alpha,m)} \alpha"] & X \arrow[d, tail] \\
\coprod\limits_{(\alpha,m)\in T(X)} G \arrow[r] & I_1(X)
\end{tikzcd}$$

Here $I_1(X)$ has the extension property with respect to

$$\begin{tikzcd}
& X \arrow[dr, tail] & \\
M \arrow[ur, "\exists"] \arrow[rr] \arrow[d, tail] & & I_1(X) \\
G & &
\end{tikzcd}$$

We proceed inductively and obtain a sequence

$$X \rightarrowtail I_1(X) \rightarrowtail I_2(X) = I_1(I_1(X)) \rightarrowtail \cdots \rightarrowtail I_n(X) \rightarrowtail \cdots$$

$$\cdots \rightarrowtail I_\alpha(X) \rightarrowtail I_{\alpha+1}(X) \rightarrowtail \cdots$$

for some limit ordinals α, so $I_\alpha(X) = \operatorname{co}\lim\limits_{\beta<\alpha} I_\beta(X)$. For α large enough, any map

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & I_\alpha \\
G &
\end{tikzcd}$$

factors via I_β where $\beta < \alpha$. ☐

Corollary 1.5.9. Every Grothendieck category $\mathcal{A}$ admits an injective cogenerator I: $\mathbf{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{\text{op}} \to \mathbf{Ab}$ is faithful and exact.

Proof. Consider $\coprod\limits_{G\twoheadrightarrow Z} Z \rightarrowtail I$ for injective cogenerator I. ☐

Example 1.5.10. In $\mathbf{Ab}$, $\mathbb{Q}/\mathbb{Z}$ is a cogenerator, given by $M \rightarrowtail \prod\limits_{f:M\to\mathbb{Q}/\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.

Remark 1.5.11. Grothendieck categories have injective hulls (envelopes): consider monomorphism $X \rightarrowtail I$ with I injective such that for all subobject Y along with $Y \rightarrowtail I$ such that $X \cap Y = 0$, i.e. having pullback at $X \cap Y$

$$\begin{tikzcd}
X \arrow[r, tail] & I \\
X \cap Y \arrow[u, tail] \arrow[r, tail] & Y \arrow[u, tail]
\end{tikzcd}$$

then $Y = 0$. We say $X \rightarrowtail I$ is essential.

Theorem 1.5.12 (Freyd's Adjoint Theorem)**.** Suppose we have $F : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ is Grothendieck, and such that F preserves colimits ,then F has a right adjoint G.

Proof. Consider Kan Formula: for $B \in \mathcal{B}$, we take $G(B) = \operatorname{co}\lim\limits_{(A,\beta)\in(F\downarrow\mathcal{B})} A$, where $F \downarrow \mathcal{B}$ is the comma category, with object pairs $(A \in \mathcal{B}, \beta : FA \to B)$ and morphism between $(A, FA \xrightarrow{\beta} B)$ and $(A', FA' \xrightarrow{\beta} B)$ is given by $\alpha : A \to A'$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F\alpha"] \arrow[dr, "\beta"'] & FA' \arrow[d, "\beta'"] \\ & B \end{tikzcd}$$

commutes. In particular, the mapping $(F \downarrow \mathcal{B}) \to \mathcal{A}$ is given by $(A, \beta) \mapsto A$. $\square$

Example 1.5.13. Consider the constant diagonal functor $\Delta : \mathcal{A} \to \mathcal{A}^S$ where S is a set, and note that Δ preserves colimits, then it has a right adjoint, given by the product functor $\prod\limits_S : \mathcal{A}^S \to \mathcal{A}$.

$$\begin{tikzcd} \mathcal{A} \arrow[d, shift right, "\Delta"'] \\ \mathcal{A}^S \arrow[u, shift right, dashed, "\prod_S"'] \end{tikzcd}$$

Therefore, $\mathcal{A}$ has all limits.

Theorem 1.5.14 (Freyd's Limit Theorem)**.** Any functor F that commutes with limits has a left adjoint.

Theorem 1.5.15 (Gabriel-Popesch Embedding Theorem)**.** Let $\mathcal{A}$ be Grothendieck and G be a generator. Let $R = \mathbf{End}_{\mathcal{A}}(G)$, then the left exact functor $\mathbf{Hom}_{\mathcal{A}}(G, -) : \mathcal{A} \hookrightarrow$ **Mod-R** is fully faithful with an exact left adjoint $(R \to G)$.

Remark 1.5.16. This implies that $\mathcal{A}$ is equivalent to a certain Gabriel quotient of $\mathbf{Mod}-\mathbf{R}$.

However, note that if I is an injective cogenerator and $S = \mathbf{End}_{\mathcal{A}}(I)$, then $\mathbf{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{\text{op}} \to S$-**Mod** is faithful and exact, but may not be full in general.

There is also another embedding theorem, described below:

Theorem 1.5.17 (Freyd-Mitchell Embedding Theorem)**.** Let $\mathcal{Q}$ be a small Abelian category, then there exists a ring R and a fully faithful exact functor $F : \mathcal{Q} \to \mathbf{R}$-**Mod**.

Remark 1.5.18. This describes the extent to allow elementwise diagram chasing. It uses the following construction:

Consider $\mathcal{L} \subseteq \mathbf{Fun}_{\mathbf{Add}}(\mathcal{Q}^{\text{op}}, \mathbf{Ab})$ which is the collection of left-exact functors $\mathcal{Q}^{\text{op}} \to \mathbf{Ab}$. Then there is a fully faithful exact functor $h : \mathcal{Q} \to \mathcal{L}$ that sends $X \mapsto \mathbf{Hom}_{\mathcal{A}}(X, -)$

Note that $\mathcal{L}$ is not an Abelian subcategory of the Abelian category $\mathbf{Fun}_{\mathbf{Add}}(\mathcal{Q}^{\text{op}}, \mathbf{Ab})$:

$$\begin{tikzcd} Q \arrow[r, hook, "f"] \arrow[dr, hook, "g"'] & \mathbf{Fun}_{\mathbf{Add}}(\mathcal{Q}^{\text{op}}, \mathbf{Ab}) \arrow[d, shift right, two heads, "a"'] \\ & \mathcal{L} \arrow[u, shift right, hook, "h"'] \end{tikzcd}$$

Here f is not exact, and g and h are fully faithful. Via the localization a, we obtain a Grothendieck category $\mathcal{L}$.

Using embedding $\mathbf{Hom}(-, I)$ for $k : \to$**S-Mod** (note that this is not full), we can check that $k \circ h$ is still full.

1.6 Exactness of Functors

Definition 1.6.1 (Exact Functor)**.** An additive functor $F : \mathcal{A} \to \mathcal{B}$ between Abelian categories is called exact if it preserves exact sequences: given a sequence $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$ exact at A_2, we may obtain another sequence $FA_1 \xrightarrow{Ff} FA_2 \xrightarrow{Fg} FA_3$ that is exact at FA_2.

Exercise 1.6.2. F is exact if and only if F preserves short exact sequences if and only if F preserves kernels and cokernels if and only if F preserves bicartesian squares.

Additivity of functor would follow from kernel preservation since it preserves direct sums.

Example 1.6.3. Let $\alpha : R \to S$ be a ring homomorphism, then the restriction of scalars, $\mathbf{Res}_{\alpha} : S$-**Mod** $\to T$-**Mod** given by $M \mapsto M$ but with $r \cdot m = \alpha(r) \cdot m$, is exact.

In a similar fashion, consider $\mathbb{Z} \to S$. The forgetful functor $U : S$-**Mod** $\to \mathbf{Ab}$ is exact and detects exactness (creates kernels and cokernels).

Note that many functors are only exact on one side: the sheafification functor is exact, but the forgetful functor from sheaves to presheaves is not.

Definition 1.6.4 (Left Exact, Right Exact)**.** An additive covariant functor $F : \mathcal{A} \to \mathcal{B}$ is called left exact if it preserves kernels: $\ker_{\mathcal{B}}(Ff) \cong F(\ker_{\mathcal{A}}(f))$ in $\mathcal{B}$ for all $f \in \mathcal{A}$. It is called right exact if it preserves cokernels: $\operatorname{coker}_{\mathcal{B}}(Ff) \cong F(\operatorname{coker}_{\mathcal{A}}(f))$ in $\mathcal{B}$ for all $f \in \mathcal{A}$.

Exercise 1.6.5. F is left exact if and only if for all exact sequences

$$0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0$$

the sequence

$$0 \longrightarrow FA_1 \xrightarrow{Ff} FA_2 \xrightarrow{Fg} FA_3$$

is exact in $\mathcal{B}$ if and only if for all exact sequences

$$0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$$

the sequence

$$0 \longrightarrow FA_1 \xrightarrow{Ff} FA_2 \xrightarrow{Fg} FA_3$$

is exact in $\mathcal{B}$.

Similar property holds true for right exact functors.

Remark 1.6.6. The convention for contravariant functors $F : \mathcal{A} \to \mathcal{B}$ is to be viewed as the covariant functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$. Therefore, for a contravariant functor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ to be left exact, for any exact sequence

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

we know the sequence

$$0 \longrightarrow FA_3 \longrightarrow FA_2 \longrightarrow FA_1$$

is exact. Similarly, for exact sequences

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

we know the sequence

$$0 \longrightarrow FA_3 \longrightarrow FA_2 \longrightarrow FA_1$$

is exact.

Example 1.6.7. Let M be an R-module. Then $M \otimes_R - : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is right exact. It is exact only when M is (right flat) over R. For example, let $R = \mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z}$, then the monomorphism $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ being monomorphism does not mean the same is true for $(0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}$.

Remark 1.6.8. A left exact functor is exact if and only if it preserves epimorphism. A right exact functor is exact if and only if it preserves monomorphism.

Example 1.6.9.

1. $\mathbf{Hom}_R(M, -) : \mathbf{Mod}\text{-}R \to \mathbf{Ab}$ is left exact. It is right exact only when M is projective: for all epimorphisms $X \to Y$ in $\mathbf{Mod}\text{-}R$ we have $\mathbf{Hom}(M, X) \to \mathbf{Hom}(M, Y)$ as an epimorphism.

$$\begin{tikzcd} & X \arrow[d, two heads] \\ M \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

2. $\mathbf{Hom}_R(-, M) : (\mathbf{Mod}\text{-}R)^{\mathrm{op}} \to \mathbf{Ab}$ is right exact. It is left exact only when M is injective: for all monomorphisms $X \to Y$ in $\mathbf{Mod}\text{-}R$ we have $\mathbf{Hom}(Y, M) \to \mathbf{Hom}(X, M)$ as a monomorphism.

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail] & M \\ Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

3. Let X be a topological space. Take $\mathcal{A}$ to be the category of R-modules, or just a Grothendieck category. The functor $\Gamma(X, -) : \mathbf{Shv}_{\mathcal{A}}(X) \to \mathcal{A}$ by sending $P \mapsto P(X)$ is only left exact.

Remark 1.6.10. Let $\mathcal{C}, \mathcal{D}$ be categories. An adjunction pair $F \dashv G$ consists of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with a natural isomorphism

$$\mathbf{Mor}_{\mathcal{D}}(FC, D) \cong \mathbf{Mor}_{\mathcal{C}}(C, GD)$$

and is natural for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$. This also produces a unit-counit pair, with unit $\eta : \mathbf{id} \to GF$ and counit $\varepsilon : FG \to \mathbf{id}$ that satisfies the triangle identities $\varepsilon F \circ F\eta = \mathbf{id}_F$ and $G\varepsilon \circ \eta G = \mathbf{id}_G$. Then the adjunction isomorphism is

$$\begin{tikzcd} & \mathbf{Mor}_{\mathcal{C}}(GFC, GD) \arrow[dr, "\eta_C^* = - \circ \eta_C"] & \\ \mathbf{Mor}_{\mathcal{D}}(FC, D) \arrow[ur, "G"] & & \mathbf{Mor}_{\mathcal{C}}(C, GD) \arrow[dl, "F"] \\ & \mathbf{Mor}_{\mathcal{D}}(FC, FGD) \arrow[ul, "(\varepsilon_D)_* = \varepsilon_D \circ -"] & \end{tikzcd}$$

Note that the adjunction of additive functors is just the same (the adjunction bijection is automatically an isomorphism of Abelian groups).

Proposition 1.6.11. Let $F : \mathcal{A} \leftrightarrows \mathcal{B} : G$ be an adjunction $F \dashv G$ between Abelian categories. Therefore, F preserves cokernels. Then F is right exact. Similarly, G is left exact.

Proof. Recall the general fact that left adjoints preserve colimits and right adjoints preserve limits. Then left adjoints are right exact and right adjoints are left exact. □

Example 1.6.12. There are some well-known adjunctions on sheaves, like the sheafification functor and the forgetful functor between presheaves and sheaves, i.e. $a \dashv U$. There is one between $\mathbf{Shv}(X)$ and $\mathbf{Shv}(U)$ where $U \subseteq X$. Given $j : U \hookrightarrow X$ open, we have an adjunction $j^* \dashv j_*$ where $j^* = \mathbf{Res}_U : \mathbf{Shv}(X) \to \mathbf{Shv}(U)$ and j_* is defined by $(j_*Q)(V) = Q(U \cap V)$ for any open $V \subseteq X$.

Example 1.6.13. Consider the (S, R)-bimodule ${}_SM_R$. We have an adjunction $M \otimes_R - \dashv \mathbf{Hom}_S(M, -)$ between R-modules and S-modules: $M \otimes_R : R\text{-}\mathbf{Mod} \leftrightarrows S\text{-}\mathbf{Mod} : \mathbf{Hom}_S(M, -)$. Therefore, $M \otimes_R -$ is right exact, and $\mathbf{Hom}_S(M, -)$ is left exact. As for $\mathbf{Hom}_S(-, M)$, we have

$$\begin{tikzcd} R\text{-}\mathbf{Mod} \arrow[d, dashed, shift right, "(\mathbf{Hom}_R(-,M))^{\mathrm{op}}"'] \\ (S\text{-}\mathbf{Mod})^{\mathrm{op}} \arrow[u, shift right, "\mathbf{Hom}_S(-,M)"'] \end{tikzcd}$$

Exercise 1.6.14. Show that $\mathbf{Hom}_R(-, M)$ yields the left adjoint $(\mathbf{Hom}_R(-, M))^{\mathrm{op}} : (R\text{-}\mathbf{Mod})^{\mathrm{op}} \to S\text{-}\mathbf{Mod}$. Therefore, $\mathbf{Hom}_R(-, M)$ is left exact. But this does not mean $\mathbf{Hom}_R(-, M)$ is right exact: we only know $(R\text{-}\mathbf{Mod})^{\mathrm{op}}$ is right exact, which means $\mathbf{Hom}_R(-, M)$ is left exact (again).

Proposition 1.6.15. Let R be a commutative ring and $S \subseteq R$ is a multiplicative subset (closed under multiplication and contains unit). Generally, we can also ask for a Ore subset $S \subseteq R$ if R is non-commutative. Then the localization $S^{-1} : R\text{-}\mathbf{Mod} \to (S^{-1}R)\text{-}\mathbf{Mod}$ is exact.

Proof. Check $S^{-1}(-) \cong (S^{-1}R) \otimes_R -$. The bijection is given by $\frac{m}{s} \leftrightarrow \frac{1}{s} \otimes m$ and $\frac{a \cdot m}{s} \leftarrow \frac{a}{s} \otimes m$. Therefore, $S^{-1}(-)$ is right exact automatically. Also, we have

$$\begin{tikzcd} R\text{-}\mathbf{Mod} \arrow[d, shift right, "S^{-1}(-)"'] \\ (S^{-1}R)\text{-}\mathbf{Mod} \arrow[u, shift right, "inc"'] \end{tikzcd}$$

where the inclusion functor inc is the restriction of scalars. Therefore, it is enough to show that $S^{-1}(-)$, i.e. preserves monomorphisms. If we have $f : M \rightarrowtail N$, then $(S^{-1})(f) : S^{-1}M \to S^{-1}N$ that sends $\frac{m}{s} \mapsto \frac{f(m)}{s}$, and if $\frac{f(m)}{s} = 0$, then there exists $t \in S$ such that $t \cdot f(m) = 0$. Therefore, $f(tm) = 0$ and so $tm = 0$. Hence, $\frac{m}{s} = \frac{tm}{ts} = 0$ in $S^{-1}M$. $\square$

1.7 Localization and Gabriel Quotient

Definition 1.7.1 (Abelian Subcategory)**.** An Abelian subcategory of an Abelian category $\mathcal{B}$ is a full subcategory $\mathcal{A} \subseteq \mathcal{B}$ which is Abelian such that $\mathbf{inc} : \mathcal{A} \hookrightarrow \mathcal{B}$ is exact, i.e. $\mathcal{A}$ is closed in $\mathcal{B}$ under taking kernels and cokernels.

Definition 1.7.2 (Serre Subcategory)**.** A Serre subcategory $\mathcal{A}$ of an Abelian category $\mathcal{B}$ is a full subcategory closed under:

- Subobjects: Suppose $B \rightarrowtail A$ in $\mathcal{B}$ (i.e. is a subobject) and $A \in \mathcal{A}$, then $B \in \mathcal{A}$.
- Quotients: Suppose $A \twoheadrightarrow B$ in $\mathcal{B}$ and $A \in \mathcal{A}$, then $B \in \mathcal{A}$.
- Extensions: Suppose $A \rightarrowtail B \twoheadrightarrow A'$ is exact in $\mathcal{B}$ and $A, A' \in \mathcal{A}$, then $B \in \mathcal{A}$.

Example 1.7.3. Let R be a (commutative) ring and $S \subseteq R$ be a multiplicative subset (central if R is not commutative, or Ore). Consider $R \to S^{-1}R$ localization and $\mathcal{B} = R$-**Mod**. Let $\mathcal{A} = S$-**torsion** R-**Mod**. An element in $\mathcal{A}$ is just $M \in \mathcal{B}$ such that for all $m \in M$, there exists $s \in S$ such that $sm = 0$. Note $\mathcal{A}$ is also the kernel of the localization functor $S^{-1}(-) : R\text{-}\mathbf{Mod} \to S^{-1}R\text{-}\mathbf{Mod}$, and this functor is exact.

Therefore, we have a more general example.

Example 1.7.4. Let $F : \mathcal{B} \to \mathcal{C}$ be an exact functor. Then $\ker(F) = \{B \in \mathcal{B} \mid F(B) \cong 0\}$ is a Serre subcategory of $\mathcal{B}$.

One may wonder: is the converse true? Given $\mathcal{A} \subseteq \mathcal{B}$ is a Serre subcategory, would there exist $\mathcal{C}$ and $F : \mathcal{B} \to \mathcal{C}$ such that $\mathcal{A} = \ker(F)$? The answer is yes, and this is called a Gabriel quotient, also known as a localization.

Definition 1.7.5 (Gabriel Quotient)**.** Let $\mathcal{A} \subseteq \mathcal{B}$ be a Serre subcategory. We want $\mathcal{B}/\mathcal{A}$ as the Gabriel Quotient. Define an exact functor $Q : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ (note $\mathcal{B}/\mathcal{A}$ is Abelian) such that $Q(\mathcal{A}) = 0$ and is initial among those exact functor $F : \mathcal{B} \to \mathcal{C}$ such that $F(\mathcal{A}) = 0$:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{C} \\ \mathcal{B}/\mathcal{A} \arrow[ur, dashed, "\exists! \bar{F}"'] \end{tikzcd}$$

In fact, $\bar{F}$ is exact.

Remark 1.7.6. A morphism $f : X \to Y$ in $\mathcal{B}$ is an isomorphism if and only if $\ker(f) = \operatorname{coker}(f) = 0$. So when we modulo out $\mathcal{A} \subseteq \mathcal{B}$, we expect new isomorphisms: $X \xrightarrow{f} Y$ in $\mathcal{B}$ such that $\ker(f), \operatorname{coker}(f) \in \mathcal{A}$.

Conversely, if we invert them, i.e. if $F : \mathcal{B} \to \mathcal{C}$ is exact and such that $Ff = 0$ for all f's with kernels and cokernels in $\mathcal{A}$, then $F(\mathcal{A}) = 0$ because $f : 0 \to A \in \mathcal{A}$.

In mathematics, localization of a category consists of adding to a category inverse morphisms for some collection of morphisms, constraining them to become isomorphisms. This is formally similar to the process of localization of a ring; it in general makes objects isomorphic that were not so before. In homotopy theory, for example, there are many examples of mappings that are invertible up to homotopy; and so large classes of homotopy equivalent spaces[clarification needed]. Calculus of fractions is another name for working in a localized category.

Definition 1.7.7 (Localization)**.** A localization $Q : \mathcal{B} \to S^{-1}\mathcal{B} = \mathcal{B}[S^{-1}]$ of categories with respect to a class S of morphisms of $\mathcal{B}$ (if it exists) is the universal (initial) functor out of B such that $Q(s)$ is an isomorphism for all $s \in S$. In other words, we want to "replace S by $Q^{-1}(\text{isomorphism}) = \bar{S}$:

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{C} \\
S^{-1}B \arrow[ur, dashed, "\exists!\bar{F}"'] \arrow[r, equal] & \bar{S}^{-1}B
\end{tikzcd}
$$

such that the diagram commutes, where F is such that $F(s)$ is an isomorphism for all $s \in S$.

The old and brutal solution to this is the "long zig-zags".

Remark 1.7.8. Define $S^{-1}B$ with the same objects, where $X \to Y$ is an equivalence class if there exists a sequence of zig-zag arrows

$$X \longleftarrow \cdot \longleftrightarrow \cdot \longrightarrow \cdots \longleftarrow \cdot \longrightarrow Y$$

where the leftward arrows are in S. This is not limited to the number of intermediate steps. Note that there may be set theory issues, that the morphisms are not a set's worth. The equivalence is genreated by amplification:

$$
\begin{tikzcd}
\cdots & \cdot \arrow[l, "s"'] \arrow[r, "f"'] & \cdots \sim \cdots & \cdot \arrow[l, "st"'] \arrow[r, "ft"'] & \cdots \\
& \cdot \arrow[ul] \arrow[u, "t"] \arrow[ur] & & &
\end{tikzcd}
$$

and

$$\cdots \xrightarrow{f} \cdot \xleftarrow[\mathbf{id}]{} \cdot \xrightarrow{g} \cdots \sim \cdots \xrightarrow{gf} \cdots$$

where every arrow with left inclination is from S.

This is much better when there is a calculus of fraction (Ore condition): whenever there is

$$\cdot \xrightarrow{g} \cdot \xleftarrow[t]{} \cdot$$

where $t \in S$, then there exists a commutative square

$$
\begin{tikzcd}
& \cdot & \\
\cdot \arrow[ur, "g"] & & \cdot \arrow[ul, "t"'] \\
& \cdot \arrow[ul, "s"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

where $s, t \in S$ and

$$
\begin{tikzcd}
& & & \cdot \arrow[dlll] \arrow[dll, "s"'] \arrow[dr, "f"] \arrow[drr] & & \\
\cdot & \cdot \arrow[l] \arrow[r, "g"'] & \cdot & \cdot \arrow[l, "t"] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

(Again, with arrows inclining leftwards to be in S.) We still have set theory issues here.

Remark 1.7.9. We say that $Q : \mathcal{B} \to \bar{\mathcal{B}}$ is a localization, if it is with respect to $S = Q^{-1}(\text{isomorphism})$.

Exercise 1.7.10. If $\mathcal{A} \subseteq \mathcal{B}$ is a Serre subcategory, then $S = \{s : x \to y, s \in \mathcal{B} \mid \ker(s), \operatorname{coker}(s) \in \mathcal{A}\}$ satisfies calculus of fractions.

Proposition 1.7.11. Let

$$
\begin{tikzcd}
\mathcal{B} \arrow[d, shift right, "Q"'] \\
\mathcal{C} \arrow[u, shift right, "R"']
\end{tikzcd}
$$

be an adjunction of categories. The following are equivalent:

1. Q is a localization with respect to Q^{-1}(isomorphism), i.e. self-dual.
2. R is fully faithful.
3. The

Proof. See Gabriel-Zisman (1967). □

Proposition 1.7.12. Let

$$\begin{tikzcd} \mathcal{B} \arrow[d, shift right, "Q"'] \\ \mathcal{C} \arrow[u, shift right, "R"'] \end{tikzcd}$$

be an adjunction of Abelian cateogries. Suppose Q is exact and R is fully faithful.

1. Q is a localization with respect to $\{f \in \mathcal{B} \mid \ker(f), \operatorname{coker}(f) \in \mathcal{A}\}$ where $\mathcal{A} = \ker(Q) = \{x \mid Q(x) = 0\}$. Hence, $Q : \mathcal{B} \to \mathcal{C}$ realizes $\mathcal{B}/\mathcal{A}$.
2. A functor $\bar{F} : \mathcal{C} \to \mathcal{D}$ between Abelian categories is exact if and only if $F = \bar{F} \circ Q : \mathcal{B} \to \mathcal{D}$ is exact.

Proof. We focus on the proof of the second part. If $\bar{F} : \mathcal{C} \to \mathcal{D}$ such that $F = \bar{F} \circ Q : \mathcal{B} \to \mathcal{D}$ is exact, then $\bar{F} = \bar{F} \circ Q \circ R = F \circ R$. So $\bar{F}$ is left exact. Why does $\bar{F}$ preserves epimorphism? Take $g : X \twoheadrightarrow Y$ in $\mathcal{C}$. Then $RX \xrightarrow{Rg} RY \twoheadrightarrow \operatorname{coker}(Rg) = Z$ in $\mathcal{B}$. Then $QRg : X \twoheadrightarrow Y \to QZ = 0$ for $Z \in \mathcal{A}$. Therefore, $FZ = \bar{F}QZ = 0$ and so $FRX \to FRY \to FZ = 0$ exact in $\mathcal{D}$. Therefore, we have

$$\begin{tikzcd} FRX \arrow[r, two heads] \arrow[d, equal] & FRY \arrow[d, equal] \\ \bar{F}X \arrow[r, two heads, "\bar{F}g"'] & \bar{F}Y \end{tikzcd}$$

is an epimorphism. □

Example 1.7.13. Consider

$$\begin{tikzcd} R\text{-}\mathbf{Mod} \arrow[d, shift right, "S^{-1}(-)=S^{-1}R\otimes_R -"'] \\ (S^{-1}R)\text{-}\mathbf{Mod} \arrow[u, hook, shift right, "\mathrm{res}"'] \end{tikzcd}$$

where the restrion functor is fully faithful. Therefore, $S^{-1}(-)$ is a Gabriel quotient: $(S^{-1}R)$-**Mod** $= R$-**Mod**$/S$-**torsion Mod**.

Example 1.7.14. In sheaves, we have

$$\begin{tikzcd} \mathbf{Pre}(X) \arrow[d, shift right, "a"'] \\ \mathbf{Shv}(X) \arrow[u, hook, shift right, "inc"'] \end{tikzcd}$$

where the inclusion functor is fully faithful.

Therefore, sheafification is a Gabriel Quotient.

Problem 3 (Exam Problem 3)**.** Let $j : U \hookrightarrow X$ be open in topological space. Then

$$\begin{tikzcd} \mathbf{Shv}(X) \arrow[d, shift right, "j^*"'] \\ \mathbf{Shv}(U) \arrow[u, shift right, "j_*"'] \end{tikzcd}$$

fits in the above setting: $\mathbf{Shv_{Ab}}(U)$ is a Gabriel Quotient (i.e. localization) of $\mathbf{Shv_{Ab}}(X)$. Here j^* is the restriction and j_* takes QV to $Q(U \cap V) = j^{-1}(V)$.

Theorem 1.7.15 (Gabriel, 1962)**.** Let $\mathcal{A} \subseteq \mathcal{B}$ be a Serre subcategory. The quotient $Q : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ can be done as follows:
Let $\mathcal{B}/\mathcal{A}$ has the same objects as in $\mathcal{B}$. The morphisms $\mathbf{Mor}_{\mathcal{B}/\mathcal{A}}(X, Y)$ are equivalence classes of

$$\begin{tikzcd}
X & Y \arrow[d, two heads, "\cdot"] \\
X' \arrow[u, tail, "\cdot"] \arrow[r] & Y'
\end{tikzcd}$$

where the $\operatorname{coker}(X' \rightarrowtail X) \in \mathcal{A}$ and $\ker(Y \twoheadrightarrow Y') \in \mathcal{A}$, under amplification

$$\begin{tikzcd}
X & Y \arrow[d, two heads, "\cdot"] \arrow[dd, bend left, two heads, "\cdot"] \\
X' \arrow[u, tail, "\cdot"] \arrow[r, "f"] & Y' \arrow[d, two heads, "u"', "\cdot"] \\
X'' \arrow[u, tail, "\cdot", "t"'] \arrow[uu, bend left, tail, "\cdot"] \arrow[r, "uft"'] & Y''
\end{tikzcd}$$

and composition

$$\begin{tikzcd}
X \arrow[rr, dashed] & & Y \arrow[dl] \arrow[rr, dashed] & & Z \arrow[d, two heads, "\cdot"] \arrow[dd, bend left, two heads, "\cdot"] \\
\cdot \arrow[u, tail, "\cdot"] \arrow[r] & \cdot & & \cdot \arrow[ul, tail, "\cdot"] \arrow[r] \arrow[dl, "\cdot"] & \cdot \arrow[d, two heads, "\cdot"] \\
\cdot \arrow[u, tail, "\cdot"] \arrow[uu, bend left, tail, "\cdot"] \arrow[rr, tail] \arrow[rrrr, bend right] & & \cdot \arrow[ul, "\cdot"] \arrow[rr] & & \cdot
\end{tikzcd}$$

where the bottom left square is a pushback, the bottom right square is a pushout and the central square is an epi-mono factorization.

