MATH 191 Notes

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1 CATEGORIES

To start any mathematical discussions, one must first define the basic objects of the study. In this section, we introduce the concept of categories.

Before we get there, we should know the motivations for the invention of category theory.

In their work in algebraic topology, Eilenberg and Mac Lane needed to make precise what it means for a family of maps to be ”natural”.

Colloquially, ”natural” means what it sounds like – something is natural if it is ”canonical” or defined without making arbitrary choices. Here are a few examples:

Example 1.1.

1. Let X be a set and let $\mathcal{P}(X) = \{A \mid A \subseteq X\}$ be the power set of X, i.e. the set of all subsets of X. Then there is a natural map $X \to \mathcal{P}(X)$ that sends each $x \in X$ to the $\{x\} \subseteq X$.
2. For any sets X and Y, let $Y^X = \{f : X \to Y\}$ be the set of all functions from X to Y. Then there is a natural bijection $\mathcal{P}(X) \xrightarrow{\cong} \{0,1\}^X$ that sends $A \subseteq X$ to its characteristic function $\mathcal{X}_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$.
3. For any sets X and Y, let $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$ be the Cartesian product of X and Y. Then there is a natural bijection $\tau : X \times Y \to Y \times X$ that sends (x,y) to (y,x).

This may seem pretty trivial, but Eilenberg and Mac Lane had deeper applications in mind, and they needed to make this intuitive notion precise to do the mathematics.

So what is the problem?

- A natural isomorphism is a kind of mapping, so it needs a domain and codomain.
- This necessitated the definition of a functor. These are, informally, ”constructions” which serve as the sources and targets of natural mappings.
- Functors are also a kind of mapping, so they also need domains and codomains.
- This necessitated the definition of categories, which are, informally, collections of objects and maps that can serve as input or output of functors.

For the sake of logic, we must start with categories, even though our first goal is to make ”naturality” precise. So without further ado, here is the definition of a category:

Definition 1.2 (Category)**.** A category $\mathscr{C}$ consists of:

(a) a collection of objects $A, B, C, \cdots$

(b) a collection of morphisms $f, g, h, \cdots$

such that

(i) each morphism has a domain and codomain object; we write $f : A \to B$ as shorthand for "f is a morphism with domain A and codomain B" and we write $\mathscr{C}(A, B)$ for the collection of all morphisms $f : A \to B$.

(ii) each object A has an identity morphism $1_A : A \to A$.

(iii) for any pair of morphisms g and f with $textdomain(g) = \text{codomain}(f)$, there is a composite morphism $g \circ f$ with $\text{domain}(g \circ f) = \text{domain}(f)$ and $\text{codomain}(g \circ f) = \text{codomain}(g)$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

These data are subject to two axioms:

1. Associativity: for any $f : A \to B, g : B \to C, h : C \to D$, $(h \circ g) \circ f = h \circ (g \circ f)$, i.e .composite is associative.
2. Unitality: for any $f : A \to B$, $f \circ 1_A = f = 1_B \circ f$, i.e. the identity morphisms are identities for compositions.

As usual with an associative operation, we shall usually omit parentheses when specifying composite morphisms.

Example 1.3. 1. The category of sets consists of the collection of all sets and all functions. We denote it $\mathbf{Set}$.

2. A pointed set is a pair (X, x), where X is a set and $x \in X$ is a distinguished element. A morphism $f : (X, x) \to (Y, y)$ of pointed sets is a function $f : X \to Y$ such that $f(x) = y$. These data define a category, which we denote $\mathbf{Set}_*$.

3. A monoid is a tuple $(M, \cdot, e)$ such that M is a set, $\cdot : M \times M \to M$ is a binary operation on M, and $e \in M$ is a distinguished element that satisfies 1) associativity: for any $x, y, z \in M$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and 2) unitality: for any $x \in M, 1 \cdot x = x = x \cdot 1$.

   A monoid homomorphism $f : (M, \cdot_M, e_M) \to (N, \cdot_N, e_N)$ is a function $f : M \to N$ such that $f(x \cdot_M y) = f(x) \cdot_N f(y)$ for all $x, y \in M$, and $f(e_M) = e_N$. These data assemble into a category $\mathbf{Mon}$.

4. A group is a quadruple $(G, \cdot, e, (-)^{-1})$ such that G is a set, $\cdot : G \times G \to G$ is a binary operation, $e \in G$ is a distinguished element, and $(-)^{-1} : G \to G$ is a unary operation such that 1) $(G, \cdot, e)$ is a monoid, and 2) for any $x \in G, x^{-1} \cdot x = e = x \cdot x^{-1}$.

   A group homomorphism $f : G \to H$ is a function such that a) for all $x, y \in G$, $f(x \cdot y) = f(x) \cdot f(y)$, b) $f(e) = e$, and c) $f(x^{-1}) = f(x)^{-1}$.

   It is easy to show that a) implies b) and c), so a) alone is often taken as the definition of a group homomorphism. These data assemble into a category $\mathbf{Grp}$.

5. A preorder is a pair $(P, \leqslant)$ such that P is a set and $\leqslant$ is a binary relation on P such that 1) reflexivity: for any $x \in P$, $x \leqslant x$, and 2) transitivity: for any $x, y, z \in P$, if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.

   A morphism of preorders $f : P \to Q$ is a function such that for any $x, y \in P$, if $x \leqslant y$, then $f(x) \leqslant f(y)$. These data assemble into a category, denoted $\mathbf{Preord}$.

Just as sets have subsets, categories have subcategories.

Definition 1.4 (Subcategory)**.** Let $\mathscr{C}$ be a category. A subcategory $\mathscr{D}$ consists of a collection of objects of $\mathscr{C}$ and a collection of morphisms of $\mathscr{D}$ such that

1. Closed under domain/codomain: if $f : A \to B$ is in $\mathscr{D}$, so too are A and B.
2. Closed under $\circ$: if $f : A \to B$ and $g : B \to C$ are in $\mathscr{D}$, then so is $g \circ f$.
3. Contains identities: if $A \in \mathscr{D}$, then so is 1_A.

If $\mathscr{D}$ is a subcategory of $\mathscr{C}$, then the category structure on $\mathscr{C}$ restricts to a category structure on $\mathscr{D}$ with the same identity morphisms and composition operation.

Example 1.5.
1. The collection of all finite sets and all set maps between them forms a category $\mathbf{FinSet}$. It is a subcategory of $\mathbf{Set}$.
2. A commutative monoid is a monoid $(M, \cdot, e)$ such that $\cdot$ is commutative, i.e. for any $x, y \in M$, $x \cdot y = y \cdot x$. The collection of all commutative monoids, together with all monoid homomorphisms between them forms a subcategory $\mathbf{CMon}$ of $\mathbf{Mon}$.
3. An Abelian group is a group $(G, \cdot, e, (-)^{-1})$ such that $\cdot$ is commutative. The collection of all Abelian groups and group homomorphisms between them forms a subcategory $\mathbf{Ab}$ of $\mathbf{Grp}$.
4. A poset is a preorder $(P, \leqslant)$ such that $\leqslant$ is antisymmetric, i.e. for any $x, y \in P$, if $x \leqslant y$ and $y \leqslant x$, then $x = y$. The collection of all posets and order-preserving maps between them forms a subcategory $\mathbf{Pos}$ of $\mathbf{Preord}$.