Example 1.7.16. If R is a Noetherian ring, let $\mathcal{A}$ be finitely generated R-modules, $\mathcal{B}$ be finitely generated $S^{-1}R$-modules, then the finitely generated $S^{-1}R$-modules is still a Gabriel quotient, although there is no right adjoint to the localization below:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook] \arrow[d, shift right, "S^{-1}"'] & R\text{-}\mathbf{Mod} \arrow[d, shift right, "S^{-1}"'] \\
\mathcal{B} \arrow[r, hook] \arrow[u, dashed, shift right, "\nexists"'] & S^{-1}R\text{-}\mathbf{Mod} \arrow[u, hook, shift right]
\end{tikzcd}$$

1.8 Injectives and Projectives

Throughout this section, $\mathcal{A}$ is an Abelian category.

Definition 1.8.1 (Projective, Injective)**.** An object $P \in \mathcal{A}$ is called projective if $\mathbf{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{Ab}$ is exact.

Since $\mathbf{Hom}_{\mathcal{A}}(X, -)$ is always left exact, it is equivalent to say $\mathbf{Hom}_{\mathcal{A}}(P, -)$ preserves epimorphisms, or just right exact. That is, for every $f : X \twoheadrightarrow Y$ epimorphism in $\mathcal{A}$, for every $g : P \to Y$ there exists $\hat{g} : P \to X$ such that $f \circ \hat{g} = g$:

$$\begin{tikzcd}
 & X \arrow[d, two heads, "f"] \\
P \arrow[ur, dashed, "\exists \hat{g}"] \arrow[r, "g"'] & Y
\end{tikzcd}$$

Dually, an object I is called injective if $I \in \mathcal{A}^{op}$ is projective, i.e. $\mathbf{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{\text{op}} \to \mathbf{Ab}$ is (right) exact. Equivalently, $\mathbf{Hom}(f, I)$ is an epimorphism for all monomorphism f in $\mathcal{A}$:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, tail, "f"'] & I \\ Y \arrow[ur, dashed, "\exists \hat{g}"'] & \end{tikzcd}$$

We write $\mathbf{Proj}(\mathcal{A})$ as the full subcategory of projectives in $\mathcal{A}$, and $\mathbf{Inj}(\mathcal{A})$ as the full subcategory of injectives in $\mathcal{A}$.

Exercise 1.8.2. $\mathbf{Proj}(\mathcal{A})$ is closed under all coproducts that exist in $\mathcal{A}$ (in particular, direct sum).

$\mathbf{Inj}(\mathcal{A})$ is closed under all products that exist in $\mathcal{A}$.

Both are closed under direct summands: if $X \oplus Y$ is in the category, then both X and Y are in the same category.

Proposition 1.8.3. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{A}$.

(a) If A is injective, then the sequence is split exact.

(b) If C is projective, then the sequence is split exact.

Example 1.8.4. Let $\mathcal{A}$ be the category of R-modules. Then P is projective if and only if P is a direct summand of a free R-module: there exists Q such that $P \oplus Q$ is free. (Free indicates projective here: $R^{(I)} \in \mathbf{Proj}(R)$ for all sets I because $\mathbf{Hom}_R(R, M) \cong M$ as Abelian groups.)

Any $M \in \mathcal{A}$ is a quotient of free module, i.e. $R^{(M)} = \coprod\limits_{m \in M} R \twoheadrightarrow M$ with $(a_m)_{m \in M} \leftrightarrow \sum\limits_{m \in M} a_m \cdot m$ where the latter is a finite summation.

Remark 1.8.5. These notions depend on the ambient category. For example, let $R = \mathbb{Z}$ and $R \twoheadrightarrow F$ where F is some residue field like $\mathbb{Z}/p\mathbb{Z}$. Then in the category of F-modules (simply just F-vector spaces) we have

$$\begin{aligned} \mathbf{Spec}(\pi_*) : F\text{-}\mathbf{Mod} &\hookrightarrow R\text{-}\mathbf{Mod} \\ V &\mapsto V \end{aligned}$$

given by the restriction of scalars, and is fully faithful. But in the category of F-modules, all sequences split, which is equivalent to having every object as projective and injective.

In general, V is neither injective nor projective in the category of R-modules, like in $\mathbb{Z}/p\mathbb{Z}$.

Proposition 1.8.6. Let R be a ring and $I \in R$-**Mod**, then I is injective if and only if it has the extension property with respect to monomorphisms of the form $J \hookrightarrow R$ for left ideals J, i.e. $\mathbf{Hom}_R(I, -)$ maps those to epimorphisms.

The following is the sketch of the proof.

Proof. Use Zorn's lemma to try to build an extension of a general monomorphism $M \rightarrowtail M'$ (as a gradual directed system to I). We then can reduce to the situation

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & M' = M + R \cdot m \\ I & \end{tikzcd}$$

We then use the pushout square

$$\begin{tikzcd} & R \arrow[d, two heads, "\cdot m"] \\ M \cap R \cdot m \arrow[r, hook] \arrow[d, hook] \arrow[dd, bend right] & R \cdot m \arrow[d, hook] \arrow[ddl, dashed] \\ M \arrow[r] & M + R \cdot m \arrow[dl, dashed] \\ I & \end{tikzcd}$$

(here the dashed arrow from $R \cdot m$ to I is the hypothesis we have) and reduce to the setting of principal module as submodule

$$\begin{tikzcd}
M'' \arrow[r, hook] \arrow[d] & R \cdot m \arrow[dl, dashed] \\
I &
\end{tikzcd}$$

and the pushout square given by J:

$$\begin{tikzcd}
J \arrow[r, hook] \arrow[d] \arrow[dd, bend right] & R \arrow[d, two heads] \arrow[ddl, dashed] \\
M'' \arrow[r, hook] \arrow[d] & R \cdot m \arrow[dl, dashed, "\exists"] \\
I &
\end{tikzcd}$$

where the dashed arrow from R to I exists by the hypothesis above. ☐

Corollary 1.8.7. Consider $R = \mathbb{Z}$ and so the R-modules are exactly $\mathbf{Ab}$. (In general, we can take any PID R and an R-module I.) An Abelian group I is injective in $\mathbf{Ab}$ if and only if it is divisible, i.e. $\forall x \in I$, $\forall a \in \mathbb{Z}$, $a \neq 0$, there exists $y \in I$ such that $a \cdot y = x$. In particular, $\mathbb{Q}$ is injective and $\mathbb{Q}/\mathbb{Z}$ is injective.

Proof. Consider

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times"] \arrow[d, hook, "\cdot a"'] & I \\
\mathbb{Z} \arrow[ur, dashed, "y"'] &
\end{tikzcd}$$

where $a\cdot$ is the map given by $a\mathbb{Z} \hookrightarrow \mathbb{Z}$. ☐

Definition 1.8.8 (Enough Injectives, Enough Projectives)**.** An Abelian category $\mathcal{A}$ has enough injectives if every $X \in \mathcal{A}$ admits a monomorphism $X \hookrightarrow I$ into I injective. Dually, $\mathcal{A}$ has enough projectives if for all $X \in \mathcal{A}$, there exists $P \in \mathbf{Proj}(\mathcal{A})$ such that $P \twoheadrightarrow X$.

Example 1.8.9. R-**Mod** has enough projectives.

Exercise 1.8.10. If $\mathcal{A}$ has enough injectives (respectively, projectives), an object I (respectively, P) is injective (respectively, projective) if and only if every sequence $I \rightarrowtail X \twoheadrightarrow Y$ (respectively, $X \rightarrowtail Y \twoheadrightarrow P$) splits.

Proposition 1.8.11. The category $\mathbf{Ab} = \mathbb{Z}$-**Mod** has enough injectives.

Proof. If M is an Abelian group and $0 \neq x \in M$, then there exists $f : M \to \mathbb{Q}/\mathbb{Z}$ such that $f(x) \neq 0$.

If $\mathbf{Ann}_{\mathbb{Z}}(m) = 0$, i.e. no torsion, then we have the map $[\frac{1}{2}] : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ injective, sending $1 \mapsto [\frac{1}{2}] \neq 0$:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "{[\frac{1}{2}]}"] \arrow[d, hook, "\cdot m"'] & \mathbb{Q}/\mathbb{Z} \\
M \arrow[ur, dashed, "\exists f"'] &
\end{tikzcd}$$

Note that the map $\cdot m$ sends 1 to m, and the map sends f from x to $[\frac{1}{2}] \neq 0$.

If $\mathbf{Ann}_{\mathbb{Z}}(m) \neq 0$, i.e. has torsion, then $\mathbf{Ann}_{\mathbb{Z}}(m) = n\mathbb{Z}$ as it has to be an ideal in $\mathbb{Z}$ for some $0 \neq n \in \mathbb{Z}$. Therefore, the map $\frac{1}{n} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is injective, sending $[1] \mapsto [\frac{1}{n}] \neq 0$:

$$\begin{tikzcd}
\mathbb{Z}/n\mathbb{Z} \arrow[r, "\frac{1}{n}"] \arrow[d, hook, "\cdot x"'] & \mathbb{Q}/\mathbb{Z} \\
M \arrow[ur, dashed, "\exists f"'] &
\end{tikzcd}$$

Similarly, $\cdot x$ sends $[1]$ to x.

Therefore, the map $M \to \prod_{f \in \mathbf{Hom}(M, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$ is a well-defined monomorphism sending x to $(f(x))_f$, because $\mathbb{Q}/\mathbb{Z}$ is injective, and so the product is also an injective object. ☐

Proposition 1.8.12. Let

$$\begin{tikzcd} \mathcal{A} \arrow[d, shift right, "F"'] \\ \mathcal{B} \arrow[u, shift right, "G"'] \end{tikzcd}$$

be an adjunction between Abelian categories.

1. Suppose that F is exact, then its right adjoint preserves injectives.
2. Suppose that G is exact, then its left adjoint preserves projectives.

Proof. It suffices to prove the first part. Let $J \in \mathbf{Inj}(\mathcal{B})$. We want to show $GJ \in \mathbf{Inj}(\mathcal{A})$. Note that $\mathbf{Hom}_{\mathcal{A}}(-, GJ) \cong \mathbf{Hom}_{\mathcal{B}}(F-, J) = \mathbf{Hom}_{\mathcal{B}}(-, J) \circ F-$. Here $F-$ is exact and $\mathbf{Hom}_{\mathcal{B}}(-, J)$ is exact because J is injective, and so GJ is injective as desired. $\square$

Alternatively, suppose $F : \mathcal{A} \to \mathcal{B}$ is exact, then left adjoint of F has to preserve projectives, and right adjoint of F has to preserve injectives.

The question is, can we use this to prove that $\mathcal{A}$ has enough injectives/projectives from the same property for $\mathcal{B}$? The answer is clearly no; just take zero functor, since it loses too much information. We want to say F has to "preserve information".

Remark 1.8.13. A faithful functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories satisfies $F\mathcal{A} = 0 \Rightarrow \mathcal{A} = 0$ since $\mathbf{id}_{\mathcal{A}}$ is being faithful.

If F is also exact, this implies that F is conservative (Ff being an isomorphism implies f is an isomorphism) by considering kernels and cokernels. Conversely, if $F : \mathcal{A} \to \mathcal{B}$ is exact between Abelian categories and is conservative, then F is faithful by applying $\mathbf{im}(f)$.

Theorem 1.8.14. Let $\mathcal{A}$ and $\mathcal{B}$ Abelian categories and $F : \mathcal{A} \to \mathcal{B}$ be exact and faithful (i.e. conservative).

(a) Suppose that $\mathcal{B}$ has enough projectives and that F has a left adjoint G. Then $\mathcal{A}$ has enough projectives. More precisely, if $X \in \mathcal{A}$, consider $FX \in \mathcal{B}$ and $\pi : Q \twoheadrightarrow FX$ where $Q \in \mathbf{Proj}(\mathcal{B})$, then we get an epimorphism

$$GQ \xrightarrow{G\pi} GFX \xrightarrow{\varepsilon} X$$

where ε is the counit of the adjunction.

(b) Suppose that $\mathcal{B}$ has enough injectives and that F has a right adjoint, then $\mathcal{A}$ has enough injectives.

Proof. Again, it suffices to prove the first proposition.

Consider

$$GQ \xrightarrow{G\pi} GFX \xrightarrow{\varepsilon} X$$

We have $G(\mathbf{Proj}(\mathcal{B})) \subseteq \mathbf{Proj}(\mathcal{A})$, then $G \dashv F$, preserves epimorphisms, hence $G\pi$ is an epimorphism.

Therefore, it suffices to prove that $\varepsilon_X : GFX \to X$ is an epimorphism. This is true because F is faithful. (It is equivalent to $f : X \to Y$ such that $Ff = 0$ with

$$\begin{tikzcd} GFX \arrow[r, "\varepsilon"] \arrow[d, "GFf=0"'] & X \arrow[d, "f"] \\ GFY \arrow[r, "\varepsilon"'] & Y \end{tikzcd}$$

which commutes by counit.)

Consider its cokernel:

$$GFX \xrightarrow{\varepsilon_X} X \longrightarrow Z \longrightarrow 0$$

Because F is exact,

$$\begin{tikzcd} FGFX \arrow[r, "F\varepsilon_X"] & FX \arrow[l, bend left, "\varepsilon_{FX}"] \arrow[r] & FZ \arrow[r] & 0 \end{tikzcd}$$

where we have a split epimorphism η_{FX} with respect to $F\varepsilon_X$. Therefore, by the unit-counit adjunction, $F\varepsilon_X \circ \eta_{FX} = \mathbf{id}_{FX}$. Hence, $FZ = 0$, then since F is faithful, so $Z = 0$. Therefore, ε_X is an epimorphism as desired. ☐

Corollary 1.8.15. The category R-**Mod** has enough injectives and projectives.

Proof. Apply the forgetful functor, we have

$$\begin{tikzcd} R\text{-}\mathbf{Mod} \arrow[d, "F"'] \\ \mathbb{Z}\text{-}\mathbf{mod} = \mathbf{Ab} \arrow[u, dashed] \end{tikzcd}$$

Note that $\mathbf{Ab}$ has enough injectives ($\mathbb{Q}/\mathbb{Z}$), and because of forgetfulness we have $F = R \otimes_R -$ with the $\mathbb{Z}, R$-bimodule ${}_{\mathbb{Z}}R_R$. Therefore, ${}_{\mathbb{Z}}R \otimes_R M \cong_{\mathbb{Z}} M$ as an Abelian group.

We now know by tensor-hom adjunction that the right adjoint is $\mathbf{Hom}_{\mathbb{Z}}(R_R, -)$:

$$\begin{tikzcd} R\text{-}\mathbf{Mod} \arrow[d, shift right, "X\otimes_R -"'] \\ S\text{-}\mathbf{Mod} \arrow[u, shift right, "\mathbf{Hom}_S(X,-)"'] \end{tikzcd}$$

for the bimodule ${}_SX_R$. ☐

Remark 1.8.16. Both proofs are explicit enough.

Exercise 1.8.17. Describe an injective I receiving an R-module M.

Example 1.8.18. Let G be a discrete group, and K is a field. Then the category $\mathcal{A}$ of K-linear representations of G has enough injectives and projectives. This is the category $\mathcal{A} = KG$-**Mod**.

Problem 4 (Exam Problem 4)**.** Let G be a finite group and K be a field of coefficients. Let $\mathcal{A} = KG$-**Mod**, the representations of G on K-vector spaces.

(a) Give explicit formulas for injective pre-envelops (in terms of M) $M \rightarrowtail I \in \mathbf{Inj}(\mathcal{A})$ where I is the pre-envelope of M. (Note that an envelope is a minimal pre-envelope.) Same with projective pre-envelopes.

(b) Show injectives and projectives coincide. (And yet KG is not semisimple in general. Refer to Maschke's Theorem in posoitive characteristic)

Proposition 1.8.19. Let X be a topological space, then the Abelian category of sheaves over Abelian groups on X, i.e. $\mathbf{Shv}_{\mathbf{Ab}}(X)$, (which can be replaced by $\mathbf{Shv}_{\mathcal{A}}(X)$ for any Grothendieck category $\mathcal{A}$,) has enough injectives.

Proof. For $x \in X$ we have an adjunction

$$\begin{tikzcd} \mathbf{Shv}_{\mathcal{A}}(X) \arrow[d, shift right, "(j_x)^*"'] \\ \mathcal{A} \arrow[u, shift right, "(j_*)_*"'] \end{tikzcd}$$

where $j_x : x \hookrightarrow X$ is the inclusion map. Therefore, j_x^* is an exact and non-faithful functor, sending $\mathcal{F}$ to $\mathcal{F}_x$, which is the stalk at x, i.e. $\operatorname{co}\lim_{U \ni x} \mathcal{F}(U)$. Also, $(j_*)_*$ describes the skyscrapper sheaves, i.e. $((j_*)_*A)(U) = \begin{cases} A, & \text{if } x \in U \\ 0, & \text{otherwise} \end{cases}$. Note that germs of sections are equivalent when restricting further is possible, that is, for $U \supseteq V \ni x$, we have a restriction functor $\mathcal{F}(U) \to \mathcal{F}(V)$. This gives

$$\begin{tikzcd} \mathbf{Shv}_{\mathcal{A}}(X) \arrow[d, shift right, "((j_*)^*)_x"'] \\ \prod_{x\in X} \mathcal{A} \arrow[u, shift right, "\prod_{x\in X}(j_*)_*"'] \end{tikzcd}$$

where $((j_*)^*)_x$ is exact and conservative. □

Remark 1.8.20 (Motivation of Complexes)**.** We consider the fundamental problem of the subject of Homological Algebra: how to handle the lack of exactness of most interesting additive functors? For example, the hom functors $\mathbf{Hom}(M, -)$ and $\mathbf{Hom}(-, M)$, the tensor functors $M \otimes -$ and so on.

Observe that if $F : \mathcal{A} \to \mathcal{B}$ is additive (e.g. exact on one side) and $A_1 \rightarrowtail A_2 \twoheadrightarrow A_3$ is split exact, then $FA_1 \rightarrowtail FA_2 \twoheadrightarrow FA_3$ is split exact.

Here is an idea to resolve this. In an ideal, boring world, all short exact sequences split, i.e. all objects are injective and projective. Therefore, we try to approximate general objects by injectives and projectives.

Suppose $\mathcal{A}$ has enough injectives. Let $M \in \mathcal{A}$. We consider an injection $M \rightarrowtail I^0 \in \mathbf{Inj}(\mathcal{A})$. The question is how far would I^0 control M, since this is just an approximation we want. Take $M^1 = \operatorname{coker}(M \rightarrowtail I^0)$. If M^1 is injective, then we have $M \rightarrowtail I^0 \twoheadrightarrow M^1$ where I^0 and M^1 are injectives. If not, we just repeat, and get $M^i \rightarrowtail I^i \in \mathbf{Inj}(\mathcal{A})$. Now we consider $M^{i+1} = \operatorname{coker}(M^i \rightarrowtail I^i)$, then we get

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, tail] & I^0 \arrow[r, dashed] \arrow[d, two heads] \arrow[rr, bend left, "0"] & I^1 \arrow[r, dashed] \arrow[d, two heads] & I^2 \arrow[r, dashed] & \cdots \\ & & M' \arrow[ur, tail] & M' \arrow[ur, tail] & & \end{tikzcd}$$

Note that the composition $M^1 \to M^2$ is the zero map, and so the map from $I^0 \to I^2$ is also the zero map, and so on. In particular, we get an exact sequence here, and can generate a long exact sequence. By rewriting, we then have

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \end{tikzcd}$$

and this is exact at $I^1, I^2, \cdots$. Note that we can consider the top sequence as a single object M, and the bottom sequence as a sequence of injectives. We then call the two sequences together as a complex of injectives (where M, although not injective itself, is described by the object I^0 below), where the mapping between two sequences is given by something called a quasi-isomorphism, which is really just an isomorphism for the concept of homology.

2 Derived Functor

2.1 Complex

Definition 2.1.1 ((Chain) Complex)**.** Let $\mathcal{A}$ be an additive category. A chain complex in $\mathcal{A}$ is a diagram (in homological indexing)

$$\cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \xrightarrow{d_{i-1}} A_{i-2} \longrightarrow \cdots$$

Figure 2: Chain Complex in Homological Indexing

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. Here the index i is called the degree and the d_i's are called differentials or boundaries. Note that we do not really care about the indices, and so we have $d^2 = 0$.

Similarly, we have a cohomological indexing

$$\cdots \longrightarrow A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d_{i+1}} A^{i+2} \longrightarrow \cdots$$

Figure 3: Chain Complex in Cohomological Indexing

Therefore, we can pass from one to another by using $A^i = A_{-i}$.

A morphism $f.$ of complex $(A., d_.^{A.}) \to (B., d_.^{B.})$ is a collection of morphisms $f_i : A_i \to B_i$ for all $i \in \mathbb{Z}$ such that $d \circ f = f \circ d$, i.e.

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r, "d^A"] \arrow[d, "f_i"'] & A_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & B_i \arrow[r, "d^B"'] & B_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

with obvious composition. This forms a category of chain complexes in $\mathcal{A}$, denoted $\mathbf{Ch}(\mathcal{A})$.

Definition 2.1.2 (Homotopy). Given two morphisms of complexes $f., g. : A. \to B.$, a homotopy between $f.$ and $g.$, denoted $\varepsilon. : f. \sim g.$, is a collection of morphisms $\varepsilon_i : A_i \to B_{i+1}$ (not assumed to commute with d) such that

$$f = g + d\varepsilon + \varepsilon d :$$

$$\begin{tikzcd}
& A_i \arrow[r, "d"] \arrow[dl, "\varepsilon"'] \arrow[d, "f_i - g_i"] & A_{i-1} \arrow[dl, "\varepsilon"] \\
B_{i+1} \arrow[r, "d"'] & B_i &
\end{tikzcd}$$

If this is the case, we say that $f.$ and $g.$ are homotopic.

Example 2.1.3. If $f \sim g$, then $f \circ h \sim g \circ h$ and $k \circ f \sim k \circ g$ for all appropriate morphisms h and k.

If $f \sim g$ and $f' \sim g'$ and are all morphisms between A and B, then $f + f' \circ g + g'$.

Note $\sim$ induces an equivalence relation.

Therefore, we can build a category $\mathbf{K}(\mathcal{A})$, the homotopy category of (chain) complexes in the additive category $\mathcal{A}$, whose objects are the same as $\mathbf{Ch}(\mathcal{A})$ and morphisms are homotopy classes of morphisms $[f.]_\sim : A. \to B.$.

Note that the mappings from $\mathcal{A}$ to $\mathbf{Ch}(\mathcal{A})$ and from $\mathcal{A}$ to $\mathbf{K}(\mathcal{A})$ are both fully faithful.

Proposition 2.1.4. Suppose $\mathcal{A}$ is moreover Abelian, then $\mathbf{Ch}(\mathcal{A})$ is Abelian, with degreewise kernels and cokernels.

Remark 2.1.5. On the other hand, $\mathbf{K}(\mathcal{A})$ is not Abelian in general. For instance, $\mathbf{K}(\mathbf{Ab})$ is not Abelian.

Exercise 2.1.6. Show that only monomorphisms in $\mathbf{K}(\mathcal{A})$ are the split monomorphisms. For instance, the morphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is not a monomorphism in $\mathbf{K}(\mathcal{A})$, but is a monomorphism in $\mathbf{Ch}(\mathcal{A})$.

Note that when we speak of objects in $\mathcal{A}$ as complexes, we put the objects in degree 0 and give a fully faithful embedding.

Definition 2.1.7 (Homology). Let $A.$ be a complex in an Abelian category $\mathcal{A}$. For every $i \in \mathbb{Z}$, we have

$$\begin{tikzcd}
A_{i+1} \arrow[rr, "d_{i+1}"] \arrow[d, two heads] & & A_i \arrow[rr, "d_i"] \arrow[drr, two heads] \arrow[dr, two heads] & & A_{i-1} \\
\operatorname{im}(d_{i+1}) \arrow[r, tail, dashed, "\exists!"'] \arrow[urr, tail] & \ker(d_i) \arrow[r, two heads] \arrow[ur, tail] & H_i(A.) \arrow[r, tail] & \operatorname{coker}(d_{i+1}) \arrow[r, two heads, dashed, "\exists!"'] & \operatorname{im}(d_i) \arrow[u, tail]
\end{tikzcd}$$

Figure 4: Homology

The cokernel of the canonical map $\operatorname{im}(d_{i+1} \rightarrowtail \ker(d_i)$ is called the homology of $A.$ in degree i.

Exercise 2.1.8. Equivalently, we have $H_i \cong \ker(\operatorname{coker}(d_{i-1}) \twoheadrightarrow \operatorname{im}(d_i)$.

Remark 2.1.9. $H_i(A.) = 0$ if and only if $A.$ is exact at A_i (i.e. at degree i). So we can think of $H_i(A)$ as a measure of exactness.

Proposition 2.1.10. $H_i : \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$ is a well-defined functor that passes to $\mathbf{K}(\mathcal{A})$:

$$f \sim g \Rightarrow H_i(f) = H_i(g).$$

Proof. To show the second part, consider

$$\begin{tikzcd}
A_{i+1} \arrow[rr, "d_{i+1}"] \arrow[d, two heads] \arrow[dd, bend right, red] & & A_i \arrow[rr, "d_i"] \arrow[dr, two heads] \arrow[drr, two heads] \arrow[dd, bend right, red] & & A_{i-1} \arrow[dd, bend right, red] \\
\operatorname{im}(d_{i+1}) \arrow[r, tail, dashed, "\exists!"'] \arrow[urr, tail] \arrow[dd, bend right, red] & \ker(d_i) \arrow[r, two heads] \arrow[ur, tail] \arrow[dd, bend right, red] & H_i(A.) \arrow[r, tail] \arrow[dd, bend right, red] & \operatorname{coker}(d_{i+1}) \arrow[r, dashed, two heads, "\exists!"'] \arrow[dd, bend right, red] & \operatorname{im}(d_i) \arrow[u, tail] \arrow[dd, bend right, red] \\
A'_{i+1} \arrow[rr, "d'_{i+1}"] \arrow[d, two heads] & & A'_i \arrow[rr, "d'_i"] \arrow[dr, two heads] \arrow[drr, two heads] & & A'_{i-1} \\
\operatorname{im}(d'_{i+1}) \arrow[r, tail, dashed, "\exists!"'] \arrow[urr, tail] & \ker(d'_i) \arrow[r, two heads] \arrow[ur, tail] & H_i(A'.) \arrow[r, tail] & \operatorname{coker}(d'_{i+1}) \arrow[r, dashed, two heads, "\exists!"'] & \operatorname{im}(d'_i) \arrow[u, tail]
\end{tikzcd}$$

As for the second part, let $y \in H_i(A.)$. Let $x \in A_i$ be a lift of y with $dx = 0$. By definition, $f(y)$ is represented by $f(x) \in A'_i$. Now, we have $f(x) = d_{i+1}(\varepsilon_i(x)) + \varepsilon_{i-1}(d_i(x)) = d_{i+1}(\varepsilon_i(x))$ since $dx = 0$. But this shows that $f(x) \in \operatorname{im}(d_{i+1})$, and thus maps to 0 in $H_i(A'.)$. $\square$

Definition 2.1.11 (Homotopy Equivalence)**.** A morphism of complexes $f : A \to B$ which is an isomorphism in $\mathbf{K}(\mathcal{A})$ is called a homotopy equivalence; it means that there exists $g : B \to A$ such that $[f]_\sim \circ [g]_\sim = [\mathbf{id}]_\sim$ and $[g]_\sim \circ [f]_\sim = [\mathbf{id}]_\sim$, i.e. $f \circ g \sim \mathbf{id}$ and $g \circ f \sim \mathbf{id}$. Also, g is called a homotopy inverse.

Corollary 2.1.12. For an Abelian category $\mathcal{A}$, if f is a homotopy equivalence, then $H_i(f)$ is an isomorphism for all $i \in \mathbb{Z}$.

Definition 2.1.13 (Quasi-isomorphism)**.** A morphism $f : A. \to B.$ is a quasi-isomorphism if it is an H_*-isomorphsim, i.e. $H_i(f) : H_i(A.) \to H_i(B.)$ is an isomorphism in $\mathcal{A}$ for all $i \in \mathbb{Z}$.

Example 2.1.14. Suppose we have an integer $n \neq 0, \pm 1$. Then

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\cdot n"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism, because H_* maps it to

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

However, it is not a homotopy equivalence because we necessarily have

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

but there are no n-torsions in $\mathbb{Z}$. Neither complex is 0-homotopic (i.e. isomorphic to 0 in $\mathbf{K}(\mathcal{A})$, as H_0 shows.

Example 2.1.15. Note that

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism, but not a homotopy equivalence.

Exercise 2.1.16. Given an exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{A}$, show that

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

are quasi-isomorphisms. They are homotopy equivalences if and only if $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{A}$ is split exact.