In each of these examples, we take a subcollection of objects and all morphisms between them. Such subcategories have a name.

Definition 1.6 (Full). A subcategory $\mathscr{D} \subseteq \mathscr{C}$ is full if for any $A, B \in \mathscr{D}$, every morphism $f : A \to B$ in $\mathscr{C}$ is also in $\mathscr{D}$.

Note that note all subcategories have this form.

Example 1.7. Let $\mathbf{Set}_{\text{inj}}$, $\mathbf{Set}_{\text{surj}}$ and $\mathbf{Set}_{\text{bij}}$ be the categories whose objects are all sets, and whose morphisms are injective, surjective, and bijective functions, respectively. The three categories are subcategories of $\mathbf{Set}$.

These three categories contain all objects of $\mathbf{Set}$. Such subcategories also have a name.

Definition 1.8 (Wide). A subcategory $\mathscr{D}$ of $\mathscr{C}$ is wide if $\mathscr{D}$ contains all objects of $\mathscr{C}$.

Of course, not all subcategories are full or wide, e.g. $\mathbf{FinSet}_{\text{bij}}$ is neither.

Most of the categories we have considered thus far are collections of structured sets, together with the structure-preserving functions between them. Thus, it makes sense to ask whether we can encode familiar properties of functions in categorical terms. The theory would be of limited use if we could not do this and here are some relevant definitions:

Definition 1.9 (Isomorphism). Suppose $\mathscr{C}$ is a category. A morphism $f : A \to B$ in $\mathscr{C}$ is an isomorphism if it has a two-sided inverse, i.e. if there is a morphism $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Two objects $A, B \in \mathscr{C}$ are isomorphic if there is an isomorphism between them, in which case one writes $A \cong B$.

Example 1.10.
1. In each of the categories $\mathbf{Set}, \mathbf{Set}_*, \mathbf{Mon}, \mathbf{CMon}, \mathbf{Grp}, \mathbf{Ab}$, an isomorphism is a bijective set map that preserves all structures (operations and distinguished elements).
2. In each of the categories $\mathbf{Preord}$ and $\mathbf{Pos}$, an isomorphism $f : P \to Q$ is a bijective set map such that $x \leqslant y$ if and only if $f(x) \leqslant f(y)$.

Thus, in these cases, the purely categorical conditions that a morphism $f : A \to B$ has a two-sided inverse encodes the fact that f sets up a bijective correspondence between the elements of A and B, which identifies the structure on A and B.

There are also categorical notions of injections and surjections.

Definition 1.11 (Monomorphism, Epimorphism). Let $\mathscr{C}$ be a category. A morphism $f : A \to B$ in $\mathscr{C}$ is a monomorphism if for any $T \in \mathscr{C}$ and $h, k : T \to A$, if $fh = fk$, then $h = k$.

Similarly, a morphism $f : A \to B$ in $\mathscr{C}$ is an epimorphism if for any $T \in \mathscr{C}$ and $h, k : B \to T$, if $hf = kf$, then $h = k$.

Example 1.12. Suppose $f : A \to B$ is a function between two sets A and B, i.e. a morphism in $\mathbf{Set}$. Then f is a monomorphism if and only if f is injective, and f is an epimorphism if and only if f is surjective.

We can actually say a bit more in this case. If $r : A \to B$ is an epimorphism in $\mathbf{Set}$, then there is a function $s : B \to A$ such that $r \circ s = 1_B$. This situation also gets a name.

Definition 1.13 (Section, Retraction, Split). Suppose $r : A \to B$ and $s : B \to A$ are morphisms such that $r \circ s = 1_B$. Then s is a section or right inverse to r and r is a retraction or left inverse to s. In general, we call a morphism $f : A \to B$ a split epimorphism if it has a section, and a split monomorphism if it has a retraction.

Every split monomorphism (respectively, epimorphism) is a monomorphism (respectively, epimorphism), but the converse is not true in general.

Every isomorphism is a split monomorphism and a split epimorphism, so every isomorphism is a monomorphism and an epimorphism, but the converse is not true in general.

Thus, we see that the algebra of compositions in a category can be used to encode familiar properties that functions can possess. In this sense, category theory provides a language for describing mathematics.

However, we can also view categories as algebraic structures in their own right. The next examples show that we may regard a number of familiar mathematical objects as special kinds of categories.

Example 1.14. 1. A discrete category is a category $\mathscr{D}$ with no non-identity morphisms. If $\mathscr{D}$ is a discrete category with only a set of morphisms, then $\mathbf{Ob}(\mathscr{D})$ is a set. If X is a set, then the category $\mathscr{X}$ with object set X and morphisms $\mathbf{Mor}(\mathscr{X}) = \{(a,a) \mid a \in X\}$ with domains and codomains, identities and compositions defined by $\text{domain}(a,a) = \text{codomain}(a,a) = a$, $1_a = (a,a)$, $(a,a) \circ (a,a) = (a,a)$ for all $a \in X$ is a discrete category. In a sense that we shall make precise later, sets and discrete categories with only a set of morphisms are ”basically the same”.

2. A preorder category is a category P with at most one morphism in $P(A,B)$ for any $A, B \in P$. If P is a preorder category with only a set's worth of morphisms, and we define $A \leqslant B$ if and only if there is a morphism $f : A \to B$ for all $A, B \in P$, then $(\mathbf{Ob}(P), \leqslant)$ is a preorder.

   Conversely, if $(P, \leqslant)$ is a preorder, and we define a category P by taking $\mathbf{Ob}(P) = P$ and $\mathbf{Mor}(P) = \{(A,B) \in P \times P \mid A \leqslant B\}$, and setting $\text{domain}(A,B) = A$, $\text{codomain}(A,B) = B$, $\mathbf{id}_A = (A,A)$, and $(B,C) \circ (A,B) = (A,C)$, we obtain a preorder category. Again, we shall later make precise a sense in which preorders and preorder categories with only a set of morphisms are ”basically the same thing”.

3. A poset category is a preorder category P such that for any objects $A, B \in P$, if $A \cong B$ then $A = B$.

   The same constructions given in the previous example convert poset categories with only a set of morphisms into posets, and convert posets into poset categories. Posets and poset categories with only a set of morphisms are ”essentially the same”.

4. A monoid category is a category $\mathscr{M}$ with a single object. If $\mathscr{M}$ is a monoid category with object $A \in \mathscr{M}$ and only a set of morphisms, then $(\mathscr{M}(A,A), \circ, \mathbf{id}_A)$ is a monoid. Conversely, if $(M, \cdot, e)$ is a monoid, then the category BM with a single object $*$ and morphisms M (all of which have domain and codomain $*$) is a category with $\mathbf{id}_* = e$ and $y \circ x = y \cdot x$. Monoids and monoid categories with only a set of morphisms are ”the same thing”.