Definition 2.1.17 (Homology, Redefined)**.** Consider an Abelian category $\mathcal{A}$ and a complex $A.$ Recall that we have

$$\begin{tikzcd}
& A_{i+1} \arrow[rr, "d_{i+1}"] \arrow[dl, two heads] \arrow[d, two heads] & & A_i \arrow[rr, "d_i"] \arrow[dr, two heads] \arrow[drr, two heads] & & A_{i-1} & \\
\operatorname{coker}(d_{i+2}) \arrow[r, two heads] & \operatorname{im}(d_{i+1}) \arrow[r, tail, dashed, "\exists!"'] \arrow[urr, tail] & \ker(d_i) \arrow[r, two heads] \arrow[ur, tail] & H_i(A.) \arrow[r, tail] & \operatorname{coker}(d_{i+1}) \arrow[r, two heads, dashed, "\exists!"'] & \operatorname{im}(d_i) \arrow[r, tail] \arrow[u, tail] & \ker(d_{i-1}) \arrow[ul, tail]
\end{tikzcd}$$

We defined
$$H_i(A.) = \operatorname{im}(\ker(d_i) \xrightarrow{\exists!} \operatorname{coker}(d_{i+1})$$

In tradition we define $H_i(A.) = \ker/\operatorname{im} = \operatorname{coker}(\operatorname{im}(d_{i+1} \rightarrowtail \ker(d_i))$. However, because we have $\operatorname{im}(d_i) \rightarrowtail \ker(d_{i-1}) \rightarrowtail A_{i-1}$, then computationally we can define

$$\begin{aligned}
H_i(A.) &\cong \ker(\operatorname{coker}(d_{i+1}) \twoheadrightarrow \operatorname{im}(d_i)) \\
&= \ker(\operatorname{coker}(d_{i+1}) \xrightarrow{\hat{d}_i} \ker(d_{i-1})) \\
&= \ker(\operatorname{coker}(d_{i+1}) \to A_{i-1})
\end{aligned}$$

Lemma 2.1.18. Since $d_i \circ d_{i+1} = 0$ and $d_{i-1} \circ d_i = 0$, then d_i induces a (unique) canonical morphism $\operatorname{coker}(d_{i+1}) \xrightarrow{\hat{d}_i} \ker(d_{i-1})$, fitting in an exact sequence:

$$0 \longrightarrow H_i(A.) \longrightarrow \operatorname{coker}(d_{i+1}) \xrightarrow{\hat{d}_i} \ker(d_{i-1}) \longrightarrow H_{i-1}(A.) \longrightarrow 0$$

Theorem 2.1.19 (Homology Long Exact Sequence)**.** Let

$$0 \longrightarrow A. \xrightarrow{f} B. \xrightarrow{g} C. \longrightarrow 0$$

be a short exact sequence in the Abelian category $\mathbf{Ch}(\mathcal{A})$. Then there exists a natural long exact sequence in $\mathcal{A}$

$$\cdots \xrightarrow{g_*} H_{i+1}(C.) \xrightarrow{d_{i+1}} H_i(A.) \xrightarrow{f_*} H_i(B.) \xrightarrow{g_*} H_i(C.) \xrightarrow{d_i} H_{i-1}(A.) \xrightarrow{f_*} \cdots$$

called the long exact sequence in homology. Here d_i is called the connecting homomorphism and can be made explicit in the proof.

Proof. Apply the snake lemma to every $i \in \mathbb{Z}$ and diagram

$$\begin{tikzcd}
0 \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, "d_i^A"'] & B_i \arrow[r, "g_i"] \arrow[d, "d_i^B"] & C_i \arrow[r] \arrow[d, "d_i^C"] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "f_{i-1}"'] & B_{i-1} \arrow[r, "g_{i-1}"'] & C_{i-1} \arrow[r] & 0
\end{tikzcd}$$

It yields $\ker(d_i^C) \to \operatorname{coker}(d_i^A)$, and the two exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker(d_i^A) \arrow[r] & \ker(d_i^B) \arrow[r] & \ker(d_i^C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\operatorname{coker}(d_i^A) \arrow[r] & \operatorname{coker}(d_i^B) \arrow[r] & \operatorname{coker}(d_i^C) \arrow[r] & 0
\end{tikzcd}$$

By using the canonical $\hat{d_i}^A : \operatorname{coker}(d_{i+1}) \to \ker(d_{i-1})$ as in the lemma, we can compare those exact sequences:

$$\begin{tikzcd}
 & H_i(A.) \arrow[r, "f_*"] \arrow[d, tail] & H_i(B.) \arrow[r, "g_*"] \arrow[d, tail] & H_i(C.) \arrow[d, tail] \arrow[dddll, bend left, "d_i=\delta"] & \\
 & \operatorname{coker}(d_{i+1}^A) \arrow[r, "f_*"] \arrow[d, "\hat{d_i}^A"] & \operatorname{coker}(d_{i+1}^B) \arrow[r, "g_*"] \arrow[d, "\hat{d_i}^B"] & C_1 \arrow[r] \arrow[d, "\hat{d_i}^C"] & 0 \\
0 \arrow[r] & \ker(d_{i-1}^A) \arrow[r, "f_*"] \arrow[d, two heads] & \ker(d_{i-1}^B) \arrow[r, "g_*"] \arrow[d, two heads] & \ker(d_{i-1}^C) \arrow[d, two heads] & \\
 & H_{i-1}(A.) \arrow[r, "f_*"'] & H_{i-1}(B.) \arrow[r, "g_*"'] & H_{i-1}(C.) &
\end{tikzcd}$$

Here we take the kernels and cokernels according to the lemma, and δ is the connecting homomorphism from the snake lemma. ☐

Corollary 2.1.20. Similarly, we have a version of this in cohomology: let

$$\begin{tikzcd}
0 \arrow[r] & A^{\cdot} \arrow[r, "f"] & B^{\cdot} \arrow[r, "g"] & C^{\cdot} \arrow[r] & 0
\end{tikzcd}$$

be a short exact sequence in the Abelian category $\mathbf{Ch}(\mathcal{A})$ with cohomoological indices, then we have a natural long exact sequence in $\mathcal{A}$

$$\cdots \xrightarrow{d^{i-1}} H^i(A^{\cdot}) \xrightarrow{f^*} H^i(B^{\cdot}) \xrightarrow{g^*} H^i(C^{\cdot}) \xrightarrow{d^i} H^{i+1}(A^{\cdot}) \xrightarrow{f^*} H^{i+1}(B^{\cdot}) \xrightarrow{g^*} \cdots$$

called the long exact sequence in cohomology.

Exercise 2.1.21. If $\mathcal{A}$ "has elements" (e.g. the category of R-modules), we have the "usual" description of d_i, that is, from $H_i(C)$ to $H_{i-1}(A)$. Here we take $[c] = z \in H_i(C)$, then we obtain $c \in \ker(d_i^C) \subseteq C_i$. By looking at exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A_i \arrow[r] \arrow[d] & B_i \arrow[r] \arrow[d] & C_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r] \arrow[d] & B_{i-1} \arrow[r] \arrow[d] & C_{i-1} \arrow[r] & 0 \\
 & A_{i-2} \arrow[r] & B_{i-2} & &
\end{tikzcd}$$

By surjectivity, we pull $c \in C_i$ back to some $b \in B_i$. Note that b gets mapped down to $d(b) \in B_{i-1}$, then both $d_i(b) \in B_{i-1}$ and $c \in C_i$ can be mapped to $0 \in C_{i-1}$. Now on the bottom square, we have another lift that sends $d(b)$ back to $a \in A_{i-1}$. Now a gets mapped down to $d(a) \in A_{i-2}$. Also, $d(b) \in B_{i-1}$ gets mapped down to $d^2(b) \in B_{i-2}$, but we have $d^2(b) = 0$. Now by injectivity, we pull $0 \in B_{i-2}$ back to $d(a) = 0$ in A_{i-2}. Hence, $a \in \ker(d_{i-1}^{A.})$, we get to define $d_i(a) = [a]$ in $H_{i-1}(A.)$.

2.2 Projective and Injective Resolutions

Throughout this section, $\mathcal{A}$ is an Abelian category and often has enough projectives/injectives.

Definition 2.2.1 (Projective Resolution, Injective Resolution)**.** A projective resolution of an object $A \in \mathcal{A}$ is an exact complex

$$\cdots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\xi} A \longrightarrow 0$$

where all P_i's are projective.

An injective resolution of an object $A \in \mathcal{A}$ is an exact complex

$$0 \longrightarrow A \xrightarrow{\xi} I^0 \xrightarrow{d_1} I^1 \xrightarrow{d_2} \cdots \longrightarrow I^n \xrightarrow{d_{n+1}} I^{n+1} \longrightarrow \cdots$$

where all I^i's are injective.

Sometimes we just write $P. \to A$ and $A \to I^{\cdot}$ because we think of them as quasi-isomorphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\xi"] & 0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Example 2.2.2. The projective resolution of $\mathbb{Z}/n\mathbb{Z}$ is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

The injective resolution of $\mathbb{Z}$ is

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

The injective resolution of $\mathbb{Z}/n\mathbb{Z}$ is

$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{[\frac{1}{n}]} \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Proposition 2.2.3. (a) If $\mathcal{A}$ has enough projectives, then every object has a projective resolution.

(b) If $\mathcal{A}$ has enough injectives, then every object has an injective resolution.

(c) Let $P. \xrightarrow{\xi} A$ and $Q. \xrightarrow{\eta} B$ be two projective resolutions in $\mathcal{A}$. Let $f : A \to B$ be a morphism, then there exists a morphism of complexes $\hat{f}. : P \to Q$

$$\begin{tikzcd}
P. \arrow[r] \arrow[d, dashed, "\exists \hat{f}."'] & A \arrow[d, "f"] \\
Q. \arrow[r] & B
\end{tikzcd}$$

such that the diagram commutes, i.e. $\eta \hat{f}_0 = f\xi$:

$$\begin{tikzcd}
\cdots \arrow[r] & P_0 \arrow[r, "\xi"] \arrow[d, "\hat{f}_0"'] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_0 \arrow[r, "\eta"'] & B \arrow[r] & 0
\end{tikzcd}$$

Moreover, such a lift is unique up to homotopy.

(d) Dually, morphisms of objects extend to injective resolutions, uniquely up to homotopy.

Proof. We only need to prove (c). Using the fact that P_0 is projective and η is an epimorphism, we get

$$\begin{tikzcd}
P_0 \arrow[r, "\xi"] \arrow[d, dashed, "\exists f_0"'] & A \arrow[r] \arrow[d, "f"] & 0 \\
Q_0 \arrow[r, two heads, "\eta"'] & B \arrow[r] & 0
\end{tikzcd}$$

Then by induction, we have

$$\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[rr] \arrow[dd, dashed, "\exists f_i"'] \arrow[dr, dashed, "\hat{g}"'] \arrow[ddrr, bend left, "g"] & & P_{i-1} \arrow[r] \arrow[dd, "f_{i-1}"] & P_{i-2} \arrow[r] \arrow[dd, "f_{i-2}"] & \cdots \\
& & \cdot \arrow[dr, tail] & & & \\
\cdots \arrow[r] & Q_i \arrow[rr] \arrow[ur, two heads] & & Q_{i-1} \arrow[r] & Q_{i-2} \arrow[r] & \cdots
\end{tikzcd}$$

Consider $g = f_{i-1} \circ d_i^P : P_i \to Q_{i-1}$, then $d^Q g = d_{i-1} f_{i-1} d_i = d_{i-1} d_i f_{i-2} = 0$. Therefore, there exists $\hat{g} : P_i \to \ker(d_{i-1}^Q) = \operatorname{im}(d_i^Q) \twoheadleftarrow Q_i$.

Because P_i is projective and $Q_i \twoheadrightarrow \ker)d_{i-1}^Q)$, we show existence. We now prove its uniqueness.

Suppose there are two projective resolutions

$$\begin{tikzcd}
P. \arrow[r, "\xi"] \arrow[d, "\tilde{f}{,}\tilde{f}'"'] & A \arrow[d, "f"] \\
Q. \arrow[r, "\eta"'] & B
\end{tikzcd}$$

It suffices to show $\tilde{f} \sim \tilde{f}'$. Then it is enough to show that if

$$\begin{tikzcd}
P. \arrow[r, "\xi"] \arrow[d, "f."'] & A \arrow[d, "0"] \\
Q. \arrow[r, "\eta"'] & B
\end{tikzcd}$$

then $f. \sim 0$. In particular, we have a base case

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[rr] \arrow[dd, "f_1"'] & & P_0 \arrow[r, "\xi"] \arrow[dd, "f_0"'] \arrow[dl, dashed, "\exists"] \arrow[ddll, dashed, bend right, "lift"'] \arrow[ddr, "0"] & A \arrow[r] \arrow[dd, "0"] & 0 \\
& & \overset{\exists\varepsilon_0}{\cdot} \arrow[dr, tail] & & & \\
\cdots \arrow[r] & Q_1 \arrow[rr] \arrow[ur, two heads] & & Q_0 \arrow[r, "\eta"'] & B \arrow[r] & 0
\end{tikzcd}$$

By induction, suppose we constructed $\varepsilon_i : P_i \to Q_{i+1}$ such that $f_i = d\varepsilon_i + \varepsilon_{i-1} d$ for all $i \leqslant n-1$. We then construct ε_n as follows (note that by inductive hypothesis, $f_{n-1} = d\varepsilon_{n-1} + \varepsilon_{n-2} d$):

$$\begin{tikzcd}
P_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dd, "f_{n+1}"'] & & P_n \arrow[r, "d_n"] \arrow[dd, "f_n"] \arrow[dl, dashed, "1"'] \arrow[ddll, dashed, bend right, "2"'] & P_{n-1} \arrow[r, "d"] \arrow[dd, "f_{n+1}"] \arrow[ddl, "\varepsilon_{n-1}"] & \cdots \\
& \ker(d_n) \arrow[dr, tail] & & & \\
Q_{n+1} \arrow[rr, "d_{n+1}"'] \arrow[ur, two heads] & & Q_n \arrow[r, "d_n"'] & Q_{n-1} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

We want $\varepsilon_n : P_n \to Q_{n+1}$ such that $f_n = d\varepsilon_n + \varepsilon_{n-1}d$. Consider $g = f_n - \varepsilon_{n-1}d : P_n \to Q_n$. Note that

$$\begin{aligned} d \circ g &= df_n - d\varepsilon_{n-1}d \\ &= f_{n-1}d - d\varepsilon_{n-1}d \\ &= (f_{n-1} - d\varepsilon_{n-1})d \\ &= (\varepsilon_{n-2} \circ d) \circ d \\ &= 0 \end{aligned}$$

① We have g factors via $\ker(d_n^Q) \rightarrowtail Q_n$.
② There exists a lift $\varepsilon_n : P_n \to Q_{n+1}$ of g because $Q_{n+1} \twoheadrightarrow \ker(d_n^Q)$ by exactness of Q and P_n is projective. □

Remark 2.2.4. The proof only used that P_i is projective in each degree and $Q.$ is exact. Hence, with the same proof, we have the following proposition.

Proposition 2.2.5. (a) If $\xi : P. \to A$ is a projective resolution and $\eta : Q. \to B$ is exact, then we have the following picture

$$\begin{tikzcd} P. \arrow[r, "\xi"] & A \arrow[d, "0"] \\ Q. \arrow[r, "\eta"'] & B \end{tikzcd}$$

and there exists $\tilde{f} : O. \to Q.$ such that $\eta\tilde{f} = f\xi$, unique up to homotopy.

(b) If we have

$$\begin{tikzcd} & P. \arrow[d] \arrow[dl, dashed, "\exists"'] \\ Q. \arrow[r] & B \end{tikzcd}$$

where $Q. \to B$ gives a quasi-isomorphism in $\mathbf{Ch}_+(\mathcal{A})$ (i.e. ends with zeros), then there exists a lift $P. \to Q.$.
The dual statements are true with injectives.

Corollary 2.2.6. Given $A \in \mathcal{A}$, the projective resolution of A is unique up to homotopy equivalence (which is itself unique if we keep ξ): if $\xi : P. \to A$ and $\xi' : P'_. \to A$ are two projective resolutions of A, then there exists a homotopy equivalence $\varphi : P. \to P'_.$ such that

$$\begin{tikzcd} P. \arrow[dd, "\varphi"'] \arrow[dr, "\xi"] & \\ & A \\ P'_. \arrow[ur, "\xi'"'] & \end{tikzcd}$$

commutes (i.e. $\xi'\varphi = \xi$), unique up to homotopy.

Proof. Take $\varphi = \hat{\mathbf{id}}_A$ and $\psi = \hat{\mathbf{id}}_A$, i.e. so that we have the following diagrams

$$\begin{tikzcd} P. \arrow[r, "\varphi"] \arrow[d, "\xi"'] & P'_. \arrow[r, "\psi"] \arrow[d, "\xi'"] & P. \arrow[d, "\xi"] \\ A \arrow[r, "\mathbf{id}"] & A \arrow[r, "\mathbf{id}"] & A \end{tikzcd}$$

Note that $\psi \circ \varphi : P. \to P.$ and $\mathbf{id}_{P.}$ are two lifts of $\mathbf{id}_A$. Therefore, they are homotopic, so $\psi \circ \varphi \sim \mathbf{id}_{P.}$. Similarly, $\varphi \circ \psi \sim \mathbf{id}_{P'_.}$. □

Remark 2.2.7. In other words, there is a well-defined projective resolution of A in $\mathbf{K}_{\geqslant 0}(\mathbf{Proj}(\mathcal{A}))$ (i.e. zero in negative homological degrees), which is unique.

Remark 2.2.8. Here is a diagram of (homological and cohomological) notations related to $\mathbf{Ch}(\mathcal{A})$. Similar notations work for $\mathbf{K}(\mathcal{A})$.

$$\begin{tikzcd}
 & & \mathbf{Ch}(\mathcal{A}) \arrow[dll] \arrow[drr] & & \\
\mathbf{Ch}_+(\mathcal{A}) = \mathbf{Ch}^-(\mathcal{A}) \arrow[dr] & & & & \mathbf{Ch}_-(\mathcal{A}) = \{A. \mid \exists n = n(\mathcal{A}) : A_i = 0 \, \forall i > n\} \arrow[dl] \\
 & & \mathbf{Ch}_b(\mathcal{A}) = \mathbf{Ch}^b(\mathcal{A}) & & \\
\mathbf{Ch}_{\geqslant 0}(\mathcal{A}) = \mathbf{Ch}^{\leqslant 0}(\mathcal{A}) \arrow[uu, hook] & & & & \mathbf{Ch}_{\leqslant 0}(\mathcal{A}) = \{A_i \mid A_i = 0, i > 0\} \arrow[uu, hook] \\
 & & \mathbf{Ch}_0(\mathcal{A}) = \mathbf{Ch}^0(\mathcal{A}) \cong \mathcal{A} \arrow[ull, hook] \arrow[urr, hook] \arrow[uu, hook] & &
\end{tikzcd}$$

Figure 5: Chain-related Notations

Corollary 2.2.9. (a) Suppose $\mathcal{A}$ has enough projectives, then there exists a functor $P : \mathcal{A} \to \mathbf{K}_{\geqslant 0}(\mathbf{Proj}(\mathcal{A}))$ together with a natural transformation $\xi : P \to C_0$ where

$$C_0(A) = \cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

such that ξ_A is a quasi-isomorphsim for all $A \in \mathcal{A}$.

(b) If $(P : A \to \mathbf{K}_{\geqslant 0}(\mathbf{Proj}(\mathcal{A})), \xi)$ and (P', ξ') are two pairs as in (a), then there exists a unique isomorphism of functors $\varphi : P \xrightarrow{\cong} P'$ such that $\xi' \circ \varphi = \xi$:

$$\begin{tikzcd}
P \arrow[rr, "\varphi"] \arrow[dr, "\xi"'] & & P' \arrow[dl, "\xi'"] \\
 & C_0 &
\end{tikzcd}$$

Dually, the statement holds for injectives: if $\mathcal{A}$ has enough injectives, then there exists a unique pair (up to isomorphism) (I, α) where $I : \mathcal{A} \to \mathbf{K}^{\geqslant 0}(\mathbf{Inj}(\mathcal{A}))$ and $\alpha : C_0 \to I$ acts as a functor $\mathcal{A} \to \mathbf{K}(\mathcal{A})$ gives objectwise quasi-isomorphism.

Proof. To build P, choose a projective resolution for every A in $\mathcal{A}$, with $\xi_A : P(A) \to A$ quasi-isomorphism. This is a functor into $\mathbf{K}(\mathcal{A})$ by proposition: use the unique lift up to homotopy.

$$\begin{tikzcd}
P \arrow[r, "A"] \arrow[d, "\hat{f}"] \arrow[dd, bend right, "g \hat{\circ} f"'] & A \arrow[d, "f"'] \arrow[dd, bend left, "g \circ f"] \\
Q \arrow[r] \arrow[d, "\hat{g}"] & B \arrow[d, "g"'] \\
R \arrow[r] & C
\end{tikzcd}$$

Here $g \mathbin{\hat{\circ}} f = \hat{g} \circ \hat{f}$, which acts as another lift of $g \circ f$. ☐

Definition 2.2.10 (Resolution Functor)**.** The functor $P : A \to \mathbf{K}_{\geqslant 0}(\mathbf{Proj}(\mathcal{A}))$ with $\xi : P \to C_0$ is the projective resolution functor.

The functor $I : A \to \mathbf{K}^{\geqslant 0}(\mathbf{Inj}(\mathcal{A}))$ with $\alpha : C_0 \to I$ is the injective resolution functor.

Remark 2.2.11. If we have enough projectives, we could ask how P reflects exact sequences:

$$P : A \to \mathbf{K}_{\geqslant 0}(\mathbf{Proj}(\mathcal{A})) \hookrightarrow \mathbf{K}(\mathcal{A})$$

Recall $\mathbf{K}(\mathcal{A})$ does not have interesting exact sequences (all of them splits).

Proposition 2.2.12 (Horseshoe Lemma)**.** Let

$$\begin{tikzcd} A' \arrow[r, tail, "\alpha'"] & A \arrow[r, two heads, "\alpha''"] & A'' \end{tikzcd}$$

be an exact sequence in $\mathcal{A}$. Let $P' \xrightarrow{\xi'} A'$ and $P'' \xrightarrow{\xi''} A''$ be projective resolutions. Then A admits a projective resolution $P. \xrightarrow{\xi} A$ together with lifts

$$\begin{tikzcd} P'_\cdot \arrow[r, tail, "\hat{\alpha'}"] \arrow[d, "\xi'"'] & P_\cdot \arrow[r, two heads, "\hat{\alpha''}"] \arrow[d, "\xi"] & P''_\cdot \arrow[d, "\xi''"] \\ A' \arrow[r, tail, "\alpha'"] & A \arrow[r, two heads, "\alpha''"] & A'' \end{tikzcd}$$

such that the sequence is exact in $\mathbf{Ch}(\mathcal{A})$, i.e. split exact in each degree. Dually, the same is true for injectives.

Proof. We want $P_i = P'_i \oplus P''_i$ for all i and $\hat{\alpha'_i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\alpha''_i} = \begin{pmatrix} 0 & 1 \end{pmatrix}$:

$$\begin{tikzcd} P'_i \arrow[r, tail, "\hat{\alpha'_i}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[d, "d'"'] & P'_i \oplus P''_i \arrow[r, two heads, "\hat{\alpha''_i}=\begin{pmatrix} 0 & 1 \end{pmatrix}"] \arrow[d, dashed, "\exists d"] & P''_i \arrow[d, "d''_i"] \\ P'_{i-1} \arrow[r, tail, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] & P'_{i-1} \oplus P''_{i-1} \arrow[r, two heads, "\begin{pmatrix} 0 & 1 \end{pmatrix}"'] & P''_{i-1} \end{tikzcd}$$

Note that the induced mapping d has the form $\begin{pmatrix} d' & * \\ 0 & d'' \end{pmatrix}$ with $*$ unknown. We now induct on the degrees. Start by degree 0:

$$\begin{tikzcd} P'_0 \arrow[r, tail, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[d, two heads, "\xi'_0"'] & P'_0 \oplus P''_0 \arrow[r, two heads, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] \arrow[d, two heads, "\xi_0=\begin{pmatrix} \alpha''\xi'_0 & \eta_0 \end{pmatrix}"] & P''_i \arrow[d, two heads, "\xi''_0"] \\ A' \arrow[r, tail, "\alpha'"'] & A \arrow[r, two heads, "\alpha''"'] & A'' \end{tikzcd}$$

We want $\begin{pmatrix} 0 & \alpha\eta_0 \end{pmatrix} = \begin{pmatrix} 0 & \xi''_0 \end{pmatrix}$, such an η_0 exists because α'' is onto and P''_0 is projective. By taking the kernels, we have

$$\begin{tikzcd} P'_1 \arrow[r, tail, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[d, two heads] & P'_1 \oplus P''_1 \arrow[r, two heads, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] \arrow[d, dashed, two heads, "\exists"'] & P''_1 \arrow[d, two heads] \\ \ker(\xi'_0) \arrow[r, tail] & \ker(\xi_0) \arrow[r, two heads] & \ker(\xi''_0) \end{tikzcd}$$

By the snake lemma, we know η_0 is an epimorphism. Also, the sequence gives exactness at $\ker(\xi'_0)$ because of snake lemma. We then repeat by induction, and note that we have

$$\begin{tikzcd} \cdots \arrow[d] & & \cdots \arrow[dll] \\ P_1 = P'_1 \oplus P''_1 \arrow[rr, two heads] \arrow[d, "d_1"] & & \ker(\xi_0) \arrow[dll, tail] \\ P_0 := P'_0 \oplus P''_0 \arrow[d, two heads, "\xi_0"] & & \\ A & & \end{tikzcd}$$

Therefore, $P. \to A$ is indeed a projective resolution. □

Problem 5 (Exam Problem 5)**.** Prove Schanuel's Lemma. The statement is as follows:

If $B \rightarrowtail P \twoheadrightarrow A$ and $C \rightarrowtail Q \twoheadrightarrow A$ are short exact sequences with the same end and P and Q are projectives, then $B \oplus Q \cong C \oplus P$. More generally, given two projective resolutions $P.$, $Q.$ of A, for all $n \geqslant 0$, $\ker(d_n^P) \oplus Q_n \oplus P_{n-1} \oplus Q_{n-2} \oplus \cdots \cong \ker(d_n^Q) \oplus P_n \oplus Q_{n-1} \oplus P_{n-2} \oplus \cdots$ exists in $\mathcal{A}$.

2.3 Derived Functor in Old Fashion

For the whole section, $\mathcal{A}$ and $\mathcal{B}$ are Abelian, and $\mathcal{A}$ has enough projectives (or, enough injectives).

Definition 2.3.1 (Derived Functor)**.** Let $\mathcal{A}$ be an Abelianc ategory with enough projective, and $F : \mathcal{A} \to \mathcal{B}$ be an additive functor (often a right exact one) to another Abelian category $\mathcal{B}$. Let $i \in \mathbb{N} = \{0, 1, \cdots\}$. The i-th left derived functor is

$$L_i F : \mathcal{A} \xrightarrow{\underline{P}} K_{\geqslant 0}(\mathbf{Proj}(\mathcal{A})) \xrightarrow{F} K_{\geqslant 0}(\mathcal{B}) \xrightarrow{H_i} \mathcal{B}$$

where $\underline{P}$ is the projective resolution functor $(P, \xi : P \to C_0)$ given by a quasi-isomorphism from last section (unique up to unique isomorphism), the middle F is just $\mathbf{K}(F) : \mathcal{A} \to F(\mathcal{A})$ degreewise, and H_i is the homology in $\mathcal{B}$.

Dually, if $\mathcal{A}$ have enough injectives, the i-th right derived fucntor (often a left exact one) is

$$R^i F : \mathcal{A} \xrightarrow{\underline{I}} K^{\geqslant 0}(\mathbf{Inj}(\mathcal{A})) \xrightarrow{F} K^{\geqslant 0}(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}$$

where $(\underline{I}, \alpha) : C_0 \to I)$ gives a quasi-isomorphism and is the injective resolution functor.

Remark 2.3.2. This is well-defined and choice-independent. We unpack the construction below.

For any $A \in \mathcal{A}$, we first pick a projective resolution

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\xi} A \longrightarrow 0$$

then we apply F everywhere and get

$$\cdots \xrightarrow{Fd} FP_2 \xrightarrow{Fd} FP_1 \xrightarrow{Fd} FP_0 \xrightarrow{F\xi} FA \longrightarrow 0$$

We now drop the FA term from the sequence

$$\cdots \xrightarrow{Fd} FP_2 \xrightarrow{Fd} FP_1 \xrightarrow{Fd} FP_0 \longrightarrow 0 \longrightarrow 0$$

And finally take the homology and get

$$L_i FA = \ker(FP_i \to FP_{i-1})/\mathrm{im}(FP_{i+1} \to FP_i).$$

Proposition 2.3.3. With the above assumptions, if $P' \xrightarrow{\xi'} A$ is another projective resolution, there exists a canonical isomorphism $L_i FA \xrightarrow{\cong} H_i FP'_{\cdot}$. For a morphism $f : A \to B$ and any lift $\hat{f} : P'_{\cdot} \to Q'_{\cdot}$ to any projective resolution, we have

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
P'_{\cdot} \arrow[u, "\xi'"] \arrow[r, "\hat{f}"] & Q'_{\cdot} \arrow[u, "\eta'"'] \\
PA \arrow[u] \arrow[uu, bend left] \arrow[r, "Pf"'] & QB \arrow[u] \arrow[uu, bend right]
\end{tikzcd}$$

and so the diagram commutes:

$$\begin{tikzcd}
H_i FP'_{\cdot} \arrow[r, "H_i F\hat{f}"] & H_i FQ'_{\cdot} \\
L_i FA \arrow[u, "\cong"] \arrow[r, "L_i Ff"'] & L_i FB \arrow[u, "\cong"]
\end{tikzcd}$$

Proof. Compare P' to P and Q' to Q and use uniqueness up to homotopy. Then we combine it with the fact that if $f \sim g$ in $\mathbf{Ch}(\mathcal{A})$, then $Ff \sim Fg$ in $\mathbf{Ch}(\mathcal{B})$. Indeed, $F = \mathbf{K}F : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ is well-defined. Moreover, we know F sends a homotopy equivalence to another homotopy equivalence. Finally, homotopy equivalent mappings agree in homology, and so homotopy equivalence are quasi-isomorphic in $\mathcal{B}$. □

Remark 2.3.4. It is easy to see how L_iF is natural in F, with respect to natural transformations.