5. A group category is a monoid category G such that every morphism of such that every morphism of is an isomorphism. The same constructions in the previous example convert group categories with only a set of morphisms into groups and convert groups into group categories. Groups and group categories with only a set of morphisms are ”the same thing”.

The last example justifies the following terminology:

Definition 1.15 (Groupoid)**.** A groupoid is a category in which every morphism is an isomorphism.

Many categories are not groupoids, but all categories contain a maximal sub-groupoid.

Definition 1.16 (Core)**.** The core of a category $\mathscr{C}$ is the wide subcategory of $\mathscr{C}$ whose morphisms are the isomorphisms of $\mathscr{C}$.

Example 1.17. The core of $\mathbf{Set}$ is $\mathbf{Set}_{\text{bij}}$.

We conclude this discussion with some obligatory comments about foundations.

Russell's Paradox from set theory implies that there is no set of all sets. For, if $U = \{X \mid X \text{ is a set}\}$ were a set, then $R = \{X \in U \mid X \notin X\}$ would be a set and $R \in R$ if and only if $R \notin R$. Since one of them must be true, we have a contradiction.

The upshot is that the collection of all sets is ”too large” to be a set. Similar issues appear in category theory, and this is why we used the word ”collection” in the definition of a category.

Our solution is to distinguish between ”small sets” and ”large sets”, i.e. between ”sets” and ”classes”. The collection of all sets is not a set, but it is a class.

We shall not worry too much about these fundamental issues, but we shall introduce some terminology for describing the size of a category.

Definition 1.18 (Small, Locally Small, Large). Let $\mathscr{C}$ be a category. We say that $\mathscr{C}$ is small if $\mathbf{Mor}(\mathscr{C})$ is a set, and we say that $\mathscr{C}$ is locally small if $\mathscr{C}(A,B)$ is a set for all $A, B \in \mathscr{C}$. We say that $\mathscr{C}$ is large if it is not small.

Example 1.19.

1. $\mathbf{Set}$, $\mathbf{Set}_*$, $\mathbf{Mon}$, $\mathbf{CMon}$, $\mathbf{Grp}$, $\mathbf{Ab}$, $\mathbf{Preord}$, $\mathbf{Pos}$ are locally small, but not small.
2. The discrete category associated to a set, the preorder/poset category associated to a preorder/poset, and the monoid/group category associates to a monoid/group are all small categories.

2 FUNCTORS

Thinking categorically, one should always consider a class of mathematical structures together with their structure-preserving maps. Thus, we are compelled to introduce morphisms between categories. These are called functors.

Definition 2.1 (Covariant Functor). A covariant functor $F : \mathscr{C} \to \mathscr{D}$ between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of a pair of functions $F : \mathbf{Ob}(\mathscr{C}) \to \mathbf{Ob}(\mathscr{D})$ and $F : \mathbf{Mor}(\mathscr{C}) \to \mathbf{Mor}(\mathscr{D})$ (both denoted F by the above notation) such that

1. for each $f : A \to B$ in $\mathscr{C}$, we have $Ff : FA \to FB$ in $\mathscr{D}$, i.e. $\text{domain}(Ff) = F\text{domain}(f)$ and $\text{codomain}(Ff) = F\text{codomain}(f)$,
2. for any composable morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}$, $F(g \circ f) = Fg \circ Ff$, and
3. for any $A \in \mathscr{C}$, $F(\mathbf{id}_A) = \mathbf{id}_{FA}$.

In other words, a functor $F : \mathscr{C} \to \mathscr{D}$ is a mapping that preserves domains, codomains, compositions and identities. The composite of two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ is defined by composing "object functions" and "morphism functions", and for any category $\mathscr{C}$, there is an identity functor $1_{\mathscr{C}}$ that is the identity on objects and arrows. Composition of functors is associative, and identity functors serve as identities in terms of compositions. Thus, we obtain a (large) category $\mathbf{Cat}$ of all small categories and functors between them. One can also contemplate a (very large) category of all large categories, sometimes denoted as $\mathbf{CAT}$. To make this last notion precise requires careful examination of theory, which we will sweep under the rug.

Example 2.2.

1. There is a "forgetful" functor $U : \mathbf{Set}_* \to \mathbf{Set}$ that sends a pointed set (X, x) to its underlying set X and a morphism $f : (X,x) \to (Y,y)$ to its underlying function $f : X \to Y$. This functor forgets the distinguished base point.
2. Similarly, there are forgetful functors $\mathbf{Grp} \xrightarrow{U} \mathbf{Mon} \xrightarrow{U} \mathbf{Set}$ and $U : \mathbf{Preord} \to \mathbf{Set}$.

These functors all have the property that if $f, g : A \to B$ and $Uf = Ug$, then $f = g$. Such functors have a name.

Definition 2.3 (Faithful). A functor $F : \mathscr{C} \to \mathscr{D}$ is faithful if for each $A, B \in \mathscr{C}$, $F : \mathscr{C}(A,B) \to \mathscr{D}(FA, FB)$ is injective.

Example 2.4. There is an "inclusion" functor $i : \mathbf{CMon} \to \mathbf{Mon}$ which sends a commutative monoid to itself, but regarded as an object of $\mathbf{Mon}$, and similarly for morphisms. There are analogous inclusions $\mathbf{FinSet} \xrightarrow{i} \mathbf{Set}$, $\mathbf{Ab} \xrightarrow{i} \mathbf{Grp}$, $\mathbf{Pos} \xrightarrow{i} \mathbf{Preord}$.

These inclusion functors have the property that if $g : iA \to iB$, then $g = if$ for some $f : A \to B$. Such functors also have a name.

Definition 2.5 (Full). A functor $F : \mathscr{C} \to \mathscr{D}$ is full if for all $A, B \in \mathscr{C}$, $F : \mathscr{C}(A,B) \to \mathscr{D}(FA, FB)$ is surjective.

In fact, the above inclusion functors are full and faithful.

Example 2.6.

1. There is a covariant power set functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$, that sends a set $X \in \mathbf{Set}$ to its power set $\mathcal{P}(X) \in \mathbf{Set}$ and a function $f : X \to Y$ to its direct image function $f : \mathcal{P}(X) \to \mathcal{P}(Y)$. The function $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ sends $A \subseteq X$ to $f(A) \subseteq Y$, where $f(A) = \{f(a) \mid a \in A\}$.

2. If $\mathscr{C}$ is a locally small category and $A \in \mathscr{C}$, then there is a covariant "hom functor" $\mathscr{C}(A, -) : \mathscr{C} \to \mathbf{Set}$, which sends an object $B \in \mathscr{C}$ to the "hom set" $\mathscr{C}(A, B)$ and a morphism $f : B \to C$ to the function $f_* : \mathscr{C}(A, B) \to \mathscr{C}(A, C)$ that sends $g : A \to B$ to $f \circ g : A \to C$.

   Hom functors will take on greater significance as we progress.

One very useful property of functors is the following:

Proposition 2.7. Suppose $F : \mathscr{C} \to \mathscr{D}$ is a functor.

1. If $f : A \to B$ is an isomorphism in $\mathscr{C}$, then $Ff : FA \to FB$ is an isomorphism in $\mathscr{D}$.
2. If $A, B \in \mathscr{C}$ and $A \cong B$, then $FA \cong FB$.

Proof. 1. Suppose $f : A \to B$ is an isomorphism and let $g : B \to A$ be the inverse to f. Then $Fg : FB \to FA$ is inverse to $Ff : FA \to FB$ because $Fg \circ Ff = F(g \circ f) = F(1_A) = 1_{FA}$ and $Ff \circ Fg = F(f \circ g) = F(1_B) = 1_{FB}$. Therefore, Ff is an isomorphism.

2. Suppose $A \cong B$, then there is an isomorphism $f : A \xrightarrow{\cong} B$ and by the first part, $Ff : FA \xrightarrow{\cong} FB$ is isomorphism. Therefore, $FA \cong FB$.

□

Here is a quick application.

Example 2.8. If G and H are finite groups and $|G| \neq |H|$, then G is not isomorphic to H.

Proof. If $G \cong H$, then apply the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ to an isomorphism $f : G \xrightarrow{\cong} H$ yields a set bijection $f : G \to H$. Thus, G and H have the same number of elements. Taking the contrapositive statement, we see that if $|G| \neq |H|$, then $G \ncong H$. □

This same principle can be used to great effect with more sophisticated functors.

Note that the converse to the previous proposition need not be true, i.e. if $F : \mathscr{C} \to \mathscr{D}$ is a functor, $f : A \to B$ is a morphism in $\mathscr{C}$, and $Ff : FA \to FB$ is an isomorphism, then it need not be the case that f is an isomorphism.

However, the converse is true if $F : \mathscr{C} \to \mathscr{D}$ is full and faithful.

Proposition 2.9. Suppose $F : \mathscr{C} \to \mathscr{D}$ is a full and faithful functor.

1. If $f : A \to B$ is a morphism in $\mathscr{C}$ and $Ff : FA \to FB$ is an isomorphism, then $f : A \to B$ is an isomorphism.
2. If $A, B \in \mathscr{C}$ and $FA \cong FB$, then $A \cong B$.

Proof. 1. Suppose $Ff : FA \to FB$ is isomorphism and let $g : FB \to FA$ is the inverse to Ff. By fullness, there is a morphism $h : B \to A$ such that $Fh = g$. We claim that h is inverse to f. For $F(h \circ f) = Fh \circ Ff = g \circ Ff = 1_{FA} = F(1_A)$ which implies $h \circ f = 1_A$ by the faithfulness of F. Similarly, $f \circ h = 1_B$, so h is inverse to f and f is an isomorphism.

2. Suppose $FA \cong FB$ and let $g : FA \xrightarrow{\cong} FB$ be an isomorphism. By fullness, there is a morphism $f : A \to B$ such that $Ff = g$. Now, by the first part, it follows that f is an isomorphism, and hence $A \cong B$.

□

Definition 2.10 (Conservative, Reflect Isomorphism)**.** A functor that satisfies (1) is said to be conservative or reflect isomorphisms.

There is another common flavor of functor, which reverses the direction of arrows. Before we talk about those, we isolate the process of reversing arrows.

Definition 2.11 (Opposite Category)**.** Suppose that $\mathscr{C}$ is a category. The opposite category $\mathscr{C}^{\text{op}}$ of $\mathscr{C}$ has $\mathbf{Ob}(\mathscr{C}^{\text{op}}) = \mathbf{Ob}(\mathscr{C})$ and $\mathbf{Mor}(\mathscr{C}^{\text{op}}) = \mathbf{Mor}(\mathscr{C})$, but domains, codomains, and compositions are reversed. More precisely, for any $f \in \mathbf{Mor}(\mathscr{C}^{\text{op}}) = \mathbf{Mor}(\mathscr{C})$, we have $\mathbf{domain}_{\mathscr{C}^{\text{op}}}(f) = \mathbf{codomain}_{\mathscr{C}}(f)$ and have $\mathbf{codomain}_{\mathscr{C}^{\text{op}}}(f) = \mathbf{domain}_{\mathscr{C}}(f)$. (i.e. $f : A \to B$ in $\mathscr{C}^{\text{op}}$ if and only if $f : B \to A$ in $\mathscr{C}$) For any $A \in \mathscr{C}$, we have $\mathbf{id}_{\mathscr{C}^{\text{op}},A} = \mathbf{id}_{\mathscr{C},A}$, and if $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathscr{C}^{\text{op}}$, so that $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathscr{C}$, then $g \circ_{\mathscr{C}^{\text{op}}} f = f \circ_{\mathscr{C}} g$.

Note that $\mathscr{C}^{\text{op}}$ contains precisely the same information as $\mathscr{C}$, just packaged differently.

One sometimes writes f^{op} in place of f when regarding $f \in \mathbf{Mor}(\mathscr{C})$ as a morphism in $\mathscr{C}^{\text{op}}$.

If $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor, then there is an induced covariant functor $F^{\text{op}} : \mathscr{C}^{\text{op}} \to \mathscr{D}^{\text{op}}$ defined by the same functions on objects and morphisms as F, i.e. $F^{\text{op}}A = FA$ and $F^{\text{op}}(f^{\text{op}}) = Ff$. So F^{op} contains the same data as F. Moreover, $(G \circ F)^{\text{op}} = G^{\text{op}} \circ F^{\text{op}}$ and $(1_{\mathscr{C}})^{\text{op}} = 1_{\mathscr{C}^{\text{op}}}$.

Thus, we obtain a functor $(-)^{\text{op}} : \mathbf{Cat} \to \mathbf{Cat}$ which has the property that $(-)^{\text{op}} \circ (-)^{\text{op}} = 1_{\mathbf{Cat}}$. Similarly for $\mathbf{CAT}$.

As noted above, the functor $F^{\text{op}} : \mathscr{C}^{\text{op}} \to \mathscr{D}^{\text{op}}$ contains the same information as $F : \mathscr{C} \to \mathscr{D}$. However, we obtain something new if we consider a functor $\mathscr{C}^{\text{op}} \to \mathscr{D}$.

Definition 2.12 (Contravariant Functor)**.** A contravariant functor $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor $F : \mathscr{C}^{\text{op}} \to \mathscr{D}$. Spelled out, such a functor consists of a pair of functions $F : \mathbf{Ob}(\mathscr{C}) \to \mathbf{Ob}(\mathscr{D})$ and $F : \mathbf{Mor}(\mathscr{C}) \to \mathbf{Mor}(\mathscr{D})$ such that

1. For each $f : A \to B$ in $\mathscr{C}$, we have $Ff : FB \to FA$ in $\mathscr{D}$,
2. For all $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathscr{C}$, we have $F(g \circ f) = Ff \circ Fg$, and
3. For all $A \in \mathscr{C}$, we have $F(1_A) = 1_{FA}$.

Example 2.13.