Theorem 2.3.5. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between Abelian categories. Suppose that $\mathcal{A}$ has enough projectives. Let $A \rightarrowtail B \twoheadrightarrow C$ be exact in $\mathcal{A}$. Then there exists a natural long exact sequence:

$$\begin{tikzcd}
L_{i+1}FC \arrow[r, "d_{i+1}"] & L_iFA \arrow[r] & L_iFB \arrow[r] & L_iFC \arrow[dll, rounded corners, "d_i"'] \\
& L_{i-1}FA \arrow[r] & L_{i-1}FB \arrow[r] & L_{i-1}FC \arrow[dll, rounded corners, "d_{i-1}"'] \\
& \cdots \arrow[r] & \cdots \arrow[r] & \cdots \arrow[dll, rounded corners] \\
& L_0FA \arrow[r] \arrow[d, "H_0F(\xi_A)"] & L_0FB \arrow[r] \arrow[d, "H_0F(\xi_B)"] & L_0FC \arrow[r] \arrow[d, "H_0F(\xi_C)"] & 0 \\
& FA \arrow[r] & FB \arrow[r] & FC
\end{tikzcd}$$

Moreover, if F is right exact, then $L_0F \cong F$ via $H_0F(\xi)$, so the long exact sequence ends with F:

$$\cdots \longrightarrow L_2F(C) \xrightarrow{d} L_1F(A) \xrightarrow{f*} L_1F(B) \xrightarrow{g*} L_1F(C) \xrightarrow{d} FA \xrightarrow{f*} FB \xrightarrow{g*} FC \longrightarrow 0$$

Dually, for right derived functors, if $F : \mathcal{A} \to \mathcal{B}$ is left exact, then for all short exact sequences

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

in $\mathcal{A}$, we have a long exact sequence

$$0 \longrightarrow FA \xrightarrow{f*} FB \xrightarrow{g*} FC \xrightarrow{d} R^1FA \longrightarrow \cdots \longrightarrow R^iFC \xrightarrow{d} R^{i+1}FA \longrightarrow \cdots$$

Proof. By the Horseshoe Lemma, we have

$$\begin{tikzcd}
P. \arrow[r, two heads] \arrow[d, "\xi"'] & Q. \arrow[r, two heads] \arrow[d, "\eta"] & R. \arrow[d, "\zeta"] \\
A \arrow[r, tail] & B \arrow[r, two heads] & C
\end{tikzcd}$$

Vertically, we have projective resolutions, and $P. \rightarrowtail Q. \twoheadrightarrow R.$ is a degreewise split short exact sequence.

Since F is additive, it preserves degreewise split exactness, i.e. $F(P.) \rightarrowtail F(Q.) \twoheadrightarrow F(R.)$ remains (degreewise split) exact sequence in $\mathbf{Ch}(\mathcal{B})$.

By the long exact sequence of homology for B, the long exact sequence in the statement is obtained. Note that $H_i(\cdots) = 0$ for $i < 0$ because the objects are 0's.

We now compare $F(P.) \rightarrowtail F(Q.) \twoheadrightarrow F(R.)$ in degree 0 with $FA \to FB \to FC$. Finally, if F is right exact, then for $P_1 \to P_0 \to A \to 0$ exact indicates $FP_1 \to FP_0 \to FA \to 0$ exact by applying F. Therefore, we have

$$\begin{tikzcd}
H_0(FP_1 \to FP_0 \to 0) \arrow[r, equal] \arrow[d, "\cong"] & L_0FA \arrow[d, "H_0F\xi"] \\
H_0(0 \to FA \to 0) \arrow[r, equal] & FA
\end{tikzcd}$$

□

Corollary 2.3.6. Suppose $\mathcal{A}$ has enough projectives. A right exact functor $F : \mathcal{A} \to \mathcal{B}$ is exact if and only if $L_iF = 0$ for all $i > 0$, if and only if $L_1F = 0$.

Proof. ($\Rightarrow$): If exact, then by applying F on the sequence $\cdots \to \cdots \to P_1 \to P_0 \to A \to 0$, we get an exact sequence. When computing homology, we have $H_i(F(P.))$ being 0 if $i > 0$ and being FA if $i = 0$.

($\Leftarrow$): By the long exact sequence. $\square$

Corollary 2.3.7. Dually, if $\mathcal{A}$ has enough injectives, for $F : \mathcal{A} \to \mathcal{B}$ left exact functor, it is exact if and only if $R^1F = 0$ if and only if $R^iF = 0$ for all $i > 0$.

Example 2.3.8. Let P be projective, then $L_iF(P) = 0$ for all $i > 0$. Indeed, use projective resolution $0 \to P \xrightarrow{\mathbf{id}} P \to 0$.

Example 2.3.9. If $\mathcal{A}$ have enough projectives and is hereditory (i.e a subobject of projective is still projective, e.g. $\mathbf{Ab}$ or $\mathbb{Z}-\mathbf{Mod}$), then $L_iF = 0$ for all $i \geqslant 2$, therefore we only have to check L_1F. This is true because by inducing the kernel from the projective, we obtain $0 \to K \to P_0 \to A \to 0$, then there exists a projective resolution $0 \to P_1 \to P_0 \to A \to 0$.

Dually, if every quotient of an injective is injective, e.g. in PIDs quotient of divisible is divisible, then there exists injective resolution of length 1: $0 \to A \to I^0 \to I^1 \to 0$. Hence, we know $R^iF = 0$ for all $i \geqslant 2$.

2.4 Deriving via Acyclics

We will discuss deriving left-derived functors. We can do this to right-derived functors too. Throughout the section, $\mathcal{A}$ is an Abelian category with enough projectives. (Respectively, enough injectives for the dual story.)

Definition 2.4.1 (Acyclic). Let $F : \mathcal{A} \to \mathcal{B}$ be a (right-exact) additive functor to another Abelian category $\mathcal{B}$. An object E of $\mathcal{A}$ is called (left) F-acyclic if $L_iF(E) = 0$ for all $i > 0$.

Example 2.4.2. Projectives are left F-acyclic: take themselves as projective resolutions.

Lemma 2.4.3. Let $F : \mathcal{A} \to \mathcal{B}$ be right exact and $\mathcal{A}$ has enough projectives.

(a) Let $A \rightarrowtail B \twoheadrightarrow$ be an exact sequence in $\mathcal{A}$, i.e. we have

$$0 \longrightarrow A \longrightarrow B \longrightarrow E \longrightarrow 0$$

exact, and we have E as (left) F-acyclic. Then

$$L_1F(E) = 0 \longrightarrow FA \longrightarrow FB \longrightarrow FE \longrightarrow 0$$

is also exact.

(b) Let $0 \to A \to E \to E' \to 0$ be exact in $\mathcal{A}$ such that E and E' are F-acyclic. Then A is also F-acyclic.

(c) Let $\cdots \to E_{n+1} \to E_n \to 0 \to 0 \to \cdots$ be a right bounded exact complex of F-acylics. Then the complex $FE. = (\cdots \to F(E_{n+1}) \to F(E_n) \to 0 \to \cdots)$ is exact.

(d) If $f : E. \to E'_.$ is a quasi-isomorphism of right-bounded complexes of F-acylics, then $Ff : F(E.) \to F(E'_.)$ remains a quasi-isomorphism.

Proof. (a) This is easy from the long exact sequence with $L_1F(E) = 0$.

(b) We have a long exact sequence in $\mathcal{B}$, and there is the segment

$$\cdots \longrightarrow L_{i+1}F(E') \xrightarrow{d} L_iF(A) \longrightarrow L_iF(E) \longrightarrow \cdots$$

and note that we have $L_{i+1}F(E') = 0$ if $i \geqslant 1$ since E' is F-cyclic and similarly we know $L_iF(E) = 0$ for all $i \geqslant 1$. Therefore, we must have $L_iF(A) = 0$ for all $i \geqslant 1$ and by definition A is F-acyclic.

(c) Take the exact sequence $\cdots \to E_{n+2} \to E_{n+1} \xrightarrow{d_{n+1}} E_n \to 0 \to 0 \to \cdots$, then by epi-mono factorization, we have $\ker(d_{n+1})$, which is also F-acyclic according to part b. Repeating by induction on (b), we have $\ker(d_i) \rightarrowtail E_i \twoheadrightarrow \mathrm{im}(d_i) = \ker(d_{i-1})$.

(d) For $f : E. \to E'_.$ quasi-isomorphism, the image under F is still a quasi-isomorphism?

We first consider the case where f is an epimorphism in every degree $f_n : E_n \twoheadrightarrow E'_n$ for all n.

Consider $A. = ker(f).$. We have a degreewise short exact sequence (i.e. a short exact sequence in $\mathbf{Ch}(\mathcal{A})$ $A. \rightarrowtail E. \twoheadrightarrow E'_.$. By the long exact sequence in homology, we have

$$\cdots \xrightarrow[\cong]{f_*} H_{i+1}(E') \xrightarrow{d} H_i(A) = 0 \longrightarrow H_i(E.) \xrightarrow[\cong]{f_*} H_{i-1}(E') \xrightarrow{d} \cdots$$

and the epi-mono factorization around $H_i(A) = 0$ must induce 0 as well. Therefore, f is a quasi-isomorphism, and so f_* are isomorphisms. Hence, A is acyclic. By (b), in each degree, we have $A_i \rightarrowtail E_i \twoheadrightarrow E'_i$ where $E_i \twoheadrightarrow E'_i$ is acyclic by assumption. Hence, A_i is acyclic, also we know $A.$ is right-bounded. By (c), $F(A.)$ remains exact. By (a), since E' is acyclic, then in each degree we know the sequence $0 \to FA_i \to FE_i \to FE'_i \to 0$ is exact in $\mathcal{B}$.

Therefore, we have a degreewise short exact sequence in $\mathbf{Ch}(\mathcal{B})$, namely

$$0 \longrightarrow F(A.) \longrightarrow F(E.) \xrightarrow{F(f)} F(E'_.) \longrightarrow 0$$

Then by the long exact sequence in $\mathcal{B}$, we know $H_i(F(f)) : H_i(F(E.)) \to H_i(F(E'_.))$ is an isomorphism, i.e. $F(f)$ is a quasi-isomorphism.

Now we prove the general case. Let $f : E. \to E'_.$ be a quasi-isomorphism. Consider Z_i to be

$$\cdots \longrightarrow 0 \longrightarrow E'_i \xrightarrow{\mathbf{id}} E'_i \longrightarrow 0 \longrightarrow \cdots$$

which is a split exact complex. We can also denote E' to be

$$\cdots \longrightarrow E'_{i+1} \longrightarrow E'_i \xrightarrow{d} E'_{i-1} \longrightarrow E'_{i-2} \longrightarrow \cdots$$

then we have a map $g_i : Z_i \to E'$ by taking

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & E'_i \arrow[r, "\mathbf{id}"] \arrow[d, equal] & E'_i \arrow[r] \arrow[d, "d"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & E'_{i+1} \arrow[r] & E'_i \arrow[r, "d"'] & E'_{i-1} \arrow[r] & E'_{i-2} \arrow[r] & \cdots
\end{tikzcd}$$

We then have

$$\begin{tikzcd}
E. \arrow[r, "\cong"] \arrow[dr, "f"'] & E. \oplus \bigoplus_{i\in\mathbb{Z}} Z_i \arrow[d, "g=(f,(g_n)_{n\in\mathbb{Z}})"] \\
& E'_.
\end{tikzcd}$$

where the isomorphism comes from the homotopy equivalence. By taking $\bar{E}_i = (E_i \oplus E'_i \oplus E'_{i+1})_i$ and so then $\bar{E}. = \bigoplus_{i\in\mathbb{Z}} Z_i$, we have $g : \bar{E}. \to E'_.$ as a degreewise epimorphism. Now E is still right-bounded, still F-acyclic in every degree. And since $E. \simeq E'_.$ gives a homotopy equivalence, then $H_i(E.) \cong H_i(\bar{E})$ because we have the diagram

$$\begin{tikzcd}
H.(E.) \arrow[r, "\cong"] \arrow[dr, "H.(f)"'] & H_i(\bar{E}) \arrow[d, "H.(g)"] \\
& H_i(E.)
\end{tikzcd}$$

Therefore, g is also a quasi-isomorphism. By the special case, we know $F(g) : F(\bar{E}) \to F(E')$ is a quasi-isomorphism. Again, we have

$$\begin{tikzcd} F(E) \arrow[r, "\cong"] \arrow[dr, "F(f)"'] & F(\bar{E}) \arrow[d, "F(g)"] \\ & F(E') \end{tikzcd}$$

and so $F(E) \simeq F(\bar{E})$ because the additive functor F preserves homotopy equivalence. Therefore, $F(g)$ is a quasi-isomorphism, and so $F(f)$ is a quasi-isomorphism.

□

Exercise 2.4.4. Part (c) and Part (d) fails for unbounded complexes. Let K be a field and $R = K[t]/t^2$, (e.g. if $\mathbf{char}(K) = 2$, this is the group algebra $KC_2 = K[x]/(x^2-1) \cong K[t]/t^2$ for $t = x-1$ and C_2 denotes the cyclic group of order 2).

Consider the complex

$$\cdots \xrightarrow{t} R \xrightarrow{t} R \xrightarrow{t} \cdots$$

that is exact, and of projectives (and injectives in characteristic 2), but it is not preserved by right exact functor $F = K \otimes_R - : R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$.

Remark 2.4.5. With the same notation as in the lemma,

(a) Let $0 \to B \to E \to A \to 0$ be an exact sequence with E F-acyclic. Then $L_{i+1}(F(A)) \cong L_i(F(B))$ for all $i \geqslant 1$.

(b) More generally, if we have an exact sequence $0 \to B \to E_m \to \cdots \to E_1 \to A \to 0$ for $m \geqslant 1$ and all E_i's are F-acyclic, then $L_{i+m}(F(A)) \cong L_i(F(B))$ for all $i \geqslant 1$. In particular, this holds for projective E_i's.

Theorem 2.4.6. Suppose $F : \mathcal{A} \to \mathcal{B}$ is right exact and $\mathcal{A}$ has enough projectives. Let $A \in \mathcal{A}$. Suppose that $E. \xrightarrow{\eta} A$ is a resolution of A by F-acyclic:

$$\begin{tikzcd} \cdots \arrow[r] & E_1 \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d, "\eta"] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

with a quasi-isomorphism in between, and with all E_i's are F-acyclic. Then there exists a natural isomorphism $L_i(F(A)) \xrightarrow{\cong} H_i(F(E.))$. The dual statement is true for resolution by right acyclic to compute right-derived functors.

Proof. Let $\xi : P. \to A$ be a projective quasi-isomorphism, then we have

$$\begin{tikzcd} P. \arrow[r, dashed, "\varphi"] \arrow[dr, "\xi"'] & E. \arrow[d, "\eta"] \\ & A \end{tikzcd}$$

and

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dashed] & P_0 \arrow[r] \arrow[d, "\text{lift}"] \arrow[dr] & A \arrow[d, equal] \\ \cdots \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r, two heads] & A \end{tikzcd}$$

where we can induce the mapping $P_i \to E_i$ by induction and using a lift by exactness, e.g. construct the kernel of $E_i \to E_{i-1}$ and construct the map $P_{i+1} \to E_{i+1}$.

Then φ is a quasi-isomorphism of right-bounded complexes of F-acyclics. Hence, $F(\varphi)$ remains a quasi-isomorphism by part (d) of the lemma: $H_i(F(\varphi)) : H_i(F(P.)) \cong L_i(F(A)) \xrightarrow{\cong} H_i(F(E.))$. □

Example 2.4.7. Consider the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. We have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

where $\mathbb{Z}$ is free and $\mathbb{Q}$ is not free over $\mathbb{Z}$ but it is flat, meaning the localization $\mathbb{Q} \otimes_{\mathbb{Z}} -$ is the localization, i.e. exact.

Note that this uses the derived functor $\mathbf{Tor}_i(M, N)$ and we face the problem of distinguishing $M \otimes -$ at N and $- \otimes N$ at M. In fact, they should be the same.

Anyways, we have $\mathbb{Q}$ to be F-acyclic for $F = M \otimes_{\mathbb{Z}} -$, and so one can compute $\mathbf{Tor}_1(M, \mathbb{Q}/\mathbb{Z})$ as H_1 of the map $0 \to M \otimes_{\mathbb{Z}} \mathbb{Z} \cong M \to M \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$. For example, if M is torsion, then $\mathbf{Tor}_1(M, \mathbb{Q}/\mathbb{Z}) \cong M$.

Example 2.4.8. Take $M = \mathbb{Z}/n\mathbb{Z}$. We have

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow M \longrightarrow 0$$

and by applying $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} -$, we get another sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, "\cdot n"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \\
\mathbb{Z}/n\mathbb{Z} \arrow[ur, tail, "{\cdot[\frac{1}{n}]}"'] & & &
\end{tikzcd}$$

and sends $H_1 = \mathbb{Z}/n\mathbb{Z}$ to $H_0 = 0$.

2.5 Ext and Tor

Roughly speaking, we can define the **Ext** and the **Tor** functors as the following:

Definition 2.5.1 (Ext, Tor). **Ext** is the right derived functor of $\mathbf{Hom}(-,-)$, which is left exact, and **Tor** is the left derived functor of $- \otimes -$, which is right exact.

The problem is, do we mean $L_i(M \otimes -)$ evaluated at N, or $L_i(- \otimes N)$ evaluated at M? The same problem may occur on the **Hom** functor. More generally, we want to consider this for any $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$.

Theorem 2.5.2. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three Abelian categories, where $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ be an additive functor in each variable, such that

1. $F(A, -) : \mathcal{B} \to \mathcal{C}$ is right exact for all $A \in \mathcal{A}$,
2. $F(-, B) : \mathcal{A} \to \mathcal{C}$ is right exact for all $B \in \mathcal{B}$,
3. $F(P, -) : \mathcal{B} \to \mathcal{C}$ is right exact for all $P \in \mathbf{Proj}(\mathcal{A})$,
4. $F(-, Q) : \mathcal{A} \to \mathcal{C}$ is right exact for all $Q \in \mathbf{Proj}(\mathcal{B})$,

then for all $i \in \mathbb{N}$, there exists a natural isomorphism

$$(L_i F(A, -))(B) \cong (L_i F(-, B))(A)$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The dual statement also holds.

Remark 2.5.3. The proof basically involves $F(A, -) : \mathbf{Ch}_+(B) \to \mathbf{Ch}(C)$ etc., and also the bifunctor $F(-,-) : \mathbf{Ch}_+(\mathcal{A}) \times \mathbf{Ch}_+(\mathcal{B}) \to \mathbf{Ch}(\mathcal{C})$. The meaning of the latter functor involves a "**Tot**" operation from double complex to single complex: if $A.$ in $\mathbf{Ch}(\mathcal{A})$ and $B.$ in $\mathbf{Ch}(\mathcal{B})$, then $F(A., B.)$ is a double complex:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & F(A_i, B_j) \arrow[r, "{F(d^*, B_j)}"] \arrow[d, "{F(A_i, d_j^B)}"'] & F(A_{i-1}, B_j) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F(A_i, B_{j-1}) \arrow[r] \arrow[d] & F(A_{i-1}, B_{j-1}) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

Observe that $F(A_i, B_j)$ has a total degree of $i+j$, the two adjacent spots have degree one less, and so on.

We have two ways of producing a complex out of a double complex

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_{i,j} \arrow[r, "(+)d^h"] \arrow[d, "(+)d^v"'] & C_{i-1,j} \arrow[r] \arrow[d, "(-)d^v"] & \cdots \\
\cdots \arrow[r] & C_{i,j-1} \arrow[r, "(-)d^h"'] \arrow[d] & C_{i-1,j-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

which are

$$\cdots \longrightarrow \bigoplus_{i+j=n} C_{i,j} \longrightarrow \bigoplus_{i+j=n-1} C_{i,j} \longrightarrow \cdots$$

i.e. $\mathbf{Tot}^{\coprod}(C_{\cdot,\cdot})$, or

$$\cdots \longrightarrow \prod_{i+j=n} C_{i,j} \longrightarrow \prod_{i+j=n-1} C_{i,j} \longrightarrow \cdots$$

i.e. $\mathbf{Tot}^{\prod}(C_{\cdot,\cdot})$. Note that there should be signs on the differentials, using

$$C_{i,j} \xrightarrow{\begin{pmatrix} d^h \\ (-1)^i d^v \end{pmatrix}} C_{i-1,j} \oplus C_{i,j-1}$$

and both are well-defined. Luckily, if for every total degree $n \in \mathbb{Z}$, the number of $\{(i,j) \in \mathbb{Z}^2 \mid i+j=n, c_{i,j} \neq 0\}$ is finite, then

$$\mathbf{Tot}^{\coprod}(C_{\cdot,\cdot}) = \mathbf{Tot}^{\prod}(C_{\cdot,\cdot}) = \mathbf{Tot}^{\oplus}(C_{\cdot,\cdot}).$$

Why is this true? If we assume the complex $A.$ in

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & F(A_i, B_j) \arrow[r, "F(d^*, B_j)"] \arrow[d, "F(A_i, d_j^B)"'] & F(A_{i-1}, B_j) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F(A_i, B_{j-1}) \arrow[r] \arrow[d] & F(A_{i-1}, B_{j-1}) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

is right-bounded, then as i is small enough, soon everything in column is zero: for right bounded B, we would have rows to be zero.

If (as will be the case everywhere today) A and B are right-bounded, then $F(A., B.)$, considering a line $i+j=n$, then everything below that line must be 0, and there are only a finite number of elements that are possibly not 0.

So $F : \mathbf{Ch}_+(\mathcal{A}) \times \mathbf{Ch}_+(\mathcal{B}) \to \mathbf{Ch}_+(\mathcal{C})$ is just $\mathbf{Tot}^{\oplus} \circ F$. Therefore,

$$\begin{tikzcd}
F(A.,B.)_n = \bigoplus\limits_{i+j=n} F(A_i,B_j) \arrow[dd] \arrow[dr, "\begin{pmatrix} F(d^A,B) \\ (-1)^i F(A,d^B) \end{pmatrix}"] & \\
& F(A_{i-1},B_j) \oplus F(A_i,B_{j-1}) \arrow[dl] \\
\bigoplus\limits_{i+j=n-1} F(A_i,B_j) &
\end{tikzcd}$$

Lemma 2.5.4. Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be additive in each variable, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are additive. Then $F^{\mathbf{Tot}} : \mathbf{Ch}_+(\mathcal{A}) \times \mathbf{Ch}_+(\mathcal{B}) \to \mathbf{Ch}_+(\mathcal{C})$ preserves degreewise split exact sequences in each variable.

Proof. This can be done just by linear algebra, left as an exercise. $\square$

Lemma 2.5.5. Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ as in the theorem. Let $f : A. \to A'_.$ be a quasi-isomorphism of right-bounded C (in $\mathbf{Ch}_+(\mathcal{A})$) and $Q. \in \mathbf{Ch}_+(\mathbf{Proj}(\mathcal{B}))$ be a right-bounded complex of projectives of $\mathcal{B}$. Then

$$F^{\mathbf{tot}}(f, Q.) : F^{\mathbf{tot}}(A, Q.) \to F^{\mathbf{tot}}(A'_., Q.)$$

is a quasi-isomorphism.

Proof. By (4) in the theorem, the results holds if $Q. = c_n(Q)$ (where $c_n(Q)$ is a sequence of zeros except Q at degree n, i.e. $\cdots \to 0 \to 0 \to Q \to 0 \to 0 \to \cdots$) for some $Q \in \mathbf{Proj}(\mathcal{B})$ because $F(-, Q)$ is exact.

By induction, we also get the result if $Q.$ is bounded on both sides, i.e. $Q. \in \mathbf{Ch}_b(\mathbf{Proj}(\mathcal{B}))$ say

$$Q. = (\cdots \to 0 \to Q_n \to \cdots \to Q_m \to 0 \to \cdots)$$

for $n, m \in \mathbb{Z}$, $n \geqslant m$; induction on $n - m$.

Now for the induction step, let Q' be the brutal truncation of Q below degree n, then as we have $Q. = (\cdots \to 0 \to Q_n \to \cdots \to Q_m \to 0 \to \cdots)$ and $Q'_. = (\cdots \to 0 \to Q_{n-1} \to \cdots \to Q_m \to 0 \to \cdots)$, so that we create degree split exact sequence $Q' \rightarrowtail Q \twoheadrightarrow c_n(Q_n)$ that is presented horizontally as below:

$$\begin{tikzcd}
c_n(Q_n) = \cdots \arrow[r] & 0 \arrow[r] & Q_n \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \cdots \\
Q = \cdots \arrow[r] & 0 \arrow[r] \arrow[u] & Q_n \arrow[r] \arrow[u] & Q_{n-1} \arrow[r] \arrow[u] & \cdots \arrow[r] \arrow[u] & Q_m \arrow[r] \arrow[u] & 0 \\
Q' = \cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & Q_{n-1} \arrow[r] \arrow[u, equal] & \cdots \arrow[r] \arrow[u, equal] & Q_m \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Hence, by additivity, we have

$$\begin{tikzcd}
0 \arrow[r] & F^{\mathbf{tot}}(A,Q') \arrow[r] \arrow[d, "F^{\mathbf{tot}}(f,Q')"] & F^{\mathbf{tot}}(A,Q) \arrow[r] \arrow[d, "F^{\mathbf{tot}}(f,Q)"] & F^{\mathbf{tot}}(A,c_n(Q_n)) \arrow[r] \arrow[d, "F^{\mathbf{tot}}(f,c_n(Q_n)"] & 0 \\
0 \arrow[r] & F^{\mathbf{tot}}(A',Q') \arrow[r] & F^{\mathbf{tot}}(A',Q) \arrow[r] & F^{\mathbf{tot}}(A',c_n(Q_n)) \arrow[r] & 0
\end{tikzcd}$$

are degreewise split short exact sequences of complex in $\mathcal{C}$.

By the induction hypothesis on $n - m$, we know $F^{\mathbf{tot}}(f, Q')$ and $F^{\mathbf{tot}}(f, c_n(Q_n))$ are quasi-isomorphisms. In the long exact sequence in homology associated to both rows:

$$\begin{tikzcd}
\cdots \arrow[r] & \cdot \arrow[r, "d"] \arrow[d, "\cong"'] & \cdot \arrow[r] \arrow[d, "\cong"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r, "d"] \arrow[d, "\cong"] & \cdot \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & \cdot \arrow[r, "d"'] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r, "d"'] & \cdot \arrow[r] & \cdots
\end{tikzcd}$$

By the Five Lemma, $F^{\mathbf{tot}}(f, Q.)$ is a quasi-isomorphism, then the general case follows by cocontinuity; if $Q. \in \mathbf{Ch}_+(\mathbf{Proj}(\mathcal{B}))$, then truncations can be presented as

$$
\begin{tikzcd}
\tau_{\leqslant n}Q = \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & Q_n \arrow[r] \arrow[d, equal] & Q_{n-1} \arrow[r] \arrow[d, equal] & \cdots \\
\tau_{\leqslant n+1}Q = \cdots \arrow[r] & 0 \arrow[r] & Q_{n+1} \arrow[r] \arrow[d, equal] & Q_n \arrow[r] \arrow[d, equal] & Q_{n-1} \arrow[r] \arrow[d, equal] & \cdots \\
 & & \vdots \arrow[d, equal] & \vdots \arrow[d, equal] & \vdots \arrow[d, equal] & \\
Q = \cdots \arrow[r] & 0 \arrow[r] & Q_{n+1} \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

So Q is the colimit $\mathrm{co}\overset{!}{\lim\limits_{n\to\infty}}\tau_{\leqslant n}Q$ is degreewise stationary, and therefore $F^{\mathbf{tot}}(A., Q.) = \mathrm{co}\overset{!}{\lim\limits_{n\to\infty}} F^{\mathbf{tot}}(A., \tau_{\leqslant n}Q)$ is again degreewise stationary (eventually), as F is additive and commute with the degreewise stationary sequences, and only finitely many are involved. This preserves quasi-isomorphisms, i.e. H_i commutes with such stationary colimits, and hence the result. $\square$

We now prove the theorem formally.

Proof. Let $P. \xrightarrow{\xi} A$ and $Q. \xrightarrow{\eta} B$ be projective resolutions, i.e. quasi-isomorphisms. We can check that the diagram

$$
\begin{tikzcd}
F^{\mathbf{tot}}(P., Q.) \arrow[r, "F^{\mathbf{tot}}(\xi, Q.)"] \arrow[d, "F^{\mathbf{tot}}(P., \eta)"'] & F^{\mathbf{tot}}(A, Q.) \arrow[d] \\
F^{\mathbf{tot}}(P., B) \arrow[r] & F^{\mathbf{tot}}(A, B)
\end{tikzcd}
$$

commutes. By lemma, $F^{\mathbf{tot}}(P., \eta)$ and $F^{\mathbf{tot}}(\xi, Q.)$ are quasi-isomorphisms, therefore, we have isomorphisms in homology:

$$
\begin{tikzcd}
H_n(F^{\mathbf{tot}}(P., B) \arrow[d, equal, "F(-,B)"'] & H_n(F^{\mathbf{tot}}(P., Q.) \arrow[l, "F^{\mathbf{tot}}(P.,\eta)"', "\cong"] \arrow[r, "\cong", "F^{\mathbf{tot}}(A,Q.)"'] & H_n(F(A, Q.)) \arrow[d, equal] \\
(L_n(F(-, B)))(A) & & (L_n(F(A, -)))(B)
\end{tikzcd}
$$

Note that the "total degree" term can be dropped in this proof. $\square$

Remark 2.5.6 (Applications for $\mathbf{Ext}$ and $\mathbf{Tor}$)**.** We can apply theorem in dual form to $\mathbf{Hom}_R(-,-) : (\mathbf{R}\text{-}\mathbf{Mod})^{\mathrm{op}} \times (\mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{Ab}$, where $\mathbf{R}$-**Mod** is injective.

For $M, N \in \mathbf{R}$-**Mod**, if $P. \to M$ is a projective resolution of M and $N \to I^{\cdot}$ is an injective resolution of N, then we have

$$H^n(\mathbf{Hom}(P., N)) \cong H^n(\mathbf{Hom}(M, I^{\cdot}))$$

where in degree i we have $\mathbf{Hom}(P_i, N)$ and $\mathbf{Hom}(M, Q^i)$, respectively. Therefore, we transform $P_i \to P_{i-1} \to \cdots \to P_0 \to 0$ into

$$0 \to \mathbf{Hom}(P_0, N) \to \cdots \to \mathbf{Hom}(P_i, N) \to \mathbf{Hom}(P_{i-1}, N)$$

in cohomology notations. This is what we called the extension group, $\mathbf{Ext}^n_R(M, N)$, given by $R^i\mathbf{Hom}_{\mathcal{A}}(A, -)(B) \cong R^i\mathbf{Hom}_{\mathcal{A}}(-, B)(A)$.