1. There is a contravariant power set functor $\mathcal{P} : \mathbf{Set}^{\text{op}} \to \mathbf{Set}$, which sends a set X to $\mathcal{P}(X)$ and a function $f : X \to Y$ to the inverse image function $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$. Here if $B \subseteq Y$, then $f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X$.
2. If $\mathscr{C}$ is a locally small category, then for any $B \in \mathscr{C}$, there is a contravariant hom functor $\mathscr{C}(-, B) : \mathscr{C}^{\text{op}} \to \mathbf{Set}$, which sends an object $A \in \mathscr{C}$ to $\mathscr{C}(A, B)$ and a morphism $f : A \to A'$ in $\mathscr{C}$ to $f^* : \mathscr{C}(A', B) \to \mathscr{C}(A, B)$ by taking $g : A' \to B$ to $g \circ f : A \to B$.

Besides enabling a conceptual definition of contravariant functors, the opposite category construction also has an important theoretical consequence.

Suppose we have proven a theorem of the form ”for all categories $\mathscr{C}$, [something] is true in $\mathscr{C}$.” Then, in particular, we have proven ”for all categories $\mathscr{C}$, [something] is true in $\mathscr{C}^{\text{op}}$.” However, [something] in $\mathscr{C}^{\text{op}}$ can be reinterpreted as a statement in $\mathscr{C}$, where all arrows have been reversed. This is sometimes called the dual statement and thus, every theorem proven about all categories has a dual interpretation.

We illustrate by examples.

Lemma 2.14. In any category $\mathscr{C}$, if $f : A \to B$ and $g : B \to C$ are monomorphisms, then so is $g \circ f$.

Proof. Suppose $h, k : T \to A$ and $gfh = gfk$, then $fh = fk$ because g is a monomorphism, and then $h = k$ because f is a monomorphism. Therefore, gf is a monomorphism by definition. ☐

Here is the dual result:

Lemma 2.15. In any category $\mathscr{C}$, if $f : A \to B$ and $g : B \to C$ are epimorphisms, then so is $g \circ f$.

Proof. Observe that a morphism $f : A \to B$ is epimorphism in $\mathscr{C}$ if and only if $f : B \to A$ is monomorphism in $\mathscr{C}^{\text{op}}$. Thus, if $f : A \to B$ and $g : B \to C$ are epimorphisms in $\mathscr{C}$, then $f : B \to A$ and $g : C \to B$ are monomorphisms in $\mathscr{C}^{\text{op}}$, so $f \circ_{\mathscr{C}^{\text{op}}} = g \circ_{\mathscr{C}} f$ is a monomrophism in $\mathscr{C}^{\text{op}}$, and thus $g \circ_{\mathscr{C}} f$ is an epimorphism in $\mathscr{C}$. ☐

Of course, one can always prove a dual theorem directly by reversing all arrows and repeating the proof of the original theorem, but the point is that by categorical duality, the dual theorem always comes for free.

We return to our discussion of functors, and introduce another flavor of these mappings.

Definition 2.16 (Product)**.** If $\mathscr{C}$ and $\mathscr{D}$ are categories, then the product $\mathscr{C} \times \mathscr{D}$ of $\mathscr{C}$ and $\mathscr{D}$ has $\mathbf{Ob}(\mathscr{C} \times \mathscr{D}) = \mathbf{Ob}(\mathscr{C}) \times \mathbf{Ob}(\mathscr{D})$ and $\mathbf{Mor}(\mathscr{C} \times \mathscr{D}) = \mathbf{Mor}(\mathscr{C}) \times \mathbf{Mor}(\mathscr{D})$. The domain and codomain of (f_1, f_2) are $\mathbf{domain}(f_1, f_2) = (\mathbf{domain}(f_1), \mathbf{domain}(f_2))$ and $\mathbf{codomain}(f_1, f_2) = (\mathbf{codomain}(f_1), \mathbf{codomain}(f_2))$, the identity on (A_1, A_2) is $\mathbf{id}_{(A_1, A_2)} = (\mathbf{id}_{A_1}, \mathbf{id}_{A_2})$, and if $(f_1, f_2) : (A_1, A_2) \to (B_1, B_2)$, and $(g_1, g_2) : (B_1, B_2) \to (C_1, C_2)$, then $(g_1, g_2) \circ (f_1, f_2) = (g_1 \circ f_1, g_2 \circ f_2)$.

Note that there is a projection functor $\pi : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ that sends $(A_1, A_2) \mapsto A_1$ and $(f_1, f_2) \mapsto f_1$. Similarly for $\mathscr{D}$.

Definition 2.17 (Bifunctor)**.** A bifunctor is a functor of the form $F : \mathscr{C} \times \mathscr{D} \to \mathscr{E}$, where $\mathscr{C}, \mathscr{D}, \mathscr{E}$ are categories.

Example 2.18.

1. The mapping $\times : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$, which sends a pair of sets (X, Y) to their product $X \times Y$, and a pair of functions $(f, g) : (X_1, Y_1) \to (X_2, Y_2)$ to the function $f \times g : X_1 \times Y_1 \to X_2 \times Y_1$ which sends (x, y) to $(f(x), g(y))$, is a bifunctor.

2. If $\mathscr{C}$ is a locally small category, then there is a hom bifunctor $\mathscr{C}(-, -) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$, which sends a pair (A, B) of objects of $\mathscr{C}$ to $\mathscr{C}(A, B)$, and a pair of morphisms $(f : A' \to A, g : B \to B')$ in $\mathscr{C}$ to $g \circ (-) \circ f : \mathscr{C}(A, B) \to \mathscr{C}(A', B')$ that sends h to $g \circ h \circ f$.

3 Natural Transformations

Recall that a "natural map", such as the bijection $\mathcal{P}(X) \xrightarrow{\cong} \mathbf{Set}(X, \{0, 1\}$ from A to $\mathcal{X}_A$ that sends a subset $A \subseteq X$ to its characteristic function $\mathcal{X}_A : X \to \{0, 1\}$, is a way of mapping between two constructions that can be performed on X (in this case, $X \mapsto \mathcal{P}(X)$ and $X \mapsto \mathbf{Set}(X, \{0, 1\})$).

Constructions such as these can be formalized with the notion of a functor, but one should now ask how to formalize the imprecise notion of "naturality". Eilenberg and Mac Lane's answer was to require comptability of the natural maps with the action of the functors as morphisms.

Definition 3.1 (Natural Transformation, Natural Isomorphism)**.** Suppose that $\mathscr{C}$ and $\mathscr{D}$ are categories and $F, G : \mathscr{C} \to \mathscr{D}$ are functors. A natural transformation $\eta : F \Rightarrow G$ is a tuple of morphisms $\eta_C : FC \to GC$ in $\mathscr{D}$, indexed by the objects $C \in \mathscr{C}$, such that for every morphism $f : C \to C'$ in $\mathscr{C}$, the square

$$\begin{tikzcd}
FC \arrow[r, "\eta_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\
FC' \arrow[r, "\eta_{C'}"'] & GC'
\end{tikzcd}$$

commutes, i.e. $Gf \circ \eta_C = \eta_{C'} \circ Ff$. We refer to the morphisms η_C as the components of η.