The theorem also applies (more directly) to

$$- \otimes_R - : (\mathbf{Mod}\text{-}\mathbf{R}) \times (\mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{Ab}.$$

If M is a right R-module and N is a left R-module, and $P. \to M$ and $Q. \to N$ are projective resolutions, then we can use $H_n(P. \otimes_R N) \cong H_n(M \otimes_R Q.)$ to denote $\mathbf{Tor}^R_n(M, N)$, the torsion group.

Problem 6 (Exam Problem 6)**.** Compute $\mathbf{Tor}_i^{\mathbb{Z}}(M,N)$ and $\mathbf{Ext}_{\mathbb{Z}}^i(M,N)$ for all pairs of M,N in the set $\{\mathbb{Z},\mathbb{Z}/n\mathbb{Z},\mathbb{Q},\mathbb{Q}/\mathbb{Z}\}$, where $n\in\mathbb{Z}$ is arbitrary.

Remark 2.5.7. All results related to $\mathbf{Hom}_R$ here can be applied to $\mathbf{Hom}_{\mathcal{A}}:\mathcal{A}^{\mathrm{op}}\times\mathcal{A}\to\mathbf{Ab}$. Therefore, $\mathbf{Ext}^n_{\mathcal{A}}$ can be interpretted as $R^n\mathbf{Hom}(A,-)(B)\cong R^n\mathbf{Hom}(-,B)(A)$ for all $A,B\in\mathcal{A}$.

2.6 Finite Projective/Injective Resolutions

Remark 2.6.1. We think of the following question: when do objects $M\in\mathcal{A}$ Abelian admit finite projective resolutions

$$0\to P_n\to P_{n-1}\to\cdots\to P_n\to M\to 0$$

or injective resolutions

$$0\to M\to I^0\to\cdots\to I^n\to M?$$

Proposition 2.6.2. Let $P\in\mathcal{A}$. Suppose $\mathcal{A}$ has enough projectives, then the following are equivalent:

(i) P is projective, i.e. $\mathbf{Hom}_{\mathcal{A}}(P,-)$ is exact.

(ii) $\mathbf{Ext}^i(P,M)=0$ for all $i\geqslant 1$, and for all $M\in\mathcal{A}$.

(iii) P is acyclic for $\mathbf{Hom}_{\mathcal{A}}(-,M)$ for all $M\in\mathcal{A}$.

(iv) $\mathbf{Ext}^1(P,M)=0$ for all $M\in\mathcal{A}$.

Proof. $(i)\Rightarrow(ii)$: suppose P is projective, compute right derived functor of exact functor and we get 0.

$(ii)\Rightarrow(iii)$: by remark, $(R^n\mathbf{Hom}(P,-))(M)\cong(R^n\mathbf{Hom}(-,M))(P)$. Note that the left-hand-side is 0 (ii) and the right-hand-side is 0 for all $n\geqslant 1$ for all $M\in\mathcal{A}$.

$(iii)\Rightarrow(iv)$: by the same argument, we apply it to $i=1$.

$(iv)\Rightarrow(i)$: it suffices to check exactness. Consider $M'\rightarrowtail M\twoheadrightarrow M''$, then we have an exact sequence by assumption:

$$0\to\mathbf{Hom}(P,M')\to\mathbf{Hom}(P,M)\to\mathbf{Hom}(P,M'')\to\mathbf{Ext}^1(P,M')$$

and therefore $\mathbf{Ext}^1(P,M')=0$. ☐

Proposition 2.6.3. Dually, if $\mathcal{A}$ has enough injectives, then $I\in\mathcal{A}$ is injective if and only if $\mathbf{Ext}^1(M,I)=0$ for all $M\in\mathcal{A}$.

Now, generally speaking, we consider the sequence

$$0\to A\to E_{n-1}\to\cdots\to E_0\to B\to 0$$

where $E_0,\cdots,E_{n-1}$ are acyclic (either projective or injective).

Now recall we should have the following result:

Lemma 2.6.4. Given the exact sequence above,

(a) if all E_i's are projective, then $\mathbf{Ext}^i(A,M)\cong\mathbf{Ext}^{i+n}(B,M)$ for all $M\in\mathcal{A}$ and $i\geqslant 1$.

(b) if all E_i's are injective, then $\mathbf{Ext}^i(M,B)\cong\mathbf{Ext}^{i+n}(M,A)$ for all $M\in\mathcal{A}$ and $i\geqslant 1$.

Proof. Check the kernel and the corresponding long exact sequence. ☐

Corollary 2.6.5. With the above notations:

(a) If all E_i's are projectives and $\mathbf{Ext}^{n+1}(B,M)=0$ for all $M\in\mathcal{A}$, then A is projective.

(b) If all E_i's are injectives and $\mathbf{Ext}^{n+1}(M,A)=0$ for all $M\in\mathcal{A}$, then B is injective.

Corollary 2.6.6. Let $n\geqslant 1$.

(a) Suppose $\mathcal{A}$ has enough projectives.Pick $A \in \mathcal{A}$, then A has a projective resolution of length $\leqslant n$, i.e. we have an exact sequence
$$0 \to P_n \to \cdots \to P_0 \to A \to 0$$
with P_i projectives, if and only if $\mathbf{Ext}^{n+1}(A, M) = 0$ for all $M \in \mathcal{A}$.

(b) Suppose $\mathcal{A}$ has enough projectives.Pick $B \in \mathcal{A}$, then B has an injective resolution of length $\leqslant n$, i.e. we have an exact sequence
$$0 \to B \to I^0 \to \cdots \to I^n \to 0$$
with I^i injectives, if and only if $\mathbf{Ext}^{n+1}(M, B) = 0$ for all $M \in \mathcal{A}$.

Proof. Do the projective/injective resolution up to step n. □

Corollary 2.6.7. Let $\mathcal{A}$ be Abelian with enough projectives and injectives. Then let $n \geqslant 1$ and the following are equivalent:

(i) Every object has a projective resolution of length $\leqslant n$.

(ii) Every object has an injective resolution of length $\leqslant n$.

(iii) $\mathbf{Ext}^{n+1}(A, B) = 0$ for all $A, B \in \mathcal{A}$.

Proof. It is clear that (i) and (ii) can imply (iii), and the other way around is also true, following from the corollary. □

Theorem 2.6.8. Let R be a ring and $n \geqslant 1$, then the following are equivalent:

(i) Every R-module has a projective resolution of length $\leqslant n$.

(ii) Every R-module has an injective resolution of length $\leqslant n$.

If moreover R is Noetherian (we can then consider the R-modules to be exactly the finitely-generated R-modules), then the above are also equivalent to

(iii) Every finitely generated R-module has a (finitely-generated) projective resolution of length $\leqslant n$.

(iv) Every finitely generated R-module has an (finitely-generated) injective resolution of length $\leqslant n$.

Proof. $(i) \iff (ii)$: by corollary for $\mathcal{A} = \mathbf{R}\text{-}\mathbf{Mod}$.

$(i), (ii) \iff (iii), (iv)$: By the assumption, we have $\mathbf{Ext}^{n+1} \equiv 0$, and so by the technique above we conclude (iii) and/or (iv).

$(iv) \Rightarrow (iii)$: as before, $\mathbf{Ext}^{n+1}(M, N) = 0$ for all M, N finitely-generated. If P is finitely-generated with $\mathbf{Ext}^1(P, N) = 0$ for all finitely-generated N, then P is projective.

$(iii) \Rightarrow (iv)$: use the same method to reduce, if $\mathbf{Ext}^1(M, E) = 0$ for all finitely-generated M and arbitrary E, then E is injective. This holds because E is injective if and only if it has the extension property with respect to $J \rightarrowtail R$ for ideal J such that both J and R are finitely-generated, if and only if $\mathbf{Ext}^1(R/J, E) = 0$, where R/J is also finitely-generated. □

Proposition 2.6.9. Let E be a right R-module, then the following are equivalent:

(i) E is flat, i.e. $E \otimes_R -$ is exact.

(ii) $\mathbf{Tor}_i(E, M) = 0$ for all R-modules M and for all $i \geqslant 1$.

(iii) E is $(- \otimes_R M)$-acyclic for all R-modules M.

(iv) $\mathbf{Tor}_1(E, M) = 0$ for all R-modules M.

Proposition 2.6.10. Let R be a local commutative Noetherian ring with maximal ideal M and residual field $K = R/M$. Then the ring R is (homologically) regular, i.e. every R-module has a finite projective resolution of length at most n if and only if K has a finite projective resolution of length at most n.

Proof. It suffices to prove the ($\Leftarrow$) direction.

Suppose $0 \to P_n \to \cdots \to P_0 \to K \to 0$ is a projective resolution of K by finitely-generated projective R-modules. Let M be any finitely-generated R-module. Do the projective resolution

$$0 \to N \to Q_{n-1} \to \cdots \to Q_0 \to M \to 0$$

where Q_i's are finitely-generated projectives (and also flat). We now want to show that N is projective. For every $i \geqslant 1$, $\mathbf{Tor}_i^R(N,K) \cong \mathbf{Tor}_{i+n}^R(M,K) = 0$. Hence, $\mathbf{Tor}_1^R(N,K) = 0$, i.e. N is flat and finitely-generated.

We now want to show that N is free. We hereby repeat the proof of "projective $\Rightarrow$ free". Consider $\bar{N} = K \otimes_R N = N/MN$ to be a finitely-generated K-vector space, so $\bar{N} \cong \bar{K^n}$. We can take a K-basis $\bar{e}_1, \cdots, \bar{e}_n \in N$ by using $e_i \in N$. This induces a map $\alpha : R^n \to N$ using $(e_1, \cdots, e_n)$ such that we have $\bar{\alpha} : K^n = K \otimes_R R^n \xrightarrow{\cong} \bar{N}$ given by the basis.

We now want to show that α is an isomorphism. Check the cokernel C from $R^n \xrightarrow{\alpha} N \to C \to 0$ and apply $K \otimes_R -$ (right-exact), then we have

$$\bar{K}^n \xrightarrow[\cong]{\bar{\alpha}} \bar{N} \longrightarrow \bar{C} \longrightarrow 0$$

and therefore $0 = \bar{C} = C/MC$. As N is finitely-generated, then C is finitely-generated by Nakayama Lemma, and then $C = 0$. Hence, $R^n \xrightarrow{\alpha} N$ is an epimorphism. For kernel

$$D \rightarrowtail R^n \overset{\alpha}{\twoheadrightarrow} N$$

where R is Noetherian, so D is finitely-generated. Follow the usual proof, take $K \otimes_R -$, we have

$$\mathbf{Tor}_1(K,N) \longrightarrow K \otimes_R D \longrightarrow \bar{K}^n \xrightarrow{\bar{\alpha}} \bar{N}$$

and note that $\mathbf{Tor}_1(K,N) = 0$ by assumption. Therefore, $\bar{K}^n \xrightarrow{\bar{\alpha}} \bar{N}$ is an isomorphism: above sequence is exact, and by epimorphism, we know $0 = K \otimes_R D = D/MD$. By Nakayama Lemma, $D = 0$. Therefore, α is an isomorphism. $\square$

2.7 Mapping Cone and Koszul Complex

Question: how to get a projective resolution of the residual field?

Definition 2.7.1 (Mapping Cone, Suspension). Let $f : A. \to B.$ be a morphism of complexes in $\mathcal{A}$ (additive or Abelian). The mapping cone of f is the complex

$$\mathrm{cone}(f)_n = A_{n-1} \oplus B_n$$

for all $n \in \mathbb{Z}$ and $d : \mathrm{cone}(f)_n \to \mathrm{cone}(f)_{n-1}$ which gives rise to

$$\begin{tikzcd}
\mathrm{cone}(f)_n \arrow[r, "d"] \arrow[d, equal] & \mathrm{cone}(f)_{n-1} \arrow[d, equal] \\
A_{n-1} \oplus B_n \arrow[r, "\begin{pmatrix} -d^A & 0 \\ f & d^B \end{pmatrix}"'] & A_{n-2} \oplus B_{n-1}
\end{tikzcd}$$

Remark 2.7.2. Note that if we write d^A instead of $-d^A$, then we would have

$$\begin{pmatrix} d & 0 \\ f & d \end{pmatrix}^2 = \begin{pmatrix} d^2 & 0 \\ fd+df & d^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ fd+df & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so the convention is to write $-d^A$, then we have

$$\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}^2 = \begin{pmatrix} d^2 & 0 \\ -fd+df & d^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

as $f : A. \to B.$ is a morphism of complexes.

It comes with two morphisms:

$$A. \xrightarrow{f} B. \xrightarrow{f'} \operatorname{cone}(f) \xrightarrow{f''} \sum A.$$

Figure 6: Suspension Morphism

where

$$f'_n : B_n \xrightarrow{\begin{pmatrix}0\\1\end{pmatrix}} \operatorname{cone}(f)_n = A_{n-1} \oplus B_n,$$

$$f''_n : A_{n-1} \oplus B_n \xrightarrow{\begin{pmatrix}1 & 0\end{pmatrix}} A_{n-1},$$

and

$$(\sum A.) = A_{n-1}, d^{\sum A} = -d$$

which is called the suspension/shift/translation to the left by 1 notation.

Remark 2.7.3. Note that

$$\begin{pmatrix}-d & 0\\ f & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\d\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} d$$

and

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}-d & 0\\ f & d\end{pmatrix} = \begin{pmatrix}-d & 0\end{pmatrix} = (-d) \circ \begin{pmatrix}1 & 0\end{pmatrix} = d^{\sum} \circ \begin{pmatrix}1 & 0\end{pmatrix}$$

and so we have a diagram

$$\begin{tikzcd}
& \operatorname{cone}(f) \arrow[dl, "f''"'] & \\
A \arrow[rr, "f"'] & & B \arrow[ul, "f'"']
\end{tikzcd}$$

Figure 7: Suspension Diagram

where the dot on f'' indicates the morphism is actually to the suspension of A, instead of A itself.

Exercise 2.7.4. We actually have $f' \circ f \sim 0$, $f'' \circ f' = 0$ and $\sum f \circ f'' \sim 0$.

Proposition 2.7.5. Suppose $\mathcal{A}$ is Abelian and $f : A. \to B.$. Let $D = \operatorname{cone}(f)$ and $A. \xrightarrow{f} B. \xrightarrow{f'} D. \xrightarrow{f''} \sum A.$. Then the sequence

$$\cdots \to H_{n+1}(D.) \xrightarrow{f''_*} H_n(A) = H_{n+1}(\sum A) \xrightarrow{f_*} H_n(B) \xrightarrow{f'_*} H_n(D) \xrightarrow{f''_*} \cdots$$

is exact.

Proof. We have a degreewise split exact sequence

$$B. \overset{f'}{\rightarrowtail} D. \overset{f''}{\twoheadrightarrow} \sum(A.)$$

and then (by theorem) gives the homology long exact sequence with the above homology.

Note that it is not split in complex, which should have been

$$B_n \overset{\begin{pmatrix}0\\1\end{pmatrix}}{\rightarrowtail} A_{n-1} \oplus B_n \overset{\begin{pmatrix}1 & 0\end{pmatrix}}{\twoheadrightarrow} A_{n-1}$$

One point to verify is that the map induced by connecting map $d : H_n(\sum A.) \to H_{n-1}(B.)$ is just $H_{n-1}(f)$, i.e. we have

$$\begin{tikzcd} H_n(\sum A.) \arrow[r, "d"] & H_{n-1}(B.) \\ H_{n-1}(A.) \arrow[u, "\cong"] \arrow[ur, "H_{n-1}(f)"'] & \end{tikzcd}$$

We can prove this by elementwise diagram chasing. Consider

$$\begin{tikzcd} B_n \arrow[r, tail, "f'=\begin{pmatrix}0\\1\end{pmatrix}"] \arrow[d, "d"] & D_n = A_{n-1} \oplus B_n \arrow[r, two heads, "f''=\begin{pmatrix}1 & 0\end{pmatrix}"] \arrow[d, "d=\begin{pmatrix}-d & 0\\ f & d\end{pmatrix}"] & A_{n-1} = (\sum A)_n \arrow[d, "-d"] \\ B_{n-1} \arrow[r, "\begin{pmatrix}0\\1\end{pmatrix}"'] & D_{n-1} = A_{n-2} \oplus B_{n-1} \arrow[r, "\begin{pmatrix}1 & 0\end{pmatrix}"'] & A_{n-2} \end{tikzcd}$$

Take $z \in \ker(-d : A_{n-1} \to A_{n-2})$, then there is an obvious lift of z, which is given by sending $(z, 0) \in D_n = A_{n-1} \oplus B_n$ through the f'' map. We then have

$$\begin{tikzcd} (z,0) \arrow[r, "f''"] \arrow[d] & z \\ (-dz, f(z) + d(0) = (0, f(z) \in D_{n-1}) & \end{tikzcd}$$

Note that $(0, f(z))$ comes from $f(z) \in B_{n-1}$ through the $\begin{pmatrix}0\\1\end{pmatrix}$ map and so we have

$$\begin{tikzcd} f(z) \in B_{n-1} \arrow[r, "\begin{pmatrix}0\\1\end{pmatrix}"] \arrow[d] & (0, f(z)) \\ {[f(z)] \in H_{n-1}(B.)} & \end{tikzcd}$$

In short, we have

$$\begin{tikzcd} H_n(\sum A.) \arrow[r, "d"] & H_{n-1}(B.) \\ H_{n-1}(A.) \arrow[u, "\cong"] \arrow[ur, "H_{n-1}(f)"'] & \end{tikzcd}$$

that sends $[z] \in H_{n-1}(A.)$ to $[f(z)] \in H_{n-1}(B.)$. ☐

Corollary 2.7.6. $f : A. \to B.$ is a quasi-isomorphism if and only if $\mathrm{cone}(f)$ is an exact complex.

Proof. Check the long exact sequence. This means complex $D.$ has no homology. ☐

Remark 2.7.7. Suppose $\mathcal{A}$ is Abelian and $f : A \rightarrowtail B$ is a monomorphism in $\mathbf{Ch}(\mathcal{A})$. There are two sources of long exact sequences:

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C = \mathrm{coker}(f)$$

and

$$A \xrightarrow{f} B \xrightarrow{f'} D \xrightarrow{f''} \textstyle\sum(A)$$

We can also compare them by

$$\begin{tikzcd} A. \arrow[r, "f"] \arrow[d, equal] & B. \arrow[r, "f'"] \arrow[d, equal] & D. \arrow[r, "f''"] \arrow[d, "s"] & \sum(A) \arrow[d, equal] \\ A. \arrow[r, "f"'] & B. \arrow[r, "g"'] & C \arrow[r, dashed, "?"'] & \sum(A) \end{tikzcd}$$

but the question is, what is the dashed map? A good guess would be taking

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r, dashed, "DNE"] \arrow[d] & 0 \\ A \arrow[r, tail, "f"] \arrow[d] & 0 \arrow[r, two heads, "g"] \arrow[d] & 0 \arrow[r, dashed, "DNE"] \arrow[d] & 0 \\ 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r, dashed, "DNE"] \arrow[d] & A \\ \vdots & \vdots & \vdots & 0 \end{tikzcd}$$

but this is not working because we don't have suitable maps. The answer would be to construct it degreewise. In degree n, we would have

$$\begin{tikzcd} D_n = A_{n-1} \oplus B_n \arrow[d, "\begin{pmatrix} 0 & g_n \end{pmatrix}"] \\ C_n = \mathrm{coker}(f_n : A_n \to B_n) \end{tikzcd}$$

Exercise 2.7.8. $s : \mathrm{cone}(f) \to \mathrm{cone}(f)$ is a morphism of complexes. Moreover, it makes the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "H_n(f)"] \arrow[d, equal] & H_n(B) \arrow[r, "H_n(f')"] \arrow[d, equal] & H_n(D) \arrow[r, "H_n(f'')"] \arrow[d, "H_n(s)"] & H_{n-1}(A) \arrow[r] \arrow[d, equal] & \cdots \arrow[d, equal] \\ \cdots \arrow[r] & H_n(A) \arrow[r, "H_n(f)"'] & H_n(B) \arrow[r, "H_n(g)"'] & H_n(C) \arrow[r, "d"] & H_{n-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

In particular, $H_n(S)$ is an isomorphism by Five Lemma. So $s : \mathrm{cone}(f) \to \mathrm{coker}(f)$ is an isomorphism.

Remark 2.7.9. Let R be a (commutative) ring. Recall tensor product is constructed by

$$\otimes_R = \otimes_R^{\mathbf{tot}} : \mathbf{Ch}_+(\mathbf{R}\text{-}\mathbf{Mod}) \times \mathbf{Ch}_+(\mathbf{R}\text{-}\mathbf{Mod}) \to \mathbf{Ch}_+(\mathbf{R}\text{-}\mathbf{Mod})$$

and explicitly we have $(P. \otimes Q.)_n = \bigoplus_{i+j=n,(i,j)\in\mathbb{Z}^2} P_i \otimes Q_j \xrightarrow{d} d(p \otimes q) = d(p) \otimes q + (-1)^i p \otimes dq \in P_i \otimes Q_j$.

Definition 2.7.10 (Koszul Complex)**.** Let $a \in R$, then let $\mathbf{Kos}(a) = \cdots \to 0 \to R \xrightarrow{\cdot a} R \to 0 \to \cdots$, where R's are in degree 1 and 0 in homological indexing. This is equivalent to $\mathbf{Kos}(a) = \mathrm{cone}(\cdot a : R = R[0] \to R = R[0])$.

For $a_1, \cdots, a_n \in R$, we define the Koszul complex to be

$$\mathbf{Kos}(\underline{a}) = \mathbf{Kos}(a_1, \cdots, a_n) = \mathbf{Kos}(a_1) \otimes_R \cdots \otimes_R \mathbf{Kos}(a_n).$$

Proposition 2.7.11. For every $X. \in \mathbf{Ch}_+(\mathbf{R}\text{-}\mathbf{Mod})$ and $a \in R$, we have a canonical isomorphism $\mathbf{Kos}(a) \otimes_R X. = \mathrm{cone}(X \xrightarrow{\cdot a} X)$.

Consequently, we have a long exact sequence

$$\cdots \to H_i(X) \xrightarrow{\cdot a} H_i(X) \to H_i(\mathbf{Kos}(a) \otimes X) \to H_{i-1}(X) \to \cdots$$

Proof. Hint: a starting point is $\mathbf{Kos}(a_i) = \begin{cases} R, & \text{for } i = 1, 0 \\ 0 \text{ otherwise} \end{cases}$ and $R \otimes X_i \cong X_i$. Write down everything! $\square$

Definition 2.7.12 (Regular Sequence)**.** A regular sequence in a ring R is a sequence $(a_1, \cdots, a_n)$ where a_1 is not a zero divisor, and each a_i is not a zero divisor in $R/\langle a_1, \cdots, a_n\rangle$.

Corollary 2.7.13. Let $\underline{a} = (a_1, \cdots, a_n)$ be a regular sequence in a commutative ring R. Then $\mathbf{Kos}(a_1, \cdots, a_n)$ is a projective (free) resolution of $R/\langle a_1, \cdots, a_n\rangle$.

Proof. By induction, we have $H_i(\mathbf{Kos}(\underline{a})) = R/\langle \underline{a}\rangle$ if $i = 0$, and $H_i(\mathbf{Kos}(\underline{a})) = 0$ otherwise.

Therefore, we have

$$\cdots \to 0 \to R \xrightarrow{a_1} R \to 0 \to \cdots$$

where $H_1 = 0$ corresponds to R/a_1.

If we tensor one more term, then we get a long exact sequence as in the proposition above. Apply the inductive hypothesis. $\square$

Corollary 2.7.14. If R is Noetherian local commutative ring with residual field $K = R/M$ with maximal ideal $M = \langle a_1, \cdots, a_n\rangle$ such that $a_1, \cdots, a_n$ is regular, then R is (homologically) regular: every finitely-generated R-module has a projective resolution of length $\leqslant n$.

Proof. It is enough to show that $K = R/M$ has a projective resolution of length $\leqslant n$. Use $\mathbf{Kos}(a_1, \cdots, a_n)$, this is a resolution by the previous proposition (just gives a mapping cone, then gives a long exact sequence).

An explicit description can be given. $\mathbf{Kos}(a_1, \cdots, a_n) = \operatorname{cone}(a_1) \otimes \cdots \otimes \operatorname{cone}(a_n)$ in degree i is free of $R^{\binom{n}{i}}$ with a basis of wedge products $e_{j_1} \wedge \cdots \wedge e_{j_i}$ for $1 \leqslant j_1 < \cdots < j_i \leqslant n$. Then $R^{\binom{n}{i}} = \bigwedge^i(R^n)$, i.e. the exterior power, and with differential

$$d(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{k=1}^{i} (-1)^{k+1} \cdot a_{j_k} \cdot (e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_i},$$

i.e. with e_{j_k} omitted. $\square$

2.8 Relative Projectivity

Definition 2.8.1 (Split)**.** Let $U : \mathcal{A} \to \mathcal{B}$ be an exact functor between Abelian categories. An exact sequence in $\mathcal{A}$

$$A' \rightarrowtail^{f} A \xrightarrow{g} \!\!\!\!\twoheadrightarrow A''$$

is called U-split if

$$UA' \rightarrowtail^{Uf} UA \xrightarrow{Ug} \!\!\!\!\twoheadrightarrow UA''$$

is split exact.

Example 2.8.2. U : **KG-Mod** $\to$ **K-vector space** is the forgetful functor for a field K and a finite group G. In such a case (i.e. B is semi-simple), then U-split and exactness are equivalent.

Definition 2.8.3 (Relative Projective)**.** An object $P \in \mathcal{A}$ is called U-projective, or projective relative to U, if $\mathbf{Hom}_{\mathcal{A}}(P, -)$ sends U-split exact sequences to exact sequences of Abelian groups; this boils down to a lifting property with respect to epimorphisms $A \twoheadrightarrow A''$ such that $U(g)$ has a section.

Remark 2.8.4. Every projective is a U-projective. This connects to relative homological algebra.

Proposition 2.8.5. Let

$$\begin{tikzcd} \mathcal{A} \arrow[d, shift left, "U"] \\ \mathcal{B} \arrow[u, shift left, "L"] \end{tikzcd}$$

be an adjunction of Abelian categories with U exact (so its left adjoint preserves projectives). Then

(a) For every $Y \in \mathcal{B}$, the object $L(Y)$ is U-projective in $\mathcal{A}$.

(b) For every $X \in \mathcal{A}$, the counit $\varepsilon_X : LU(X) \to X$ is a U-split-epimorphism (i.e. $U(\varepsilon_X)$ is a split epimorphism; it is an actual epimorphism if and only if U is faithful.

(c) For U faithful, the U-projectives are exactly the direct summands (i.e. the retracts) of $L(Y)$ for $Y \in \mathcal{B}$.

Proof. Left as an exercise. □

Remark 2.8.6. Suppose we have

$$\begin{tikzcd} \mathcal{A} \arrow[d, shift left, "U"] \\ \mathcal{B} \arrow[u, shift left, "L"] \end{tikzcd}$$

and $C := L \circ U : \mathcal{A} \to \mathcal{A}$ has (counit) $\varepsilon : C \to \mathbf{id}_A$ and (using the unit $\eta : \mathbf{id}_B \to UL$) $\nabla : C = LU \xrightarrow{L\eta U} LULU = C^2$ is a comultiplication with "unit" η. This means we have a coassociate

$$\begin{tikzcd} C \arrow[r, "\nabla"] \arrow[d, "\nabla"'] & C^2 \arrow[d, "C\nabla"] \\ C^2 \arrow[r, "\nabla C"'] & C^3 \end{tikzcd}$$

and counit

$$\begin{tikzcd} & C \arrow[dl, equal] \arrow[d, "\nabla"] \arrow[dr, equal] & \\ C & C^2 \arrow[l, "\varepsilon C"] \arrow[r, "C\varepsilon"'] & C \end{tikzcd}$$

which gives a comonad structure on C.

We can now iterate $C : \mathcal{A} \to \mathcal{A}$, i.e. $C^n = C \circ \cdots \circ C : \mathcal{A} \to \mathcal{A}$ as an endofunctor. We have natural transformation $C^n \to C^{n-1}$ that can put ε at several places (maybe n). For example, we have

$$C \xrightarrow{\varepsilon} C^0 = \mathbf{id}$$

$$C^2 \underset{C\varepsilon}{\overset{\varepsilon C}{\rightrightarrows}} C$$

and inductively $C^n \xrightarrow{C^i \varepsilon C^{n-i+1}} C^{n-1}$ for any i. If we define $\partial_n : C^{n+1}(-) \top C^n(-)$, e.g. $\partial_2 = \varepsilon C - C\varepsilon$, then we have

$$\partial_n = \sum_{i=0}^{n} (-1)^i C^i \varepsilon C^{n-i} : C^{n+1} \to C^n.$$

This yields a complex in $\mathcal{A}$, such that for $x \in \mathcal{A}$ we have

$$\begin{tikzcd} C(x) \arrow[r, "\varepsilon"] & x \arrow[r] & 0 \\ C^2(x) \arrow[u, "C(\varepsilon)"] \arrow[r, "\varepsilon_{C(X)}"'] & X \arrow[u, "\varepsilon"'] & \end{tikzcd}$$

and so we have a sequence ($\star$)

$$\begin{tikzcd} \cdots \arrow[r, "\partial_2"] & C^2(x) \arrow[r, "\partial_1"] \arrow[rr, dashed, bend right, "0"'] & C(x) \arrow[r, "\partial_0"] & x \arrow[r] & 0 \end{tikzcd}$$

resembling a projective resolution of C^i of x.