A natural isomorphism is a natural transformation $\eta : F \Rightarrow G$ such that each component of η is an isomorphism. In this case, we may write $\eta : F \cong G$.

It is common to see a natural transformation $\eta : F \Rightarrow G$ depicted by the "globular diagram":

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\eta \Downarrow"] & \mathscr{D}
\end{tikzcd}$$

Example 3.2.

1. Let $1_{\mathbf{Set}}, \mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ by the identity functor and the covariant power set functor, respectively. Then the functions $\sigma_X : X \to \mathcal{P}(X)$, defined by $\sigma_X(a) = \{a\}$, are the components of a natural transformation $\sigma : 1_{\mathbf{Set}} \Rightarrow \mathcal{P}$.

2. Let $\mathcal{P}, \mathbf{Set}(-, \{0, 1\}) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ be the contravariant power set functor and the contravariant hom functor associated with $\{0, 1\} \in \mathbf{Set}$. Then the set bijections $k_X : \mathcal{P}(X) \xrightarrow{\cong} \mathbf{Set}(X, \{0, 1\})$, defined by $k_X(A) = \mathcal{X}_A$ are the components of a natural isomorphism $k : \mathcal{P} \cong \mathbf{Set}(-, \{0, 1\})$.

3. Let $\times : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ be the Cartesian product bifunctor, and let $\times_{tw}$ be the "twisted product" bifunctor that sends a pair of sets (X, Y) to $Y \times X$ and a pair of functions $(f, g) : (X, Y) \to (X', Y')$ to the function $g \times f : Y \times X \to Y' \times X'$ which sends $(y, x) \mapsto (g(y), f(x))$. Then the set bijections $\tau_{X,Y} : X \times Y \xrightarrow{\cong} Y \times X$ defined by $\tau_{X,Y}(x, y) = (y, x)$ are the components of a natural isomorphism $\tau : \times \cong \times_{tw}$

4. Suppose $\mathscr{C}$ is a locally small category and $f : A \to B$ is a morphism in $\mathscr{C}$. Let $\mathscr{C}(-, A), \mathscr{C}(-, B) : \mathscr{C}^{\text{op}} \to \textbf{Set}$ be the contravariant hom functors associated to A and B. Then the set maps $f_* : \mathscr{C}(X, A) \to \mathscr{C}(X, B)$, defined by $f_*(g : X \to A) = f \circ g : X \to B$, are the components of a natural transformation $f_* : \mathscr{C}(-, A) \Rightarrow \mathscr{C}(-, B)$.

We now consider the algebra of natural transformations. Given three parallel functions $F, G, H : \mathscr{C} \to \mathscr{D}$ and natural transformations $\zeta : F \Rightarrow G, \eta : G \Rightarrow H$, we have

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=50, "\overset{\Downarrow\zeta}{F}"] \arrow[r, "G"] \arrow[r, bend right=50, "\underset{\Downarrow\eta}{H}"'] & \mathscr{D}
\end{tikzcd}$$

As suggested by the diagram, we can "paste" η and ζ together to get a new natural transformation.

Definition 3.3 (Vertical Composite). Suppose $F, G, H : \mathscr{C} \to \mathscr{D}$ are parallel functors and $\zeta : F \Rightarrow G$ and $\eta : G \Rightarrow H$ are natural transformations. The vertical composite $\eta \circ \zeta : F \Rightarrow H$ is the natural transformation where components are $(\eta \circ \zeta)_C = \eta_C \circ \zeta_C$ for all $C \in \mathscr{C}$.

Note that this is really a natural transformation: for any $f : C \to C'$ in $\mathscr{C}$, we have

$$\begin{tikzcd}
FC \arrow[r, "\zeta_C"] \arrow[d, "Ff"] & GC \arrow[r, "\eta_C"] \arrow[d, "Gf"] & HC \arrow[d, "Hf"] \\
FC' \arrow[r, "\zeta_{C'}"'] & GC' \arrow[r, "\eta_{C'}"'] & HC'
\end{tikzcd}$$

and by the naturality of ζ and η, $Hf \circ \eta_C \circ \zeta_C = \eta_{C'} \circ Gf \circ \zeta_C = \eta_{C'} \circ \zeta_{C'} \circ Ff$, which means all arrows commute.

Next, note that vertical composites is associative and every functor $F : \mathscr{C} \to \mathscr{D}$ has an identity transformation $1_F : F \Rightarrow F$ defined componentwise by $(1_F)_C = 1_{FC} : FC \to FC$.

Thus, for each pair of categories $\mathscr{C}$ and $\mathscr{D}$, there is a category of functors and natural transformations between $\mathscr{C}$ and $\mathscr{D}$.

Definition 3.4 (Functor Category). For any categories $\mathscr{C}$ and $\mathscr{D}$, the functor category $\mathscr{D}^{\mathscr{C}}$ is the category whose objects are functors $F : \mathscr{C} \to \mathscr{D}$ and whose morphisms are natural transformations between such functors. Vertical composition is the composition operation on $\mathscr{D}^{\mathscr{C}}$.

Given that a natural transformation $\eta : F \Rightarrow G$ is a morphism in $\mathscr{D}^{\mathscr{C}}$, it makes sense to ask what an isomorphism in $\mathscr{D}^{\mathscr{C}}$ is. These are precisely natural isomorphisms.

Proposition 3.5. Suppose $F, G : \mathscr{C} \to \mathscr{D}$ are functors and that $\eta : F \Rightarrow G$ is a natural transformation. Then η is a natural isomorphism if and only if η is an isomorphism in $\mathscr{D}^{\mathscr{C}}$.

As suggested by the terminology "vertical composition", there are other composition operations that we can perform on natural transformations.

First, we explain how to compose a natural transformation with a functor.

Definition 3.6 (Whiskered Transformation, Whiskered Transform). Given

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\eta \Downarrow"] & \mathscr{D} \arrow[r, "H"] & \mathscr{E}
\end{tikzcd}$$

the whiskered transformation $H\eta : HF \Rightarrow HG$ is defined by $(H\eta)_C = H(\eta_C)$.

Given

$$\begin{tikzcd}
\mathscr{B} \arrow[r, "E"] & \mathscr{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\eta \Downarrow"] & \mathscr{D}
\end{tikzcd}$$

the whiskered transformation $\eta E : FE \Rightarrow GE$ is defined by $(\eta E)_B = \eta_{EB}$ for all $B \in \mathscr{B}$.