Proposition 2.8.7. The image of complex $(\star)$ under $U : \mathcal{A} \to \mathcal{B}$ is a split exact complex.

Proof. The image admits a homotopy

$$\begin{tikzcd}
UC^{n+1}(x) \arrow[r, "\partial_n"] \arrow[d, equal] & UC^n(x) \arrow[r, "\partial_{n-1}"] \arrow[d, equal] & UC^{n-1}(x) \arrow[r, "\partial_{n-1}"] \arrow[d, equal] & \cdots \\
U(LU)^{n+1}(x) \arrow[r] & U(LU)^n(x) \arrow[r] \arrow[l, bend left, "\eta(U(LU)^n"] & U(LU)^{n-1}(x) \arrow[r] \arrow[l, bend left, "\eta(U(LU)^{n-1}"] & \cdots
\end{tikzcd}$$

where $\eta : \mathbf{id} \to UL$.

After cancellation, it suffices to compute $U\varepsilon(LU)^n \circ \eta U(LU)^n = ((U\varepsilon) \circ (\eta U))LU^n$. Note that $(U\varepsilon) \circ (\eta U)$ is $\mathbf{id}_U$ by unit-counit relation $L \dashv U$, i.e.

$$\begin{tikzcd}
U \arrow[r, "\eta U"] \arrow[rr, bend right, "\mathbf{id}"'] & ULU \arrow[r, "U\varepsilon"] & U
\end{tikzcd}$$

□

Corollary 2.8.8. If $U : \mathcal{A} \to \mathcal{B}$ is exact and faithful, then the complex $(\star)$, for every $x \in \mathcal{A}$, is a resolution (i.e. $(\star)$ is exact) by U-projectives.

Corollary 2.8.9. If $U : \mathcal{A} \to \mathcal{B}$ is exact and faithful and $\mathcal{B}$ is semi-simple (e.g. $\mathcal{B}$ is the category of K-vector spaces), then $(\star)$ is a projective resolution of x.

Example 2.8.10. Let K be a field and G be a finite group. Let $\mathcal{A} = \mathbf{KG}$-**Mod**, $\mathcal{B} = \mathbf{K}$-**Mod**, then there is an adjunction $L \dashv U$ given by

$$\begin{tikzcd}
\mathcal{A} = \mathbf{KG} \arrow[d, shift left, "U"] \\
\mathcal{B} = \mathbf{K}\text{-}\mathbf{Mod} \arrow[u, shift left, "L"]
\end{tikzcd}$$

where $L = KG \otimes_K -$ is the induction from 1 to G, also denoted as $\mathbf{Ind}_1^G$, and $U = \mathbf{Res}_1^G$ is the restriction from G to 1. Note that $L \cong U$. Also, both functors are exact: $\mathbf{Res}_1^G$ is exact because for homology of R-modules, look at the underlying Abelian group "forgets everything" and "detects exactness"; exactness of $\mathbf{ind}_1^G$ is given by tensoring free modules. Also note that no derived functor here can be done as exact. Now, the comonad $LU : \mathcal{A} \to \mathcal{A}$ is just $C = KG \otimes_K - : \mathcal{A} \to \mathcal{A}$.

Remark 2.8.11 (Frobenius). $C \cong KG \otimes_K -$ with diagonal G-action, whcih is the natural tensor on $\mathbf{KG}$-**Mod**. For any module M, we can find $\varepsilon : KG \otimes M \to M$ such that $g \otimes m \mapsto g \cdot m$, and then we have a Frobenius isomorphism:

$$\begin{aligned}
KG \otimes_K M &\xrightarrow{\sim} KG \otimes_K M \\
1 \otimes m &\leftrightarrow 1 \otimes m \\
g \otimes m &\mapsto g \otimes gm \\
g \otimes g^{-1}m &\leftarrow g \otimes m
\end{aligned}$$

The complex $(\star)$ for $M \in \mathcal{A}$ looks like

$$KG^{\otimes(n+1)} \otimes M \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} KG \otimes_K M \xrightarrow{\partial_0} M \longrightarrow 0$$

with very explicit formula for ∂_n.

For instance, for $M = K$ with a very trivial G-action, then $(KG)^{\otimes(n+1)} \otimes_K K = (KG)^{\otimes(n+1)}$ is a free KG-module (with KG acting on the leftmost tensor factor) with basis indexed by the K-basis of $(KG)^{\otimes n}$. Hence, KG-basis of $(KG)^{\otimes(n+1)}$ is

$$[g_1 \mid \cdots \mid g_n] = 1 \otimes g_1 \otimes \cdots \otimes g_n$$

for $g_1, \cdots, g_n \in G$ (with possible repetitions). The explicit formula of the differential, i.e. projective resolution of K over KG, is

$$\partial_n([g_1 \mid \cdots \mid g_n]) = g_1 \cdot [g_2 + \cdots + g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1 \mid \cdots \mid g_i g_{i+1} \mid g_{i+2} \mid \cdots] + (-1)^n [g_1 \mid \cdots \mid g_{n-1}]$$

in $(KG)^{\otimes n}$, being KG-free over $[g_1 \mid \cdots \mid g_{n-1}]$.

Hence, a very explicit projective (free) KG-resolution of K.

2.9 Group (co)Homology

For the whole section, K is a commutative ring (e.g. $K = \mathbb{Z}$ or a field) and G is a finite group.

We saw a projective (free) resolution of K (for G a finite group and K an arbitrary commutative ring) given by

$$\begin{tikzcd} P_n = (KG)^{\otimes_K (n+1)} \arrow[r] & \cdots \arrow[r] & P_0 = KG \arrow[r] \arrow[d, "\varepsilon"] & 0 \\ & 0 \arrow[r] & K \arrow[r] & 0 \end{tikzcd}$$

where P_n is free over KG with basis $G^{\times n} = \{[g_1 \mid \cdots \mid g_n] \text{ for } (g_1, \cdots, g_n) \in G^n\}$. The differential is given by $\partial_n = P_n \to P_{n-1}$ where

$$\partial_n([g_1 \mid \cdots \mid g_n]) = g_1 \cdot [g_2 + \cdots + g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1 \mid \cdots \mid g_i g_{i+1} \mid g_{i+2} \mid \cdots] + (-1)^n [g_1 \mid \cdots \mid g_{n-1}]$$

and $\varepsilon([\cdot]) = 1$, i.e. $\varepsilon : KG = P_0 \twoheadrightarrow K$ is the augmentation given by $\sum\limits_{g \in G} a_g \cdot g \mapsto \sum\limits_{g \in G} a_g$.

We have the trivial functor $\mathbf{triv} : \mathbf{K}\text{-}\mathbf{Mod} \to \mathbf{KG}\text{-}\mathbf{Mod}$ (restriction of scalars along $KG \to K$ and $g \mapsto 1$, sending V to $V_{\mathbf{triv}}$ by taking $g \cdot v = v$ for all $g \in G$.

We have adjoints

$$\begin{tikzcd} \mathbf{KG}\text{-}\mathbf{Mod} \arrow[d, bend right=60, "(-)_G"'] \arrow[d, bend left=60, "(-)^G"] \\ \mathbf{K}\text{-}\mathbf{Mod} \arrow[u, "\mathbf{triv}"] \end{tikzcd}$$

where $(-)^G$ is left exact and $(-)_G$ is right exact. Moreover, we have G-coinvariant $M_G = K \otimes_{KG} M \cong M / \{gm - m \mid g \in G, m \in M\} \twoheadleftarrow M$, i.e. acting on the trivial action $[g \cdot m] = [m]$, as well as G-invariant $M^G = \mathbf{Hom}_{KG}(K, M) \cong \{m \in M \mid g \cdot m = m \; \forall g \in G\} \rightarrowtail M$.

Remark 2.9.1 (Maschke's Theorem)**.** Let K be a field and G be a finite group, KG is semisimple, i.e. **KG-Mod** is semisimple, if and only if $|G| \in K^\times$, i.e. is invertible. More precisely, for K a commutative ring, the trivial KG-module K is projective if and only if $|G| \in K^\times$.

Exercise 2.9.2.
1. Study the surjective map $\varepsilon : KG \twoheadrightarrow K$ from KG free.
2. Take note of the averaging trick: if $|G| \in K$, $f : M \to N$ is K-linear, then $\frac{1}{|G|} \sum\limits_{g \in G} g \cdot f(g^{-1}(-))$ is K-linear.

Example 2.9.3. If p is prime, let $K = \mathbb{F}_p$ and $G = C_p$, i.e. cyclic group of order p, then $KG = KC_p = K[x]/(x^p - 1) \cong K[t]/t^p$ for $t = x - 1$, as $(x-1)^p = x^p - 1$ in the setting of characteristic p.

Every finitely-generated $\mathbb{F}_p C_p$-module is a direct sum of $M_1, \cdots, M_p$ where $M_i = K[t]/t^i$ for $1 \leqslant i \leqslant p$ and $M_p = KG$ is free. But the other ones are non-projective (and injective), hence gives a lot of non-split exact sequences (to feed $(-)^G$ and $(-)_G$) like

$$M_{p-i} \rightarrowtail M_p \twoheadrightarrow M_i$$

for instance, $1 \leqslant i \leqslant p-1$.

Definition 2.9.4 (Group Homology/Cohomology)**.** The homology of G with coefficients in a KG-module M is the left derived functor of $(-)_G$ evaluated at M. We write $H_i(G, M) = L_i((-)_G)(M)$ for $i \in \mathbb{N}$.

The cohomology of G with coefficients in a KG-module M is the right derived functor of $(-)^G$ evaluated at M. We write $H^i(G, M) = R^i((-)^G)(M)$ for $i \in \mathbb{N}$.

Note that K is missing from the notation.

Proposition 2.9.5. We have natural isomorphisms

$$H_i(G, M) \cong \mathbf{Tor}_i^{KG}(K, M)$$

and

$$H^i(G, M) \cong \mathbf{Ext}^i_{KG}(K, M)$$

for all $i \in \mathbb{N}$.

Proof. Note $(-)_G = K \otimes_{KG} -$ and $(-)^G = \mathbf{Hom}_{KG}(K, -)$. $\square$

Corollary 2.9.6. Let $\cdots \to P_2 \to P_1 \to P_0 \to K \to 0$ be a projective resolution over KG. Let M be any KG-module. Then for every $i \in \mathbb{N}$, $H_i(G, M \cong H_i(P. \otimes_{KG} M)$ and $H^i(G, M) \cong H^i(\mathbf{Hom}_{KG}(P., M))$.

Remark 2.9.7. We can use the explicit resolution mentioned above and use $\mathbf{Hom}_{KG}(KG^r, M) \cong M^r$ and $KG^r \otimes_{KG} M = M^r$.

For instance, if we let $C^n(G, M)$ be the set of functions $f : G^n \to M$, we can get a complex

$$\cdots \xrightarrow{\partial_{n-1}} C^n(G, M) \xrightarrow{\partial_n = \mathbf{Hom}_{KG}(\partial_{n+1}, M)} C^{n+1}(G, M) \to \cdots$$

that sends $f \mapsto \partial_n f$, where

$$\partial_n f(g_0, \cdots, g_n) = g_0 f(g_1, \cdots, g_n) + \sum_{i=1}^{n} (-1)^i f(g_0, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_n) + (-1)^{n+1} f(g_0, \cdots, g_{n-1})$$

Example 2.9.8.

$$H^1(G, M) = \frac{\{f : G \to M \mid f(g_1, g_2) = f(g_1) + g_1 \cdot f(g_2)\}}{\{g \mapsto g \cdot m - m \mid m \in M\}},$$

i.e. the set of crossed homomorphisms quotient by the set of principal crossed homomorphisms. Moreover, we have $H^0(G, M) = M^G$. In particular, if M has trivial G-action, then $H^1(G, M) \cong \mathbf{Hom}(G, M)$.

Problem 7 (Exam Problem 7: Description of $H^2(G, M)$)**.** Let $K = \mathbb{Z}$ and M be an Abelian group with G-action described as at the end of the problem (and so comes with a G-module structure). The elements of $H^2(G, M)$ are in one-to-one correspondence with isomorphism classes (i.e. act as the identity map on M and G) of extensions of groups

$$\begin{tikzcd} M \arrow[r, tail] \arrow[d, equal] & E \arrow[r, two heads, "\pi"] \arrow[d, "f", "\cong"'] & G \arrow[d, equal] \\ M \arrow[r, tail] & E' \arrow[r, two heads, "\pi'"'] & G \end{tikzcd}$$

Choose a set-theoretic section of π, $\sigma : G \to E$ such that $\pi \circ \sigma = \mathbf{id}$. Define $\theta \in C^2(G, M) = M^{G \times G}$ by $\theta([g \mid h]) = \sigma(g) \cdot \sigma(h) \cdot \sigma(gh)^{-1}$ in M (since θ acts on the kernel of π).

Conversely, if $\theta \in C^2(G, M)$ is a cocycle, ($d^2(\theta) = 0$), then define

$$M \rightarrowtail E_\theta \twoheadrightarrow G$$

where $E_\theta = M \times G$ with multiplication $(m, g) * (n, h) = (m + g \cdot n + \theta(g, h), g \cdot h)$.

The G-action defined on M is given by: for an element $g \in G$, it is associated to $m \in M$ by the action ${}^g m = x m x^{-1}$ where x is a lift such that $\pi(x) = g$.

Remark 2.9.9. Let K be a commutative ring and G be a group. Let M be a KG-module. Recall that we have K-modules $H_i(G,M) = \mathbf{Tor}_i^{KG}(K,M)$, i.e. derive functor of $(-)_G = K \otimes_{KG} -$, and $H^i(G,M) \cong \mathbf{Ext}^i_{KG}(K,M)$, i.e. derive functor (or fixed point functor) $(-)^G = \mathbf{Hom}_{KG}(K,-)$.

The key is that there is enough to projectively resolve K over KG.

Suppose $K \to L$ is a ring homomorphism, and M is an LG-module, then the $H_i(G,M)$ structure over K and over L are the same; the same result holds for $H^i(G,M)$.

Proposition 2.9.10. If $\mathbf{Res}_f(M) \in \mathbf{KG\text{-}Mod}$ is the KG-module M with K acting via f, then there is a canonical isomorphism

$$H_i(G, \mathbf{Res}_f(M)) \cong \mathbf{Res}_f(H_i(G,M))$$

and

$$H^i(G, \mathbf{Res}_f(M)) \cong \mathbf{Res}_f(H^i(G,M)),$$

where the left side structures are over K and the right side structures are over L.

Proof. Let $P. \to K$ be a projective resolution as a KG-module, e.g. $P_i = (KG)^{G^i} = KG^{\otimes_L (i+1)}$. Note that $L \otimes_K P_i$ is a projective resolution of L as LG-module, because $L \otimes_K KG \cong LG$ for KG free. Then $L \otimes_K P. \to L$ is still exact because we cant test without G-action and then we are talking about split exact complexes. We may compute the following over K:

$$\begin{aligned} H^i(G, \mathbf{Res}_f(M)) &= \mathbf{Ext}^i_{KG}(K, \mathbf{Res}_f(M)) \\ &= H^i(\mathbf{Hom}_{KG}(P., \mathbf{Hom}_L({}_L L_K, M))) \\ &= H^i(\mathbf{Hom}_{LG}(L \otimes_K P., M)) \\ &= \mathbf{Ext}^i_{LG}(L, M), \end{aligned}$$

here $L \otimes_K P.$ is a projective resolution of L as a LG-module. □

Corollary 2.9.11. If G is a finite group and M is a KG-module on which $|G|$ is invertible, i.e. $|G| : M \xrightarrow{\sim} M$, then

$$H_i(G,M) = H^i(G,M) = 0$$

for all $i > 0$.

Proof. Let $L = K[\frac{1}{|G|}]$. Note that $M = \mathbf{Res}(M^1)$ where $f : K \to L$ and $M' = M$ with L-action. On L, G is invertible, so L is a projective LG-module, so $H^*(G,M') = H_*(G,M') = 0$ for all $i > 0$. Then we apply the proposition. □

Example 2.9.12. Let $p = 2$ and let $G = C_2 = \langle x \rangle$ and K be a field of characteristic 2. We can find the projective resolution of K via $KC_2 = K[t]/t^2$:

$$\begin{tikzcd} \cdots \arrow[r] & KC_2 \arrow[r, "1+x"] & KC_2 \arrow[r, "1+x"] & KC_2 \arrow[r, "1+x"] \arrow[d, two heads, "\xi"'] & KC_2 \arrow[r, "\xi"] & K \arrow[r] & 0 \\ & & & K \arrow[ur, tail] & & & \end{tikzcd}$$

where we send $K \rightarrowtail KC_2$ via $1 \mapsto 1 + x$.

Exercise 2.9.13. $H^i(C_2, K) = K$ for all $i \geqslant 0$.

Remark 2.9.14. As a ring, $\mathbf{Ext}^*_{KG}(K,K) = H^*(C_2,K) = K[\xi]$ for ξ in degree 1, and ξ in $\mathbf{Ext}^1_{KC_2}(K,K)$ is $K \rightarrowtail KC_2 \twoheadrightarrow K$. However, $H^*(C_p,K) = K[\xi,\sigma]/\langle \sigma^2 \rangle$ (i.e. $\sigma^2 = 0$) for K with characteristic $p > 2$, ξ with degree 2 and σ with degree 1.

So $H^i(C_p,K) = K$ for all $i \geqslant 0$. Hence, we have $\sigma \cdot \xi^n$ in odd degrees, and ξ^n in even degrees.

Finally, there is no homology for characteristic 0.

2.10 Sheaf Cohomology

For this section, X is a topological space.

Consider (pre)sheaves of Abelian groups (or a Grothendick category). Recall that $\mathbf{Shv}(X)$ has enough injectives. For $F \in \mathbf{Shv}(X)$, we can embed

$$F \hookrightarrow \prod_{x \in X} (i_*)_* I(F_X)$$

where $I(F_X)$ is the injective pre-envelope. (For example, we can take $\prod \mathbb{Q}/\mathbb{Z}$ and get $\mathbf{Hom}(F_X, \mathbb{Q}/\mathbb{Z})$.) Moreover, we have

$$\begin{aligned} i_X : * &\xrightarrow{X} X \\ * &\mapsto X \end{aligned}$$

and $*((i_X)_*(E))(U) = \begin{cases} E, & \text{if } x \in U \\ 0, & \text{if } x \notin U \end{cases}$ for all $U \subseteq X$ open.

Definition 2.10.1 (Cohomology Group)**.** The right derived functors of $\Gamma : \mathbf{Shv}(X) \to \mathcal{A}$ that sends $F \mapsto F(X)$ are called the cohomology groups $H^i(X, F) = (R^i(\Gamma(X, -)))(F)$. In cash, we have $F \to I^\cdot$ injective resolution in $\mathbf{Shv}(X)$, then $H^i(X, F) = H^i(I^\cdot(X))$.

Note that if $E \rightarrowtail F \twoheadrightarrow G$ is a short exact sequence in $\mathbf{Shv}(X)$ then we have a long exact sequence

$$0 \to E(X) \to F(X) \to G(X) \to H^1(X, E) \to H^1(X, F) \to H^1(X, G) \to H^{i+1}(X, E) \to \cdots$$

Definition 2.10.2 (Flasque)**.** A sheaf is called flasque or flabby if every pair open $V \subseteq U$ in X, the restriction map $F(U) \to F(V)$ is surjective.

Proposition 2.10.3. (a) Injective sheaves are flasque.

(b) If $E \rightarrowtail F \twoheadrightarrow G$ is a short exact sequence in $\mathbf{Shv}(X)$ with E flasque, then the sequence $E(X) \rightarrowtail F(X) \twoheadrightarrow G(X)$ is exact.

(c) If $E' \rightarrowtail E \twoheadrightarrow F$ is a short exact sequence where E, E' are flasque, then F is flasque.

(d) Every flasque sheaf E is right $\Gamma(X, -)$-acyclic: $H^i(X, E) = 0$ for all $i > 0$.

(e) For every sheaf F, $E_F = \prod_{x \in X} (i_x)_*(F_X)$ (without $I(\cdots)$) is flasque, and $F \to E_F$ is a monomorphism that sends $F(U) \mapsto E_F(U) = \prod_{x \in U} F_X$ and $s \mapsto (s_x)_{x \in U}$. So every sheaf embeds into a flasque, then we have enough flasque. Hence, we can compute the cohomology group using flasque resolutions.

Proof. (a) Take $\mathbb{Z}_U = (j_U)_! \underline{\mathbb{Z}}$ for all $U \subseteq X$ open. This is the sheafification of the presheaf

$$\mathbb{Z}_U^{\mathbf{Pre}} : X \supseteq W \mapsto \begin{cases} \mathbb{Z}, & \text{if } W \subseteq U \\ 0, & \text{otherwise} \end{cases}$$

For $V \subseteq U$, we get $\mathbb{Z}_V \hookrightarrow \mathbb{Z}_U$. Also, $\mathbf{Hom}_{\mathbf{Shv}}(\mathbb{Z}_U, F) \cong \mathbf{Hom}_{\mathbf{Pre}}(\mathbb{Z}_U^{\mathbf{Pre}}, F) \cong F(U)$. We write, for I injective, the extension property against $\mathbb{Z}_V \hookrightarrow \mathbb{Z}_U$, gives $I(U) \to I(V)$ is onto.

(b) Exercise on sheaves: extend local lifts by correcting the "error" (on pairwise intersection) which lives in E, but can be extended (to both open).

(c) For $V \subseteq V$, we have

$$\begin{tikzcd} E'(U) \arrow[r, tail] \arrow[d, two heads, "\mathbf{Res}_{\mathbf{Shv}}"'] & E(U) \arrow[r, two heads] \arrow[d, two heads, "\mathbf{Res}_{\mathbf{Shv}}"'] & F(U) \arrow[d] \\ E'(V) \arrow[r, tail] & E(V) \arrow[r, two heads] & F(V) \end{tikzcd}$$

Then by the snake lemma, $F(U) \twoheadrightarrow F(V)$ is an onto map.

(d) If E is flasque, we put it into injectives $E \rightarrowtail I \twoheadrightarrow F$, where F is the cokernel and flasque. Then on the cohomology we have
$$I(X) \twoheadrightarrow F(X) \to H^1(X,E) = 0 \to H^1(X,I) = 0 \to 0 \to \cdots$$
where the first map is onto by part (b). We then do induction on i, using F flasque.

(e) $E_F(U)$ is flasque by construction, so the projection is surjective. It is injective because we can test everything stalkwise.

□

2.11 Yoneda Ext Group

The goal of this section is to give a description of the Abelian groups $\mathbf{Ext}^n_{\mathcal{A}}(A,B)$ for $n \geqslant 1$ and an Abelian category $\mathcal{A}$ (with enough projectives/or not). We therefore have

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

where A is at degree 0 and B is at degree n.

We consider exact sequences of length n:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & E_{n-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] \arrow[d] & A \arrow[r] & 0 \\ & & & & \cdots \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

with equivalence relation on those, generated by of such sequences that are identity on A and B:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Note that not all maps are necessarily isomorphisms. Then extensions with E's are equivalent ($\sim$) with extensions with F's (two steps will do).

For $n = 1$, these are the isomorphisms of extensions

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\sim"] & A \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\ 0 \arrow[r] & B \arrow[r] & F \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

For $n \geqslant 1$, there is the split exact seuqence

$$0 \to B \xrightarrow{\mathbf{id}} B \to 0 \to \cdots \to 0 \to A \xrightarrow{\mathbf{id}} A \to 0$$

and this becomes $0 \to B \xrightarrow{\binom{1}{0}} B \oplus A \to A \to 0$ when $n = 1$. This will correspond to 0 in the group $\mathbf{Ext}^n(A,B)$.

Definition 2.11.1 (Yoneda Ext Group)**.** The Yoneda Ext group $\mathbf{Ext}^n(A,B)$ is the set of equivalence classes of extensions of A by B, of length n.

Remark 2.11.2. It is not very clear how to do addition on Yoneda Ext groups. What about $\oplus$? For example, can we get $0 \to B \oplus B \to E. \oplus F. \to \cdots \to A \oplus A \to 0$?

Remark 2.11.3 (Functoriality)**.** The Yoneda Ext group has functoriality. Given $f : A \to A'$ and an extension $f^*(E.) \mapsto E.$, we have

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[dd, equal] & E_{n-1} \arrow[r] \arrow[dd, equal] & \cdots \arrow[r] & E_1 \arrow[rr] \arrow[dd, equal] \arrow[dr, two heads] & & E_0 \arrow[rr] & & A' \arrow[r] & 0 \\
& & & & & \cdot \arrow[ur, tail] \arrow[dr, tail] & & & & \\
0 \arrow[r] & B \arrow[r] & E_{n-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[rr] \arrow[ur, two heads] & & E_0' \arrow[rr] \arrow[uu] \arrow[uurr, phantom, "\urcorner"] & & A \arrow[r] \arrow[uu, "f"'] & 0
\end{tikzcd}$$

and E_0' acts as a pullback to A'. This is a well-defined map $\mathbf{Ext}^n(A', B) \xrightarrow{f^*} \mathbf{Ext}^n(A, B)$. Similarly, for $g : B \to B'$ and an extension $E. \mapsto g_*(E.)$, we have

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[rr] \arrow[dd, "g"'] \arrow[ddrr, phantom, "\ulcorner"] & & E_{n-1} \arrow[rr] \arrow[dd] \arrow[dr, two heads] & & E_{n-2} \arrow[r] \arrow[dd, equal] & \cdots \arrow[r] & E_0 \arrow[r] \arrow[dd, equal] & A \arrow[r] \arrow[dd, equal] & 0 \\
& & & & \cdot \arrow[ur, tail] \arrow[dr, tail] & & & & & \\
0 \arrow[r] & B' \arrow[rr] & & E_{n-1}' \arrow[rr] \arrow[ur, two heads] & & E_{n-2} \arrow[r] & \cdots \arrow[r] & E_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

and that yields a well-defined map $\mathbf{Ext}^n(A, B) \xrightarrow{g_*} \mathbf{Ext}^n(A, B')$. Using that, we can recover the addition on $\mathbf{Ext}^n(A, B)$, where $\oplus$ gives

$$\begin{tikzcd}
\mathbf{Ext}^n(A, B) \otimes \mathbf{Ext}^n(A, B) \arrow[r, "\oplus"] \arrow[dr, "+"'] & \mathbf{Ext}^n(A \oplus A, B \oplus B) \arrow[d, "(\Delta^*, \Delta_*)"] \\
& \mathbf{Ext}^n(A, B)
\end{tikzcd}$$

and then use $A \xrightarrow{\Delta = \binom{1}{1}} A \oplus A$ as well as $B \oplus B \xrightarrow{\nabla = (1\ 1)} B$.

Exercise 2.11.4. Unpack this argument with R-modules.

Proposition 2.11.5. There is a canonical isomorphism of Abelian group between $\mathbf{Ext}^n_{\mathcal{A}}(A, B)$ and Yoneda's extension group $\mathbf{Ext}^n_{\mathcal{A}}(A, B)$ above, say, when $\mathcal{A}$ has enough projectives or enough injectives.

Problem 8 (Exam Problem 8)**.** Prove the proposition.

Remark 2.11.6. We do the version with enough projectives here. Suppose $\mathcal{A}$ has enough projectives. Pick the projective resolution of A

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0,$$

then we have

$$\begin{aligned}
\mathbf{Ext}^n_{\mathcal{A}}(A, B) &= H(\cdots \to \mathbf{Hom}(P_{n-1}, B) \xrightarrow{-\circ d} \mathbf{Hom}(P_n, B) \xrightarrow{-\circ d_{n+1}} \mathbf{Hom}(P_{n+1}, B) \to \cdots) \\
&= \frac{\{f : P_n \to B \mid f \circ d = 0 : P_{n+1} \to B\}}{\{f' \circ d \mid f' : P_{n-1} \to B\}}
\end{aligned}$$

We now construct the two mappings. First, given $[f]$ for $f : P_n \to B$ such that $f \circ d_{n+1} = 0$, we have the following map that is exact at P_n and at B, where the row above is the projective resolution, and the row below is in $\mathbf{Ext}^n(A, B)$:

$$\begin{tikzcd}
P_{n+1} \arrow[r, "d_{n+1}"] \arrow[ddr, "0"'] & P_n \arrow[rr, "d_n"] \arrow[dd, "f"] \arrow[dr, two heads] & & P_{n-1} \arrow[rr] \arrow[dd] \arrow[dr, two heads] & & P_{n-2} \arrow[r] \arrow[dd, equal] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[dd, equal] & A \arrow[r] \arrow[dd, equal] & 0 \\
& & C \arrow[ur, tail] \arrow[dl, dashed, "\exists \bar{f}"] & & \cdot \arrow[ur, tail] \arrow[dr, tail] & & & & & \\
0 \arrow[r] & B \arrow[rr] & & E_{n-1} \arrow[rr] \arrow[ur, two heads] & & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Note that E_{n-1} pulls back from C.

Conversely, given an extension $E.$ in the Yoneda extension group $\mathbf{Ext}^n(A,B)$, we have another construction from the projective resolution to the extension that forms an exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "d"] \arrow[d] & P_n \arrow[r] \arrow[d, "\exists f"] \arrow[dl, dashed, "\varepsilon"] & P_{n-1} \arrow[r] \arrow[d] \arrow[dl, dashed, "\varepsilon"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\exists f"] & P_0 \arrow[r] \arrow[d, "\exists f"] & A \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & B \arrow[r] & E_{n-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

where $\varepsilon = f' \circ d$ as denoted before. Now there exists $f. : P. \to$ "$E.$" with $E_n = B$, lifting the identity $\mathbf{id}_A$ and is unique up to homotopy. This gives $f = f_n : P_n \to B$ such that $f \circ d = 0$.

If we attempt to change f up to homotopy, note that the change is killed by the quotient.

Remark 2.11.7 (Motivation for $\mathbf{D}(\mathcal{A})$, the derived category of $\mathcal{A}$)**.** What happens if we cannot get interesting complex maps? We can always construct quasi-isomorphisms between sequences such that they form a complex:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & B \arrow[r] \arrow[u] \arrow[d, equal] & E_{n-1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[u] \arrow[d] & E_0 \arrow[r] \arrow[u] \arrow[d] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & B \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So $\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})[\{\text{ quasi-isomorphisms }\}^{-1}]$, by Grothendieck. This gives an isomorphism $\mathbf{Hom}_{D(\mathcal{A})}(A, B[n]) \cong \mathbf{Ext}^n(A,B)$, where $B[n]$ indicates B shifted. Note that there is no need to require enough injectives/surjectives.

Remark 2.11.8. The computation requires a technique called "splicing" upon Yoneda product/composition: $\mathbf{Ext}^m(A,B) \times \mathbf{Ext}^n(B,C) \to \mathbf{Ext}^{n+m}(A,C)$, i.e. we may obtain a sequence that looks like

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] \arrow[d, two heads] & E_{m-1} \arrow[r] & \cdots \arrow[r] & E_0 \arrow[r] & A \arrow[r] & 0 \\
& & & & B \arrow[ur, tail] & & & & &
\end{tikzcd}$$

For example, let K be a field of characteristic $p > 0$. Set $G = C_p = \langle x \mid x^p = 1 \rangle$ cyclic. We define $H^*(G,K) = \mathbf{Ext}^*_{KG}(K,K)$, i.e. acting as the trivial G-action. We now can identify group algebra $KG = KC_p = K[x]/(x^p - 1) \cong K[t]/t^p$ for $t = x - 1$. Therefore, there are basic indecomposible KC_p-modules (i.e. $R = K[t]/t^p$) or R-modules. We denote $[i] = K[t]/t^i$ for $1 \leqslant i \leqslant p$, i.e. $[1] = k, \cdots, [p] = R$. Now the projective resolution of K (note that it is 2-periodic for odd p) is of the form

$$\cdots \xrightarrow{t} R \xrightarrow{t^{p-1}} R \xrightarrow{t} R \xrightarrow{t^{p-1}} R \xrightarrow{t} R \xrightarrow[\varepsilon]{1} K \longrightarrow 0$$

where ε is the augmentation. From that, it is easy to see that $\mathbf{Ext}^n_{KC_p}(K,K) \cong K$ for all $n \geqslant 0$ via the generator $\varepsilon : R \to K$ for R at degree n. We can now trace in terms of extensions, by considering the generator of $\mathbf{Ext}^n_R(K,K)$ in degree i. For example, in degree 1, we have ε_1:

$$0 \longrightarrow [1] \xrightarrow{t} [2] \longrightarrow [1] \longrightarrow 0$$

and for degree 2, we have ε_2:

$$\begin{tikzcd}
0 \arrow[r] & {[1]} \arrow[r, "t"] & {[p]} \arrow[r] \arrow[d, two heads] & {[p]} \arrow[r] & {[1]} \arrow[r] & 0 \\
& & {[p-1]} \arrow[ur, tail] & & &
\end{tikzcd}$$

Observe that $[p] = [2]$ in degree 2! In general, we see that $\varepsilon_2 = \varepsilon_1 \circ \varepsilon_1$ with the composition as the Yoneda composition: the number is just the dimension!

Now, in degree n, for $p = 2$, the generator is ε_1^n. In particular, we have $H^*(C_2, K) = K[\xi]$ as a ring for ξ in degree 1; for p odd, in degree $2n$, we have ξ_2^n as a generator:

$$0 \longrightarrow [1] \longrightarrow [p] \xrightarrow{t^{p-1}} [p] \xrightarrow{t} [p] \xrightarrow{t^{p-1}} \cdots \longrightarrow [p] \xrightarrow{t^{p-1}} [p] \xrightarrow{t} [1] \longrightarrow 0$$

in odd degrees, $\xi_1\xi_2^n = (-1)^{1\cdot 2n}\xi_2^n\xi_1 = \xi_2^n\xi_1$ is the generator. Note $\xi_1^2 = 0$ for odd p. Therefore, we have

$$\begin{tikzcd}
 & & {[1]} \arrow[rd, tail] & & & \\
0 \arrow[r] & {[1]} \arrow[r] \arrow[d, equal] & {[2]} \arrow[r, "t"] \arrow[u, two heads] \arrow[d, "t^{p-2}"] & {[2]} \arrow[r, "1"] \arrow[d, "\binom{t^{p-3}}{1}"] & {[1]} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & {[1]} \arrow[r, "t^{p-1}"] & {[p]} \arrow[r, "\binom{1}{0}"] & {[p-1] \oplus [1]} \arrow[r, "(0\ 1)"] & {[1]} \arrow[r] & 0 \\
0 \arrow[r] & {[1]} \arrow[r, "1"'] \arrow[u, equal] & {[1]} \arrow[r, "0"'] \arrow[u, "t^{p-1}"] & {[1]} \arrow[r] \arrow[u, "\binom{0}{1}"] & {[1]} \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Note that the last row is the split extension in $\mathbf{Ext}^2$, and the previous rows are not split. In particular, we have

$$H^*(C_p, K) = K[\xi, \eta]/\left\langle \xi^2 \right\rangle$$

as commutative graded ring ($(-1)^{|a|\cdot|b|}a \cdot b = b \cdot a$), where $\xi = \xi_1$ in degree 1 and $\eta = \xi_2$ in degree 2.

3 Spectral Sequences

3.1 Introduction

The idea is to put many long exact sequences together.

Definition 3.1.1 (Homologically-indexed Spectral Sequence)**.** A homologically-indexed spectral sequence in an Abelian category $\mathcal{A}$ (e.g. the category of R-modules) is a collection of objects $E_{p,q}^r$ for $(p, q) \in \mathbb{Z}^2$ and page $r \geqslant r_0$, usually $r_0 = 0, 1$ or 2 (and is called the starting page), along with differentials $d_{p,q}^r : E_{p,q}^r \to E_{p-r,q+r-1}^r$ within a system of the form

$$\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
\cdots & E_{0,1}^r & E_{1,1}^r & E_{2,1}^r & \cdots \\
\cdots & E_{0,0}^r & E_{1,0}^r & E_{2,0}^r & \cdots \\
 & \vdots & \vdots & \vdots &
\end{array}$$

such that $d^r \circ d^r = 0$ and an isomorphism

$$E_{p+q}^{r+1} \cong \mathbf{Homology}(E_{p+r,q-r-1}^r \xrightarrow{d^r} E_{p,q}^r \xrightarrow{d^r} E_{p-r,q+r-1}^r)$$

for all $r \geqslant r_0$ and all $(p, q) \in \mathbb{Z}^2$. In particular, E_{p+q}^{r+1} is a subquotient of $E_{p,q}^r$.

Remark 3.1.2. In particular, every entry $E_{p,q}^r$ is a subquotient of $E_{p,q}^{r_0}$ where r_0 is the starting page. Therefore, we have

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots & E^1_{0,1} \arrow[l, "d^1"'] & E^1_{1,1} \arrow[l, "d^1"'] & E^1_{2,1} \arrow[l, "d^1"'] & \cdots \arrow[l, "d^1"'] \\
\cdots & E^1_{0,0} \arrow[l, "d^1"'] & E^1_{1,0} \arrow[l, "d^1"'] & E^1_{2,0} \arrow[l, "d^1"'] & \cdots \arrow[l, "d^1"'] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

as page 1 and

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots & E^2_{0,1} & E^2_{1,1} & E^2_{2,1} & \cdots \\
\cdots & E^2_{0,0} & E^2_{1,0} & E^2_{2,0} \arrow[ull, "d^2"] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

as page 2.

Definition 3.1.3 (Cohomologically-indexed Spectral Sequence)**.** A cohomologically-indexed spectral sequence in an Abelian category is a collection of objects and differentials

$$(E_r^{p,q}, d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

for $r \geqslant r_0$, $(p, q) \in \mathbb{Z}^2$ and $E_{r+1}^{p,q} \cong H(\cdot \xrightarrow{d} E_r^{p,q} \xrightarrow{d} \cdots)$.

Remark 3.1.4. A common misconception is that the data of the r-th page does not describe the $(r+1)$-th page. Note that only objects are, not d^{r+1}.

Remark 3.1.5. Note that we can also write $E^r_{\cdot,\cdot} = Z^r_{\cdot,\cdot}/B^r_{\cdot,\cdot}$ where we have the sequence

$$0 \subseteq B^{r_0+1} \subseteq \cdots \subseteq B^r \subseteq B^{r+1} \subseteq \cdots \subseteq Z^{r+1} \subseteq Z^r \subseteq \cdots \subseteq Z^{r_0+1} \subseteq E^{r_0}_{\cdot,\cdot}.$$

Here the Z's are called the cycles and converges leftwards to a limit, and the B's are called boundaries and converges rightwards to a colimit. By defining $B^\infty_{p,q} = \bigcup_{r\geqslant r_0} B^r_{p,q}$ and $Z^\infty_{p,q} = \bigcap_{r\geqslant r_0} Z^r_{p,q}$, we may define $E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}$.

Very often, (as an important condition), this stops for given $(p, q) \in \mathbb{Z}^2$ after some large enough $r >> r_0$. Typically, we should have all $d^r = 0$ for differentials into and out of $E^r_{p,q}$. This is related to vanishing in E^1, E^2, etc. For example, we can have a page r_0 where $E_{p,q} = 0$ for all p, q such that $p < 0$ or $q < 0$.

Definition 3.1.6 (Convergence)**.** For a given page $r = r_0 = 0, 1$ or 2, we say the spectral sequence $E^r_{p,q}$ converges weakly to H_n along $p + q = n$ if

1. there is a spectral sequence $E^{\cdot}_{\cdot,\cdot}, d^{\cdot})$ for all pages, and

2. H_n admits a filtration
$$\cdots \subseteq J_{p-1,n} \subseteq J_{p,n} \subseteq J_{p+1,n} \subseteq \cdots \subseteq H_n$$
such that $H_n = \bigcup_p J_{p,n}$ and isomorphism
$$J_{p,n}/J_{p-1,n} \cong E^{\infty}_{p,n-p}$$
for $p+q=n$.

In addition, we say the spectral sequence $E^r_{p,q}$ converges to H_n along total degree $p+q=n$ if

1. the spectral sequence converges to H_n weakly, and
2. $\bigcap_p J_{p,n} = 0$, i.e. we have a Hausdorff filtration.

Remark 3.1.7. Knowing an object H, like above, via a filtration
$$\cdots \subseteq J_p \subseteq J_{p+1} \subseteq \cdots$$
in order to define $H = \bigcup_p J+p$ and even separated ($\bigcap_p J_p = 0$) can be rather void: we can consider the sequence
$$\cdots \subseteq 2^n\mathbb{Z} \subseteq \cdots \subseteq 4\mathbb{Z} \subseteq 2\mathbb{Z}/ \subseteq \mathbb{Z} = \mathbb{Z} = \cdots = \mathbb{Z},$$
where we have $\bigcap_n 2^n\mathbb{Z} = 0$ but $2^n\mathbb{Z}/2^{n+1}\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$.

Remark 3.1.8. A very common and useful condition is to assume $E^r_{p,q}$ bounded below: for every total degree n, there exists $p_0 = p_0(n)$ such that $E^s_{p,n-p} = 0$ for all $p \leqslant p_0(n)$, i.e. $E^r_{p,n-p} = 0$ for all $r \geqslant s$.

The question is, how to build a spectral sequence?

Definition 3.1.9 (Exact Couple)**.** An exact couple is an exact sequence

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

Figure 8: Exact Couple

Then the differentiation is $d := \beta \circ \gamma : E \to D \to E$ as $d^2 = \beta\gamma\beta\gamma = 0$.
A derived exact couple is

$$\begin{tikzcd} D' \arrow[rr, "\alpha'"] & & D' \arrow[dl, "\beta'"] \\ & E' \arrow[ul, "\gamma'"] & \end{tikzcd}$$

Figure 9: Derived Exact Couple

where $E' = H(E \xrightarrow{d} E \xrightarrow{d} E) = \ker(\beta\gamma)/\operatorname{im}(\beta\gamma)$ and $D' = \operatorname{im}(\alpha) \subseteq D$. We can then denote $\alpha' = \alpha \mid^{D'}_{D'}$, β' as a map on elements that sends $\alpha(d) \mapsto [\beta(d)] \in E'$, where $\alpha(d) \in D' = \operatorname{im}(\alpha)$ and $[\beta(d)]$ as an equivalence class in E', and finally γ' is induced by γ with elementwise mapping that sends $[e] \mapsto \gamma(e)$ for $e \in \ker(d) \subseteq E$. This is derived from the exact couple as

$$\begin{tikzcd} & D' \arrow[r, hook] & D \arrow[r, "\alpha"] & D \arrow[dl, "\beta"] \\ E' \arrow[ur, dashed, "\gamma'"] & \ker(\beta\gamma) \arrow[l, two heads] \arrow[u, dashed] \arrow[r, hook] & E \arrow[u, "\gamma"] & \end{tikzcd}$$

Problem 9 (Massey, Exam Problem 9)**.** Prove that the derived exact couple

$$\begin{tikzcd} D' \arrow[rr, "\alpha'"] & & D' \arrow[dl, "\beta'"] \\ & E' \arrow[ul, "\gamma'"] & \end{tikzcd}$$

is well-defined and exact.

Remark 3.1.10. Note that we can do this repeatedly: given

$$\begin{tikzcd} D^1 \arrow[rr, "\alpha^1"] & & D^1 \arrow[dl, "\beta^1"] \\ & E^1 \arrow[ul, "\gamma^1"] \arrow[loop left, "d^1=\beta^2\circ\gamma^1"] & \end{tikzcd}$$

we can derive it for $r-1$ times and obtain

$$\begin{tikzcd} D^r \arrow[rr, "\alpha^r"] & & D^r \arrow[dl, "\beta^r"] \\ & E^r \arrow[ul, "\gamma^r"] \arrow[loop left, "d^r=\beta^r\circ\gamma^r"] & \end{tikzcd}$$

where (E^r, d^r) then forms a spectral sequence.

Remark 3.1.11 (Bigrading)**.** Suppose we have an exact couple with bidegrees

$$\begin{tikzcd} D^r \arrow[rr, "(1,-1)", "\alpha^r"'] & & D^r \arrow[dl, "\beta^r"', "(-r+1,r-1)"] \\ & E^r \arrow[ul, "\gamma^r"', "(-1,0)"] \arrow[loop left, "d^r", "(-r,r+1)"'] & \end{tikzcd}$$

then we can derive

$$\begin{tikzcd} D^{r+1} \arrow[rr, "(1,1)", "\alpha^{r+1}"'] & & D^{r+1} \arrow[dl, "\beta^{r+1}"', "(-r,r)"] \\ & E^{r+1} \arrow[ul, "\gamma^{r+1}"', "(-1,0)"] & \end{tikzcd}$$

via bigrading.

Theorem 3.1.12. Let $(D^{r_0}_{p,q}, E^{r_0}_{p,q}, \alpha, \beta, \gamma)$ be a homologically-index exact couple as above with $r_0 = 0, 1$ or 2, and α, β, γ have bidgrees $(1,-1)$, $(-r_0+1, r_0-1)$ and $(-1,0)$, respectively. We thereby obtain a sequence of exact couples $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$ for all $r \geqslant r_0$. Moreover, we assume that $(D, E, \alpha, \beta, \gamma)$ is bounded below, that is, for all n, there exists some $p_0 = p_0(n)$ such that $D^{r_0}_{p,q} = 0$ for all $(p.q)$ such that $p < p_0(n)$. If this is the case, then the spectral sequence is bounded below and converges towards the colimit

$$H_n = co \lim_{p\to\infty} D_{p,n-p}$$

for $n = p+q$. This induces the same idea as $E^r_{p,q} \xrightarrow{p+q=n} H_n$.

3.2 Constructing Spectral Sequences

Recall a spectral sequence looks like $E^r_{p,q} \xrightarrow{d^r} E^r_{p-r,q+r-1}$ such that $E^{r+1}_{\cdot,\cdot} = \ker(d^r_{\cdot,\cdot})/\mathrm{im}(d^r_{\cdot,\cdot})$ for $(p,q) \in \mathbb{Z}^2$ and $r \geqslant r_0 = 0, 1$ or 2, and can be obtained through exact couples

$$\begin{tikzcd} D^r \arrow[rr, "(1,-1)", "\alpha^r"'] & & D^r \arrow[dl, "\beta^r"', "(-r+1,r-1)"] \\ & E^r \arrow[ul, "\gamma^r"', "(-1,0)"] \arrow[loop left, "d^r", "(-r,r+1)"'] & \end{tikzcd}$$

This is derived in filter complex, which then can be extened to a double complex. Note that the double complex also gives rise to the concept of Grothendieck spectral sequence and the derived functors of $G \circ F$ from those G and F.

Remark 3.2.1 (Spectral Sequence of Filtered Complex)**.** Let $\mathcal{A}$ be an Abelian category. Suppose $C.$ is a complex with a tower of subcomplexes $F_p = F_{p,\cdot} \subseteq C.$:

$$\cdots \subseteq F_{p-1,\cdot} \subseteq F_{p,\cdot} \subseteq F_{p+1,\cdot} \subseteq \cdots \subseteq C.$$

Then there is a short exact sequence of complexes $F_{p-1} \rightarrowtail F_p \twoheadrightarrow F_p/F_{p-1}$ and gives a long exact sequence in homology H_*:

$$\cdots \to H_*(F_{p-1}) \to H_*(F_p) \to H_*(F_p/F_{p-1}) \to H_{*-1}(F_{p-1})$$

This gives an exact couple

$$\begin{tikzcd}
D^1_{\cdot,\cdot} \arrow[rr, "(1{,}-1)", "\alpha^r"'] & & D^1_{\cdot,\cdot} \arrow[dl, "\beta^r"', "(0{,}0)=(r-1{,}-r+1)"] \\
& E^1_{\cdot,\cdot} \arrow[ul, "\gamma^r", "(-1{,}0)"'] \arrow[loop left, "d^r", "(-r{,}r+1)"'] &
\end{tikzcd}$$

and thereby $r = 1$ and so we have

$$\cdots \to H_{*,p+q}(F_{p-1}) = D^{r_0}_{p-1,q+1} \xrightarrow{\alpha:(1,-1)} H_{*,p+q}(F_p) = D^{r_0}_{p,q} \xrightarrow{\beta:(0,0)}$$

$$H_{*,p+q}(F_p/F_{p-1}) = E^{r_0}_{p,q} \xrightarrow{\gamma:(-1,0)} H_{*,p+q-1}(F_{p-1}) = D^{r_0}_{p-1,q}$$

Applying the last theorem from the previous section, we know the following: if the filtration ($\bigcup_p F_p = C$) $F_{p-1} \subseteq F_p \subseteq \cdots \subseteq C$ is bounded below (for all $n \in \mathbb{Z}$, there exists $p_0 = p_0(n)$ such that $F_{p,n} = 0$ for all $p < p_0$), then the spectral sequence associated to the above (bounded below) exact couple converges to $H_n(C)$:

$$E^1_{p,q} = H_{p+q}(F_p/F_{p-1}) \xrightarrow{p+q=n} H_n(C).$$

Remark 3.2.2 (Spectral Sequence of Double Complex)**.** Let $C_{\cdot,\cdot}$ be such that

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d, "d^v"'] & C_{p,q} \arrow[l, "d^h"'] \arrow[d, "d^v"] & \cdots \arrow[l] \\
\cdots & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l, "d^h"] \arrow[d] & \cdots \arrow[l] \\
& \vdots & \vdots &
\end{tikzcd}$$

such that we have $d^h \circ d^h = 0$, $d^v \circ d^v = 0$, and $d^h \circ d^v = d^v \circ d^h$. Moreover, we have the boundedness condition, i.e. for all $n \in \mathbb{Z}$, there exists $p_0 = p_0(n) \geqslant 1$ such that $C_{p,n-p} = 0$ for all $|p| > p_0$.

Therefore, the total complex is denoted $\mathbf{Tot}^{\oplus}(C_{\cdot,\cdot}) = \mathbf{Tot}^{\prod}(C_{\cdot,\cdot}) = \mathbf{Tot}^{\coprod}(C_{\cdot,\cdot}) = (\bigoplus_{p+q=n} C_{p,q})_n$ with $d = (d^n_{p,q} + (-1)^p d^v_{p,q})$.

There are two notions of filtrations on $C_{\cdot,\cdot}$, namely by considering the total degree, we can check for the filtrations of $\mathbf{Tot}(C_{\cdot,\cdot})$:

$${}^{\mathrm{I}}E^1_{p,q} = H^v_q(C_{p,\cdot}) \xrightarrow{n=p+q} H_{n=p+q}(\mathbf{Tot}(C))$$

where the homology has ${}^{\mathrm{I}}d^1 = d^h$ acting as the horizontal differential, and

$$ {}^{\mathrm{II}}E^1_{p,q} = H^h_p(C_{\cdot,q}) \xrightarrow{n=p+q} H_{n=p+q}(\mathbf{Tot}(C)) $$

where the homology has ${}^{\mathrm{II}}d^1 = d^v$ acting as the vertical differential.

Therefore, on the next page, we can do something similar:

$$ {}^{\mathrm{I}}E^2_{p,q} = H^h_p(H^v_q(C_{\cdot,\cdot})) \xrightarrow{n=p+q} H_{n=p+q}(\mathbf{Tot}(C)) $$

and

$$ {}^{\mathrm{II}}E^2_{p,q} = H^v_q(H^h_p(C_{\cdot,\cdot})) \xrightarrow{n=p+q} H_{n=p+q}(\mathbf{Tot}(C)) $$

3.3 Grothendieck Spectral Sequence

Suppose $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ are functors between Abelian categories (with enough projectives). Suppose G and F are right exact. We want to somehow relate L_*G, L_*F and $L_*(G \circ F)$. A good hypothesis is that $F(\mathbf{Proj}(\mathcal{A})) \subseteq G$-cyclics.

Remark 3.3.1 (Cartan-Eilenberg Resolutions)**.** Suppose $\mathcal{A}$ is Abelian with enough projectives. Let $C_\cdot \in \mathbf{Ch}(\mathcal{A})$ be a complex

$$
\begin{tikzcd}
 & \vdots \arrow[d] & & & \vdots \arrow[d] & & \\
 & \vdots \arrow[d] & & & P_{p+1,0} \arrow[lll, dashed, "?"] \arrow[d] & & \\
 & P_{p-1,0} \arrow[dd] & & & P_{p,0} \arrow[lll, dashed, "?"] \arrow[dl] \arrow[dd] & \vdots \arrow[dd] & \\
 & & \cdot \arrow[ul] \arrow[dd] & \cdot \arrow[l] \arrow[dd] & & & \\
\cdots & C_{p-1} \arrow[l] \arrow[dd] & & & C_p \arrow[lll] \arrow[dl, two heads] \arrow[dd] & C_{p+1} \arrow[l, "d"] \arrow[dd] & \cdots \arrow[l] \\
 & & Z_{p-1} \arrow[ul, tail] & B_{p-1} \arrow[l, tail] & & & \\
 & 0 & & & 0 & 0 &
\end{tikzcd}
$$

Then there exists a double complex $P_{\cdot,\cdot}$ with all $P_{p,q}$ projective, together with the morphism of complexes

$$ P_{\cdot,\cdot} \to C_\cdot $$

such that each vertical complex is a projective resolution, and $P_{p,q} = 0$ for all $q < 0$:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & & \\
\cdots & P_{p-1,1} \arrow[l] \arrow[d] & P_{p,1} \arrow[l] \arrow[d] & \cdots \arrow[l] & \cdots \arrow[l] \\
\cdots & P_{p-1,0} \arrow[l] \arrow[d] & P_{p,0} \arrow[l] \arrow[d] & \cdots \arrow[l] & \cdots \arrow[l] \\
\cdots & C_{p-1} \arrow[l] \arrow[d] & C_p \arrow[l] \arrow[d] & C_{p+1} \arrow[l] & \cdots \arrow[l] \\
 & 0 & 0 & &
\end{tikzcd}
$$

but moreover the horizontal complexes are not necessarily exact but split complexes: for each $q \geqslant 0$, all

$$Z^h_{p,q} = \ker(d^h : P_{p,q} \to P_{p-1,q}),$$

$$B^h_{p,q} = \mathrm{im}(d^h : P_{p+1,q} \to P_{p,q})$$

and $H^h_{p,q} = Z^h_{p,q}/B^h_{p,q}$ are all projective, (hence, all

$$Z^h_{p,q} \rightarrowtail P_{p,q} \twoheadrightarrow B^h_{p-1,q}$$

$$B^h_{p,q} \rightarrowtail Z_{p,q} \twoheadrightarrow H^h_{p,q}$$

are split exact sequences,) and finally the vertical sequences with Z's and B's and H's are projective resolutions. In particular, there is a projective resolution

$$\cdots \to H^h_{p,q}(P_{\cdot,\cdot}) \to H^h_{p,q-1}(P_{\cdot,\cdot}) \to \cdots \to H^h_{p,1}(P_{\cdot,\cdot}) \to H^h_{p,0}(P_{\cdot,\cdot}) \to H_p(C.) \to 0$$

Proof. Horseshoe Lemma tells us that from chosen projective resolution of $H_p(C.)$ and $B_p(C.)$ applied to the short exact sequence. By taking choosing sequences $R_{p,\cdot}$ and $Q_{p,\cdot}$ where R is degreewise split, we can form $S_{p,\cdot}$ such that

$$\begin{tikzcd}
R_{p,\cdot} \arrow[r] \arrow[d, two heads] & S_{p,\cdot} \arrow[r] \arrow[d, dashed, two heads] & Q_{p,\cdot} \arrow[d, two heads] \\
B_p(C.) \arrow[r, tail] & Z_p(C.) \arrow[r, two heads] & H_p(C.)
\end{tikzcd}$$

then we have degreewise split $S_{p,\cdot}$ that forces

$$\begin{tikzcd}
S_{p,\cdot} \arrow[r, tail] \arrow[d, two heads] & P_{p,\cdot} \arrow[r, two heads] \arrow[d, dashed, two heads] & R_{p-1,\cdot} \arrow[d, two heads] \\
Z_p(C.) \arrow[r, tail] & C. \arrow[r, two heads] & B_{p-1}(C.)
\end{tikzcd}$$

and we have a sequence

$$\begin{tikzcd}
P_{p-1,q} & S_{p-1,q} \arrow[l, tail, "split"'] & R_{p-1,q} \arrow[l, tail, "split"'] & P_{p,q} \arrow[l, "split"'] \arrow[lll, bend left, "d^n"] & S_{p,q} \arrow[l]
\end{tikzcd}$$

$\square$

We now consider the Grothendieck spectral sequences.

Theorem 3.3.2. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be two right exact functors between Abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Suppose F maps projectives to G-acyclics. Let $A \in \mathcal{A}$. There exists a convergent first-quarter spectral sequence (i.e. all terms at position (p,q) are 0 for $p < 0$ or $q < 0$)

$$E^2_{p,q} = L_qG \circ L_pF(A) \xrightarrow{n=p+q} L_n(G \circ F)(A)$$

in $\mathcal{C}$.

Proof. Let $P. \to A$ be a projective resolution of A in $\mathcal{A}$. Consider $C. = F(P.)$ in $\mathcal{B}$: it is a complex $(H_p(F(P.)) = L_p(F(A)))$. Applying Cartan-Eilenberg to this complex in $\mathcal{B}$ to get a double complex

$$\begin{tikzcd}
& Q_{\cdot,\cdot} \arrow[d] & & \\
& \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & Q_{p,1} \arrow[l] \arrow[d] & Q_{p+1,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & Q_{p,0} \arrow[l] \arrow[d] \arrow[dd, bend left] & Q_{p+1,0} \arrow[l] \arrow[d] \arrow[dd, bend left] & \cdots \arrow[l] \\
\cdots & F(P_p) \arrow[l] & F(P_{p+1}) \arrow[l] & \cdots \arrow[l] \\
& 0 & 0 &
\end{tikzcd}$$

We can now consider a spectral sequence for $G(Q_{\cdot,\cdot})$ in $\mathcal{C}$, by using

$$^{\mathrm{I}}E^2_{p,q} = H^h_p H^v_q(G(Q_{\cdot,\cdot})) \xrightarrow{n=p+q} H_n(\mathbf{Tot}(G(Q_{\cdot,\cdot})))$$

and

$$^{\mathrm{II}}E^2_{p,q} = H^v_q H^h_p(G(Q_{\cdot,\cdot})) \xrightarrow{n=p+q} H_n(\mathbf{Tot}(G(Q_{\cdot,\cdot})))$$

and so

$$\begin{tikzcd}
& GQ_{\cdot,\cdot} \arrow[d] & & \\
& \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & GQ_{p,1} \arrow[l] \arrow[d] & GQ_{p+1,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & GQ_{p,0} \arrow[l] \arrow[d] & GQ_{p+1,0} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
& \vdots & \vdots &
\end{tikzcd}$$

Therefore, $H^h_p(G(Q_{\cdot,\cdot})) = G(H^h_p(Q_{\cdot,\cdot}))$ for $^{\mathrm{II}}E$ since $Q_{\cdot,\cdot}$ is horizontally split. We then have

$$\begin{aligned}
H^v_q(H^v_p(G(Q_{\cdot,\cdot}))) &= H^v_q(G(H^h_p(Q_{\cdot,\cdot}))) \\
&= L_qG(H_p(C.)) \\
&= L_qG(L_pF(A.)).
\end{aligned}$$

As for $^{\mathrm{I}}E$, we have $H^v_q(G(Q_{p,\cdot})) = L_qG(F_p(P_p))$. Since P_p is projective, so $F_p(P_p)$ is G-acyclic. In particular,

$$\begin{aligned}
H^v_q(G(Q_{p,\cdot})) &= L_qG(F_p(P_p)) \\
&= \begin{cases} 0, & \forall q \neq 0 \\ GF(P_p), & q = 0 \end{cases}
\end{aligned}$$

Therefore, ${}^{\mathrm{I}}E$ is degenerate and only contains the $q = 0$ row. In particular,

$$
{}^{\mathrm{I}}E_{p,q} = \begin{cases} H_p^h(GF(P_p)) = L_p(GF)(A), & \text{for } q = 0 \\ 0, & \text{for } q \neq 0 \end{cases}
$$

Hence, it converges to $H_n(\mathbf{Tot}(G(Q_{\cdot,\cdot})))$ along $n = p + q$. We conclude that ${}^{\mathrm{I}}E^2_{p,q} = {}^{\mathrm{I}} E^{\infty}_{p,q}$ by examining the page

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
L_{p-2}F(A) & L_{p-1}F(A) & L_pF(A) & L_{p+1}F(A) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}
$$

Therefore, $H_n(\mathbf{Tot}(G(Q_{\cdot,\cdot}))) \cong L_n(GF)(A)$. We now replace that with ${}^{\mathrm{II}}E$. Then we get

$$
\begin{tikzcd}
 & \mathbf{K}_+(\mathbf{Proj}(\mathcal{A})) \arrow[d, hook] \arrow[dr, "F"] & & \mathbf{K}_+(\mathbf{Proj}(\mathcal{B})) \arrow[d, hook] \arrow[dr, "G"] & \\
\mathcal{A} \arrow[ur, "P."] \arrow[r] \arrow[drr, "L_*F"'] & \mathbf{K}_+(\mathcal{A}) \arrow[r, "F"] & \mathbf{K}(\mathcal{B}) \arrow[ur, "P."] \arrow[r, equal] \arrow[d, "H_*"] & \mathbf{K}_+(\mathcal{B}) \arrow[r, "G"] & \mathbf{K}_+(\mathcal{C}) \arrow[d, "H_*"] \\
 & & \mathcal{B} \arrow[rr, "L_*G"'] & & \mathcal{C}
\end{tikzcd}
$$

where the diagram commutes with quasi-isomorphisms $\mathbf{K}_+(\mathbf{Proj}(\mathcal{A})) \Rightarrow (A \to \mathbf{K}_+(\mathcal{A})$ as well as $\mathbf{K}_+(\mathbf{Proj}(\mathcal{B})) \Rightarrow (\mathbf{K}(\mathcal{B}) \cong \mathbf{K}_+(\mathcal{B})$. Note that $P.$ induces $P. : B \to \mathbf{K}_+(\mathbf{Proj}(\mathcal{B}))$ as $P. \circ H^*{-1}$, and therefore we have $L_*(F) = H_*FP.$ and $L_*G = H_*GP.$ □

However, we want a better description of "LF" without taking homology.