Next, we explain how to compare natural transformations ”horizontally”. Suppose given natural transformations

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\zeta \Downarrow"] & \mathscr{D} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\eta \Downarrow"] & \mathscr{E}
\end{tikzcd}$$

so that $\zeta_C : FC \to GC$ in $\mathscr{D}$ for all $C \in \mathscr{C}$. Then the naturality of η implies that for any $C \in \mathscr{C}$, the diagram

$$\begin{tikzcd}
HFC \arrow[r, "\eta_{FC}"] \arrow[d, "H\zeta_C"] & KFC \arrow[d, "K\zeta_C"] \\
HGC \arrow[r, "\eta_{GC}"] & KGC
\end{tikzcd}$$

commutes.
i.e. $K\zeta_C \circ \eta_{FC} = \eta_{GC} \circ H\zeta_C$. Hence, $K\zeta \circ \eta F = \eta G \circ H\zeta$.
We define the horizontal composite of η and ζ to be this common value: $\eta \circ \zeta = K\zeta \circ \eta F = \eta G \circ H\zeta$.
Note that vertical and horizontal compositions can ”interchange” in the following sense:
Given

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=80, "F"] \arrow[r, "G"] \arrow[r, bend right=80, "H"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\zeta"] \arrow[r, phantom, shift right=3ex, "\Downarrow\eta"] & \mathscr{D} \arrow[r, bend left=80, "J"] \arrow[r, "K"] \arrow[r, bend right=80, "L"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3ex, "\Downarrow\delta"] & \mathscr{E}
\end{tikzcd}$$

we have $(\delta \circ \gamma) \circ (\beta \circ \alpha) = (\delta \circ \beta) \circ (\gamma \circ \alpha)$, which means the following two diagrams

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\beta \circ \alpha \Downarrow"] & \mathscr{D} \arrow[r, bend left=50, "J"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\delta \circ \gamma \Downarrow"] & \mathscr{E}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=80, "KG"] \arrow[r, "JF"] \arrow[r, bend right=80, "LH"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\gamma\circ\alpha"] \arrow[r, phantom, shift right=3ex, "\Downarrow\delta\circ\beta"] & \mathscr{D}
\end{tikzcd}$$

are equivalent.
This can be thought of as a 2D version of association.
All told, categories, functors, and natural transformations assemble into a 2D algebraic structure called a 2-category.
The definition codifies the sort of structure we have seen, but we shall omit the precise details.

4 EQUIVALENCE OF CATEGORIES

Suppose $\mathscr{C}$ and $\mathscr{D}$ are categories. What does it mean for $\mathscr{C}$ and $\mathscr{D}$ to be "the same"? To answer this question, let us back up.

Suppose X and Y are sets. Then $X = Y$ if and only if they have the same elements. This is often what we mean when we say that X and Y are the same set.

Now suppose that X and Y are sets equipped with additional structure (e.g. operations, distinguished elements, relations). What does it mean for X and Y to be the same?

In the strictest sense, X and Y are the same if they are equal, i.e. if they have equal underlying sets and all of the corresponding pieces of structure are equal. However, that is a very restrictive notion. Experience has shown that it is often useful to regard X and Y as "the same" if they are isomorphic, i.e. if there are inverses, structure-preserving functions $f : X \to Y$ and $g : Y \to X$. In this case, X and Y have the same structural properties, even if they are not literally equal.

By analogy, one might posit that two categories $\mathscr{C}$ and $\mathscr{D}$ should be regarded as "the same" if they are isomorphic, i.e. if there is a pair of inverse functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$. This is sometimes the right notion, but it can also be too restrictive. To see why, note that F and G being inverses means that

- $GFC = C$ for all objects $C \in \mathscr{C}$
- $GFf = f$ for all morphisms $f \in \mathscr{C}$
- $FGD = D$ for all objects $D \in \mathscr{D}$
- $FGg = g$ for all morphisms $g \in \mathscr{D}$

Note that we use the equal sign here.

As explained earlier, equality is sometimes too rigid a notion of "sameness" in a category. We often want to treat isomorphic things as the same. Thus, we should relax $=$ to $\cong$ in the above list. In other words, we should not always demand that the functors F and G to be strictly inverses, i.e. that $G \circ F = 1_\mathscr{C}$ and $F \circ G = 1_\mathscr{D}$. We should only require that $G \circ F \cong 1_\mathscr{C}$ and $F \circ G \cong 1_\mathscr{D}$, i.e. that F and G are "inverses up to isomorphism". This is the notion of an equivalence of categories.

Definition 4.1 (Equivalence of Categories)**.** An equivalence of categories consists of functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ together with natural isomorphisms $\eta : 1_\mathscr{C} \cong GF$ and $\varepsilon : FG \cong 1_\mathscr{D}$. Two categories $\mathscr{C}$ and $\mathscr{D}$ are equivalent if there is an equivalence between them, in case we write $\mathscr{C} \simeq \mathscr{D}$.

Note that we have chosen the direction $\eta : 1_\mathscr{C} \cong GF$ and $\varepsilon : FG \cong 1_\mathscr{D}$ to be consistent with the notation for adjunctions, but these directions don't matter in the definition above because η and ε are natural isomorphisms.

Note that if $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are strictly inverses, then there are natural isomorphisms $1_{1_\mathscr{C}} : 1_\mathscr{C} \Rightarrow 1_\mathscr{C} = GF$ and $1_{1_\mathscr{D}} : FG = 1_\mathscr{D} \Rightarrow 1_\mathscr{D}$ and thus $(F, G, 1_{1_\mathscr{C}}, 1_{1\mathscr{D}})$ is an equivalence between $\mathscr{C}$ and $\mathscr{D}$. Thus, $\mathscr{C} \cong \mathscr{D}$ implies $\mathscr{C} \simeq \mathscr{D}$, i.e. $\simeq$ is a weaker notion of equivalence than $\cong$, which is weaker then $=$.

Example 4.2. Let $\mathbf{Set}$ be the category of sets and $\mathbf{Cat}$ be the category of small categories. Let $\mathbf{DCat} \subseteq \mathbf{Cat}$ be the full subcategory of $\mathbf{Cat}$ whose objects are the discrete categories (i.e. those categories with no non-identity morphisms). We shall construct an equivalence of categories between $\mathbf{DCat}$ and $\mathbf{Set}$.

Let $\mathbf{Ob} : \mathbf{DCat} \to \mathbf{Set}$ be the functor that sends $\mathscr{C} \in \mathbf{DCat}$ to its set of objects $\mathbf{Ob}(\mathscr{C}) \in \mathbf{Set}$ and a functor $F : \mathscr{C} \to \mathscr{D}$ in $\mathbf{DCat}$ to its object function $\mathbf{Ob}(F) : \mathbf{Ob}(\mathscr{C}) \to \mathbf{Ob}(\mathscr{D})$. Next let $(-)^{\text{disc}} : \mathbf{Set} \to \mathbf{DCat}$ be the functor that sends a set X to the discrete category X^{disc} with $\mathbf{Ob}(X^{\text{disc}}) = X$, $\mathbf{Mor}(X^{\text{disc}}) = \{(a, a) \mid a \in X\}$, and for all $a \in X$, the domain and codomain of morphism (a, a) are a, with $\mathbf{id}_a = (a, a)$, and $(a, a) \circ (a, a) = (a, a)$.

Given a functor $f : X \to Y$ in $\mathbf{Set}$, we define the functor $f^{\text{disc}} : X^{\text{disc}} \to Y^{\text{disc}}$ by

$$\begin{gathered} f^{\text{disc}}(a) = f(a) \text{ for all } a \in \mathbf{Ob}(X^{\text{disc}}) \\ f^{\text{disc}}(a, a) = (f(a), f(a)) \text{ for all } (a, a) \in \mathbf{Mor}(X^{\text{disc}}) \end{gathered}$$

This construction makes $(-)^{\text{disc}}$ into a functor.