4 Triangulated and Derived Category

Let $\mathcal{A}$ be an Abelian category. The motivation of this chapter is that, sometimes, we wished quasi-isomorphisms are actually isomorphisms up to homotopy.

Definition 4.0.1 (Derived Category)**.** The derived category of $\mathcal{A}$ is the localization of $\mathbf{Ch}(\mathcal{A})$, the Abelian category of chain complexes, with respect to quasi-isomorphisms, i.e.

$$
\begin{tikzcd}
\mathbf{D}(\mathcal{A}) \arrow[r, equal] & \mathbf{Ch}(\mathcal{A})[\{\text{ quasi-isomorphisms }\}^{-1}] \\
\mathbf{Ch}(\mathcal{A}) \arrow[u, "Q"] &
\end{tikzcd}
$$

which is universal for the property that Q sends quasi-isomorphisms to isomorphisms.

Remark 4.0.2. The homotopy equivalences are quasi-isomorphisms and we can obtain $\mathbf{Ch}(\mathcal{A})[\{\text{ homotopy equivalence}\}^{-1}] = \mathbf{K}(\mathcal{A})$ as the homotopy category. Note that $\mathbf{K}(\mathcal{A}$ has the same objects as $\mathbf{Ch}(\mathcal{A})$ and maps that are morphisms of complexes up to homotopy. If we consider the diagram

$$\begin{tikzcd} \mathbf{Ch}(\mathcal{A}) \arrow[d] \arrow[dd, bend right, "Q"'] \\ \mathbf{K}(\mathcal{A}) \arrow[d, "\text{calculus of fractions}"] \\ \mathbf{D}(\mathcal{A}) \cong \mathbf{K}(\mathcal{A})[\{\text{ quasi-isomorphisms }\}^{-1}] \end{tikzcd}$$

then $\mathbf{Ch}(\mathcal{A})$ is Abelian, $\mathbf{K}(\mathcal{A})$ is not Abelian in general. However, we can get $\mathbf{K}(\mathcal{A})$ and $\mathbf{D}(\mathcal{A})$ to always be triangulated, as we can think of the ore condition as previously mentioned. Generally, this diagram presents two sources of long exact sequence in H_*: suppose we have an exact sequence $A. \rightarrowtail B. \twoheadrightarrow C.$, then

$$\begin{tikzcd} A. \arrow[r, tail, "f"] \arrow[d] & B. \arrow[r, two heads, "g"] & C. & \\ A. \arrow[r, "f"] \arrow[d, equal] & B. \arrow[r, "g"] \arrow[d, equal] & C. \arrow[r, dashed, "d=?"] & \sum A. \arrow[d, equal] \\ A. \arrow[r, "f"] & B. \arrow[r] & \mathrm{cone}(f) \arrow[r] \arrow[u, "\text{quasi-iso}"] & \sum A. \end{tikzcd}$$

4.1 Triangulated Category

Definition 4.1.1 (Suspended Category, Pre-triangulated Category)**.** A suspended category $(\mathcal{T}, \Sigma)$ is an additive category $\mathcal{T}$ with an additive self-equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ called suspension such that $A \mapsto \Sigma A = A[1]$.

A pre-triangulated category is a suspension category with a choice (classes) of "exact triangles" (admits/distinguished), taking the form $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$[3], satisfying some axioms:

- For every object $A \in \mathcal{T}$, the triangle $0 \to A \xrightarrow{\mathbf{id}} A \to 0$ is exact (or, the triangle $A \xrightarrow{\mathbf{id}} 0 \to \Sigma A$ is exact).
- The triangle
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$$
is exact if and only if the triangle
$$B \xrightarrow{g} C \xrightarrow{h} \Sigma(A) \xrightarrow{-\Sigma(f)} \Sigma(B)$$
is exact.
- A triangle isomorphic to an exact triangle is exact, i.e.
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"', "\cong"] & B \arrow[r, "g"] \arrow[d, "\cong"', "v"] & C \arrow[r, "h"] \arrow[d, "\cong"', "w"] & \Sigma A \arrow[d, "\cong"', "\Sigma u"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma(A') \end{tikzcd}$$

(Existence Axiom) For every morphism $g : B \to C$ of $\mathcal{T}$, there exists an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ which g fits into.

(Morphism Axiom) If we have a diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r, "h"] \arrow[d, dashed, "w"] & \Sigma A \arrow[d, "\Sigma u"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma(A') \end{tikzcd}$$

[3]There are many ways one can choose to interpret this, including the commonly known Suspension Diagram Figure 2.7.3, and/or the long exact sequence given by $\cdots \to \Sigma^{-1}C \xrightarrow{\Sigma^{-1}h} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{\Sigma f} \Sigma B \to \cdots$

where the two rows are exact triangles and the left most square commutes, then there exists some (not necessarily unique) morphism $w : C \to C'$ making the diagram commutes, i.e. making the above diagram a morphism of exact triangles (u, v, w).

Remark 4.1.2. The second axiom allows us to suspend and desuspend to move around a diagram, so where we are in an exact triangle does not matter that much, and we will mostly only consider claims in one position, where the other positions will follow.

As an example of this, suppose that we have a diagram of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "g"] & C \arrow[r, "h"] \arrow[d, "w"] & \Sigma A \arrow[d, "\Sigma u"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma(A')
\end{tikzcd}$$

which has rows exact triangles and the rightmost square commutes. We can desuspend to get a diagram

$$\begin{tikzcd}
\Sigma^{-1}(C) \arrow[r, "\Sigma^{-1}(h)"] \arrow[d, "\Sigma^{-1}(w)"'] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] & C \arrow[d, "w"] \\
\Sigma^{-1}(C') \arrow[r, "\Sigma^{-1}(f')"'] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'
\end{tikzcd}$$

satisfying the hypothesis of the morphism axiom, and so we get a map from B to B' making the diagram commute. Suspending back, the arrow from B to B' makes our old diagram commute, and so we get the morphism axiom with the arrows in the triangle moved around a bit.

Example 4.1.3.

$$\begin{tikzcd}
A \arrow[r, "\binom{1}{0}"] \arrow[d, "u"'] & A \oplus B \arrow[r, "(0\ 1)"] \arrow[d, dashed, "\begin{pmatrix} u & * \\ 0 & w \end{pmatrix}"] & B \arrow[r, "0"] \arrow[d, "w"] & \Sigma A \arrow[d, "\Sigma u"] \\
A' \arrow[r, "\binom{1}{0}"'] & A' \oplus B' \arrow[r, "(0\ 1)"'] & B' \arrow[r, "0"'] & \Sigma A'
\end{tikzcd}$$

where : $B \to A'$.

Example 4.1.4.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, "h"] & \Sigma(A) \arrow[d, "\Sigma f"] \\
B \arrow[r, "\mathbf{id}"] & B \arrow[r] & 0 \arrow[r] & \Sigma A
\end{tikzcd}$$

where $\Sigma f \circ h = 0$, $h \circ g = 0$ and $g \circ f = 0$.

Example 4.1.5. Suppose we have

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d, "t"] & C \arrow[r, "h"] \arrow[d, dashed, "\exists \bar{t}"] & \Sigma A \arrow[d] \\
0 \arrow[r] & T \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

such that $t \circ f = 0$, then there exists $\bar{t} : C \to T$ such that $\bar{t} \circ g = t$. Therefore, (C, g) acts as a weak cokernel of f. Similarly, $(\Sigma^{-1}C, \Sigma^{-1}h)$ acts as a weak kernel of f.

Exercise 4.1.6. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ be an exact triangle. Show then h acts as a weak cokernel of g, that is $hg = 0$, and given any $x : C \to X$ such that $xg = 0$, there exists some (not necessarily unique) $y : \Sigma(A) \to X$ such that $yh = x$. Similarly, show that f is a weak kernel of g, where a weak kernel is defined dually.

Proof. To see $hg = 0$, suspend, and then look at a map to the exact triangle from one of the form $B \xrightarrow{\mathbf{id}} B \to 0 \to \Sigma(B)$. For the fact that $xg = 0$ implies x factors over h, try making use of the morphism axiom and the exact triangle $0 \to X \xrightarrow{\mathbf{id}} X \to 0$. $\square$

Exercise 4.1.7. Suppose that if $w : C \to C'$ is an arrow in $\mathcal{T}$ that is both monic and epic, then w is an isomorphism.

Proof. Construct an exact triangle with w in the middle using the existence axiom, and see what we can say about the other arrows in the triangle. ☐

Exercise 4.1.8. The C in the diagrams above is unique up to (non-unique) isomorphisms. That is, suppose that we have two exact triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ and $A \xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{h'} \Sigma(A)$, with the same f, then there is a non-canonical isomorphism $w : C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, "h"] \arrow[d, "w"] & \Sigma(A) \arrow[d, equal] \\
A \arrow[r, "f"'] & B \arrow[r, "C'"'] & C' \arrow[r, "h'"'] & \Sigma(A)
\end{tikzcd}$$

Here we say $f : A \to B$ fixed the object C and/or the triple (C, g, h), which is called the cone of f.

Proof. Try showing w is a monomorphism and an epimorphism, then apply the previous exercise. ☐

Remark 4.1.9. f is an isomorphism if and only if $\operatorname{cone}(f) \cong 0$.

Theorem 4.1.10. If $\mathcal{A}$ is additive, then $\mathbf{K}(\mathcal{A})$ is a triangulated category with $\Sigma A = A[1]$: $(\Sigma A)_n = A_{n-1}$, $d^{\Sigma A} = -d^A$, and the chosen exact triangles are those isomorphic (in $\mathbf{K}(\mathcal{A})$) to the form

$$A \xrightarrow{f} B \xrightarrow[\binom{0}{1}]{f'} \operatorname{cone}(f) \xrightarrow[(1\ 0)]{f''} \Sigma A$$

for all $f : A \to B$ in $\mathbf{Ch}(\mathcal{A})$.

Exercise 4.1.11. Show that if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma(A)$ are two exact triangles, then there direct sum, the triangle

$$A \oplus A' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}} C \oplus C' \xrightarrow{\begin{pmatrix} h & 0 \\ 0 & h' \end{pmatrix}} \Sigma(A \oplus A')$$

is exact.[4]

Proof. Take an exact triangle of the form

$$A \oplus A' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} B \oplus B' \xrightarrow{p} D \xrightarrow{q} \Sigma(A \oplus A')$$

and then see what can be asid about D in relation to $C \oplus C'$. ☐

Remark 4.1.12. For a pre-triangulated category $\mathcal{T}$, if f is a monomorphsim, then it is a split monomorphism. Similarly, if f is an epimorphism, then f is a split epimorphism.

To see this, make use of the exact triangle $A \xrightarrow{\binom{1}{0}} A \oplus B \xrightarrow{(0\ 1)} B \xrightarrow{0} \Sigma(A)$ which is exact by the previous exercise. This is especially useful. For example, in $K(\mathbb{Z}\text{-}\mathbf{Mod})$, if we can check the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

does not contain the split epimorphism, and so it is not an epimorphism in general.

[4]Try treating $\Sigma(A \oplus A')$ as $\Sigma(A) \oplus \Sigma(A')$ through a suitable natural isomorphism.

4.2 Derived Category and Derived Functor

Definition 4.2.1. A triangulated category $\mathcal{T}$ is a pre-triangulated category $(\mathcal{T}, \Sigma)$ with $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ such that the composition axiom holds.

Composition Axiom: for any composable morphisms f_1, f_2, there exists a diagram

$$\begin{tikzcd}
A_1 \arrow[rr, "f_1"] \arrow[rrrr, bend left, "f_3"] & & A_2 \arrow[rr, "f_2"] \arrow[dl, "g_1"] & & A_3 \arrow[dl, "g_2"] \arrow[ddll, bend left, "g_3"] \\
& C_1 \arrow[ul, "h_1"] \arrow[dr, "f_4"] & & C_2 \arrow[ul, "h_2"] \arrow[ll, "(\Sigma g_1)\circ h_2"] & \\
& & C_3 \arrow[ur, "g_4"] \arrow[uull, bend left, "h_3"] & &
\end{tikzcd}$$

Figure 10: Composition Axiom

where $A_1A_2A_3$, $A_1C_1C_3$, $A_3C_2C_3$ and $A_2C_1C_2$ commutes, and $A_1A_2C_1$, $A_2A_3C_2$ and $C_1C_2C_3$ are exact. Moreover, we have $f_4 \circ g_1 = g_3 \circ f_2$ and $(\Sigma f_1) \circ h_3 = h_2 \circ g_4$. Alternatively, we have the following diagram:

$$\begin{tikzcd}
& & & A_3 \arrow[ddll, "g_2"] \arrow[ddr, "g_3"] & \\
& & & & \\
A_2 \arrow[uurrr, "f_2"] \arrow[ddr, "g_1"] & C_2 \arrow[l, "h_2"] \arrow[dd, "h_4"] & & & C_3 \arrow[lll, "g_4"] \arrow[dl, "h_3"] \\
& & & A_1 \arrow[ulll, "f_1"] \arrow[uuu, bend left, "f_3"] & \\
& C_1 \arrow[urr, "h_1"] \arrow[uurrr, bend right, "f_4"] & & &
\end{tikzcd}$$

Remark 4.2.2. This induces diagram in the shape of

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d] & A_2 \arrow[r, "f_2"] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & \Sigma A_1 \arrow[d] \\
& & 0 \arrow[r] & C_2 \arrow[r] \arrow[d] & \Sigma A_2 \arrow[d] \\
& & & 0 \arrow[r] & \Sigma A_3
\end{tikzcd}$$

For example, we can have

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d] & A_2 \arrow[r, "f_2"] \arrow[d] & A_3 \arrow[r, "f_3"] \arrow[d] & A_4 \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & 0 \arrow[r] & C_{12} \arrow[r] \arrow[d] & C_{13} \arrow[r] \arrow[d] & C_{14} \arrow[r] \arrow[d] & \Sigma A_1 \arrow[d] \\
 & & 0 \arrow[r] & C_{23} \arrow[r] \arrow[d] & C_{24} \arrow[r] \arrow[d] & \Sigma A_2 \arrow[d] \\
 & & & 0 \arrow[r] & C_{34} \arrow[r] \arrow[d] & \Sigma A_3 \arrow[d] \\
 & & & & 0 \arrow[r] & \Sigma A_4 \arrow[d] \\
 & & & & & 0
\end{tikzcd}$$

Now, this induces the map we are looking for when considering derived category from $\mathbf{Ch}(\mathcal{A}) \to \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$:

$$\begin{tikzcd}
A. \arrow[r, tail, "f"] \arrow[d] & B. \arrow[r, two heads, "g"] & C. & \\
A. \arrow[r, "f"] \arrow[d, equal] & B. \arrow[r, "g"] \arrow[d, equal] & C. \arrow[r, dashed, "d"] & \sum A. \arrow[d, equal] \\
A. \arrow[r, "f"] & B. \arrow[r] & \mathrm{cone}(f) \arrow[r] \arrow[u, "\text{quasi-iso}"] & \sum A.
\end{tikzcd}$$

This can be done by the localization of triangulated categories.

Theorem 4.2.3 (Verdier Localization Theorem)**.** Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{C} \subseteq \mathcal{D}$ be a triangulated subcategory (not necessarily thick). Then there is a universal functor $F : \mathcal{D} \to \mathcal{T}$ with $C \in \ker(F)$. In other words, there exists a triangulated category $\mathcal{D}/\mathcal{C}$, and a triangulated functor $F_{univ} : \mathcal{D} \to \mathcal{D}/\mathcal{C}$, so that $\mathcal{C}$ is the kernel of F_{univ}, and that F_{univ} is universal with this property. If $F : \mathcal{D} \to \mathcal{T}$ is a triangulated functor whose kernel contains $\mathcal{C}$, then it factors uniquely as $\mathcal{D} \to \mathcal{D}/\mathcal{C} \to \mathcal{T}$, where the functor from $\mathcal{D}$ to $\mathcal{D}/\mathcal{C}$ is F_{univ}.

Remark 4.2.4 (Verdier Localization)**.** Let us think of the exact functor $Q : \mathcal{T} \to \mathcal{T}[S^{-1}]$ (so it preserves exact triangles), which sends $s \in S$ to isomorphisms. The kernel of Q is then $\ker(Q) = J_s = \{A \in \mathcal{T} \mid (0 \to A) \in S\}$. Therefore, we then have the map sending $A \xrightarrow{s} B \to C \to \Sigma(A)$ to $\cdot \xrightarrow{Q(s),\cong} \cdot \to (Q(c) = \mathrm{cone}(\cong) = 0) \to 0$. This does not help with the construction. Therefore, we want to think from the other way around, that is we start with a subcategory $J \subseteq \mathcal{T}$ to kill, and we want to define $\mathcal{T}/J = \mathcal{T}[S^{-1}]$, where $S(J) = S = \{s : A \to B \mid \mathrm{cone}(s) \in J\}$.

For example, for $\mathcal{T} = \mathbf{K}(\mathcal{A})$, take $J = \mathbf{K}_{ac}(\mathcal{A}) = \ker(H_*) = \{A. \in \mathbf{K}(\mathcal{A}) \mid A. \text{ exact (acyclic) }\}$.

There are also conditions required on J: that is, we want

- $0 \in J$ and $\Sigma(J) = J$,
- $A \to B \to C \to \Sigma(A)$ exact in J; any two of A, B, C in J indicates the third one also in J (this is sometimes called 2-out-of-3).
- $A \oplus B \in J$ indicates $A, B \in J$.

In short, we want a thick subcategory of a triangulated category to be closed under direct summands.

Suppose we know our J above is thick, then a useful fact is that $S(J) = \{s \in S \mid \mathrm{core}(s) \in J\}$ has calculus of fractions:

- closed under composition, and 2-out-of-3.

- for any $s \in S$ and diagram

$$\begin{tikzcd} & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

  there exists a diagram

$$\begin{tikzcd} \cdot \arrow[r, dashed, "t \in S"] \arrow[d, dashed] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

  The dual claim also holds.

- For any coequalizer $s \in S$ of f, g, there exists an equalizer $t \in S$ such that $ft = fg$. Note that the converse should also hold if we have $Q(f) = Q(g)$ in $\mathcal{T}[S^{-1}]$.

Hence, we have $Q : \mathcal{T} \to \mathcal{T}[S^{-1}]$ where $\mathcal{T}[S^{-1}]$ has the same objects as $\mathcal{T}$ and

$$\mathbf{Hom}_{\mathcal{T}[S^{-1}]}(A, B) = \{A \xleftarrow{s \in S} X \xrightarrow{f} B\}/\sim$$

where $\sim$ is generated by amplification, i.e. any commutative diagram

$$\begin{tikzcd} & \cdot \arrow[dl, "t"'] \arrow[d, "u"] \arrow[dr, "g"] & \\ \cdot & \cdot \arrow[l, "s"] \arrow[r, "f"'] & \cdot \end{tikzcd}$$

should satisfy $fs^{-1} = gt^{-1}$, and any commutative diagram below should induced the dashed morphisms in S, i.e. closed compositions:

$$\begin{tikzcd} & & \cdot \arrow[dll] \arrow[dl, dashed] \arrow[dr, dashed] \arrow[drr] & & \\ \cdot & \cdot \arrow[l, "s"] \arrow[r, "f"'] & \cdot & \cdot \arrow[l, "t"] \arrow[r, "g"'] & \cdot \end{tikzcd}$$

However, we may have set-theoretic issues again: it could be a proper class (lies in the next Grothendieck Universe). Therefore, for $Q : \mathcal{T} \to \mathcal{T}[S^{-1}] = \mathcal{T}/J$, we want to send $f : A \to B$ to $A \xleftarrow{\mathbf{id}} A \xrightarrow{f} B$. This helps us to make $\mathcal{T}/J$ into a triangulated category by taking the images of exact triangles in J under Q, and closing under isomorphisms.

A useful fact is that $\mathcal{T}/J = T[S^{-1}]$ is triangulated, and so $\mathbf{D}(\mathcal{A})$ is triangulated.

Example 4.2.5. $\mathbf{D}(K\text{-vector spaces}) \cong$ Graded K-vector spaces.

Remark 4.2.6. Just like those of $\mathbf{K}(\mathcal{A})$, there are also variants with boundedness on $\mathbf{D}(\mathcal{A})$. Note that by applying an invert quasi-isomorphism, the diagram

$$\begin{tikzcd} & \mathbf{K}(\mathcal{A}) & \\ \mathbf{K}^-(\mathcal{A}) = \mathbf{K}_+(\mathcal{A}) \arrow[ur, hook] & & \mathbf{K}^+(\mathcal{A}) = \mathbf{K}_-(\mathcal{A}) \arrow[ul, hook'] \\ & \mathbf{K}^b = \mathbf{K}_b \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

is sent to

$$\begin{tikzcd} & \mathbf{D}(\mathcal{A}) & \\ \mathbf{D}^-(\mathcal{A}) = \mathbf{D}_+(\mathcal{A}) \arrow[ur, hook] & & \mathbf{D}^+(\mathcal{A}) = \mathbf{D}_-(\mathcal{A}) \arrow[ul, hook'] \\ & \mathbf{D}^b = \mathbf{D}_b \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

Example 4.2.7. Take $R = \mathbb{Z}$. Consider the category $\mathbf{D}_b(\mathbf{Ab}) = \mathbf{D}_b(\mathbb{Z}\text{-}\mathbf{Mod})$, in which every object is a direct sum of finitely-generated Abelian groups but not in a $\mathbb{Z}$-graded way:

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z}[0] \arrow[r, equal] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \\
X \arrow[u, "\text{quasi-isomorphism}"'] \arrow[d, "f"'] \arrow[r, equal] & 0 \arrow[r] & \mathbb{Z} \arrow[u] \arrow[d, "\mathbf{id}"] \arrow[r] & \mathbb{Z} \arrow[u] \arrow[d] \arrow[r] & 0 \\
\mathbb{Z}/2\mathbb{Z}[1] \arrow[r, equal] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Remark 4.2.8. There is the fact that $\mathbf{Ext}^n_{\mathcal{A}}(A, B) \cong \mathbf{Hom}_{\mathbf{D}_b(\mathcal{A})}(A, B[\cdots])$. According to Yoneda, we have a correspondence between long sequences

$$0 \to B \to E_{n-1} \to \cdots \to E_0 \to A \to 0$$

and diagrams

$$\begin{tikzcd}
 & & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathbf{id}"] & E_{n-1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[u] \arrow[d] & E_0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & B \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 & &
\end{tikzcd}$$

Similarly, we can consider the same thing about resolutions.

Suppose $\mathcal{A}$ has enough projectives, then we have a diagram

$$\begin{tikzcd}
\mathbf{K}_+(\mathbf{Proj}(\mathcal{A})) \arrow[r, hook] \arrow[d, "\cong"'] \arrow[dr, "\cong"] & \mathbf{K}_+(\mathcal{A}) \arrow[d, "Q"] \\
\mathbf{D}_+(\mathbf{Proj}(\mathcal{A})) \arrow[r, "\cong", "\text{equivalence}"'] & \mathbf{D}_+(\mathcal{A})
\end{tikzcd}$$

Note that the isomorphism $\mathbf{K}_+(\mathbf{Proj}(\mathcal{A})) \to \mathbf{D}_+(\mathbf{Proj}(\mathcal{A}))$ is given by the quasi-isomorphisms of right-bounded complexes of projectives, which are homotopy equivalent.

Now, by taking the cone $\cdot \to P_{n+1} \to P_0 \to 0 \to \cdots$, the exactness transfers into split exactness, and we obtain backwards arrow on the diagram, with the following diagram

$$\begin{tikzcd}
P. & X \arrow[l, "\text{quasi-iso}"'] \arrow[r] & Q. \\
 & L. \arrow[ul, "\text{quasi-iso}"] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

We may obtain similar results if we have enough injectives:

$$\begin{tikzcd}
\mathbf{K}_+(\mathbf{Inj}(\mathcal{A})) \arrow[r, hook] \arrow[d, "\cong"'] \arrow[dr, "\cong"] & \mathbf{K}_+(\mathcal{A}) \arrow[d, "Q"] \\
\mathbf{D}_+(\mathbf{Inj}(\mathcal{A})) \arrow[r, "\cong", "\text{equivalence}"'] & \mathbf{D}_+(\mathcal{A})
\end{tikzcd}$$

At the end of the chapter, we revisit the derived functors. Suppose we have a functor $F : \mathcal{A} \to \mathcal{B}$ under the usual assumptions, then we have a diagram

$$\begin{tikzcd}
\mathbf{Ch}(\mathcal{A}) \arrow[r, "F"] \arrow[d] & \mathbf{Ch}(\mathcal{B}) \arrow[d] \\
\mathbf{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d] & \mathbf{K}(\mathcal{B}) \arrow[d] \\
\mathbf{D}(\mathcal{A}) \arrow[r, dashed] & \mathbf{D}(\mathcal{B})
\end{tikzcd}$$

Note that the bottom functor $F : \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ exists if and only if F is exact.

In general, one would try to left-derive or right-derive instead of applying localization:

$$\begin{tikzcd}
\mathbf{K}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"'] \arrow[dr] & \mathbf{K}(\mathcal{B}) \arrow[d, "Q"] \\
\mathbf{D}_+(\mathcal{A}) \arrow[r, "LF"'] & \mathbf{D}_+(\mathcal{B})
\end{tikzcd}$$

Therefore, we want $\lambda : LF \circ Q \Rightarrow Q \circ F$. A good guess is to take $LF = 0$, but this is not as good as we want. Note that the pair (LF, λ) is the left-derived functor on $\mathbf{D}_+$, and we have the following construction:

$$\begin{tikzcd}
& \mathbf{K}_+(\mathcal{A}) \arrow[r, "F"] \arrow[dd] \arrow[dl, shift left, "P"] & \mathbf{K}_+(\mathcal{B}) \arrow[dd, "Q"] \\
\mathbf{K}_+(\mathbf{Proj}(\mathcal{A})) \arrow[ur, hook, shift left, "\mathrm{inc}"] \arrow[dr, "\cong"'] \arrow[urr, bend left, "F|_{\mathrm{proj}}=F\circ\mathrm{inc}"] & & \mathcal{T} :\Rightarrow \\
& \mathbf{D}_+(\mathcal{A}) \arrow[r, "LF=QFP"] & \mathbf{D}_+(\mathcal{B})
\end{tikzcd}$$

Here Q is the composition of P and the isomorphism; the inclusion and P forms an adjunction inc $\dashv P$, and $\mathcal{T}$ forms a mapping from $\mathbf{D}_+(\mathcal{A})$ to $\mathbf{K}_+(\mathcal{B})$. Moreover, the map $\varepsilon : \mathrm{inc} \circ P \Rightarrow \mathbf{id}$ is a quasi-isomorphism from $P(A)$ to A as a counit, and induces $\lambda = QF(\varepsilon) : LFQ = F \circ F \circ \mathrm{inc} \circ P \Rightarrow QF$.

Under the same condition, we have $F(\mathbf{Proj}(\mathcal{A})) \subseteq G$-acyclics for $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$, which induces

$$\begin{tikzcd}
\mathbf{D}_+(\mathcal{A}) \arrow[r, "LF"] \arrow[rr, bend left, "L(G\circ F)"] & \mathbf{D}_+(\mathcal{B}) \arrow[r, "LG"] & \mathbf{D}_+(\mathcal{C})
\end{tikzcd}$$

and therefore we have $L(G \circ F) \cong LG \circ LF$ in this sense, as desired.