Note that we have that $\mathbf{Ob} \circ (-)^{\text{disc}} = 1_{\mathbf{Set}}$. On the other hand, $(-)^{\text{disc}} \circ \mathbf{Ob} \neq 1_{\mathbf{DCat}}$, but for any discrete category $\mathscr{C}$, there is a natural isomorphism $\eta_\mathscr{C} : \mathscr{C} \xrightarrow{\cong} ((\mathbf{Ob}(\mathscr{C}))^{\text{disc}}$ that is the identity on objects, and which sends $1_A \in \mathscr{C}$ to $(A, A) \in (\mathbf{Ob}(\mathscr{C}))^{\text{disc}}$ for all $A \in \mathscr{C}$. Thus, $(\mathbf{Ob}, (-)^{\text{disc}}, \eta, 1_{1_{\mathbf{Set}}})$ is an equivalence.

It takes a lot of data to specify an equivalence, but there is a one-sided formulation that is sometimes easier to check. We first make a definition.

Definition 4.3 (Essentially Surjective on Objects)**.** A functor $F : \mathscr{C} \to \mathscr{D}$ is essentially surjective on objects if for any object $D \in \mathscr{C}$, there is an object $C \in \mathscr{C}$ such that $FC \cong D$.

Theorem 4.4. A functor $F : \mathscr{C} \to \mathscr{D}$ is a part of an equivalence of categories ($F : \mathscr{C} \rightleftarrows \mathscr{D} : G, \eta : 1_{\mathscr{C}} \cong GF, \varepsilon : FG \cong 1_{\mathscr{D}}$) if and only if F is fully faithful and is essentially surjective on objects.

Proof. Following Emily's advice, we shall leave it as a (good) exercise. ☐

Let us apply this theorem to get some more equivalences of categories.

Example 4.5.

1. Let $\mathbf{Precat}$ be the full subcategory of $\mathbf{Cat}$ whose objects are small preorder categories, and let $\mathbf{Preord}$ be the category of all preorders. There is a functor $\mathbf{Ob} : \mathbf{PreCat} \to \mathbf{Preord}$ that sends
   - (a) a small preorder category $\mathcal{P}$ to the preorder $\mathbf{Ob}(\mathcal{P})$ whose underlying set is $\mathbf{Ob}(\mathcal{P})$ and whose order relation is $A \leqslant B$ if and only if there is a morphism $A \to B$ in $\mathcal{P}$.
   - (b) a functor $F : \mathcal{P} \to \mathcal{Q}$ between preorder categories to the object function $\mathbf{Ob}(F) : \mathbf{Ob}(\mathcal{P}) \to \mathbf{Ob}(\mathcal{Q})$.

   The functor $\mathbf{Ob} : \mathbf{PreCat} \to \mathbf{Preord}$ is fully faithful and essentially surjective on objects, and hence $\mathbf{PreCat} \simeq \mathbf{Preord}$. Similarly, $\mathbf{PosCat} \simeq \mathbf{Pos}$ via $\mathbf{Ob} : \mathbf{PosCat} \to \mathbf{Pos}$, where $\mathbf{PosCat} \subseteq \mathbf{Cat}$ is the full subcategory of small poset categories.

2. Let $\mathbf{MonCat} \subseteq \mathbf{Cat}$ be the full subcategory of small monoid categories. Let $\mathbf{Mon}$ be the category of monoids. There is a functor $\mathbf{Mor} : \mathbf{MonCat} \to \mathbf{Mon}$ that sends
   - (a) a small monoid category $\mathcal{M}$ with single object A to the monoid $(\mathbf{Mor}(\mathcal{M}), \circ, 1_A)$
   - (b) a functor $F : \mathcal{M} \to \mathcal{N}$ between small monoid categories to $\mathbf{Mon}(F) : \mathbf{Mon}(\mathcal{M}) \to \mathbf{Mon}(\mathcal{N})$.

   Then $\mathbf{Mor} : \mathbf{MonCat} \to \mathbf{Mon}$ is fully faithful and essentially surjective on objects, so $\mathbf{MonCat} \simeq \mathbf{Mon}$. Similarly, $\mathbf{GrpCat} \simeq \mathbf{Grp}$ via $\mathbf{Mor}$.

5 UNIVERSALITY AND THE YONEDA LEMMA

Roughly speaking, a ”universal object” is an object that uniquely gives rise to all similarly structured objects. In this note, we shall formalize this notion in several ways. But first, to get the idea, let us look at some examples.

Example 5.1.

1. The set $\{0, 1\}$, together with the subset $\{1\} \subseteq \{0, 1\}$ is a ”universal set equipped with a subset” in the following sense: given any other set X, together with a subset $A \subseteq X$, there is a unique function $f : X \to \{0, 1\}$ such that $A = f^{-1}(1)$ (namely $f = \mathcal{X}_A$), i.e. the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\exists! f"] & \{0,1\} \\
A \arrow[u, hook] & \{1\} \arrow[u, hook] \arrow[l, "f^{-1}"']
\end{tikzcd}$$

   In this sense, every other subset $A \subseteq X$ of every other set X ”comes from” $\{1\} \subseteq \{0, 1\}$ as an inverse image.

2. The set $\{0, 1\}$, together with the ordered pair $(0, 1)$ is a ”universal set equipped with an ordered pair” in the following sense: given any other set X, equipped with an ordered pair $(a, b) \in X \times X$, there is a unique function $f : \{0, 1\} \to X$ such that $(a, b) = (f(0), f(1))$.

   In this sense, every other ordered pair (a, b) of elements in any other set ”comes from” $(0, 1) \in \{0, 1\} \times \{0, 1\}$ as an image.

We shall now give a more sophisticated example, but first a review of equivalence relations and quotient sets.

Recall that if X is a set, then an equivalence relation on X is a binary relation $\sim$ such that

1. Reflexivity: for any $x \in X$, $x \sim x$.
2. Symmetry: for any $x, y \in X$, if $x \sim y$ then $y \sim x$, and
3. Transitivity: for any $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

We say x and y are equivalent if $x \sim y$.

If $\sim$ is an equivalence relation on X and $x \in X$, then the equivalence class of x, denoted $[x]$ is the set of all elements of X that are equivalent to x: $[x] = \{y \in X \mid y \sim x\}$. Therefore, $x \sim y$ if and only if $[x] = [y]$: since $x \sim x$, it follows that $x \in [x]$. Thus, if $[x] = [y]$, then $x \in [x] = [y]$, which implies $x \sim y$. Conversely, if $x \sim y$, then $z \in [x]$ if and only if $z \sim x$ if and only if $z \sim y$ if and only if $z \in [y]$, so that $[x] = [y]$.

The quotient $X/\sim$ of a set X by an equivalence relation $\sim$ is the set of all equivalence classes of elements of X: $X/\sim = \{[x] \mid x \in X\}$. There is a canonical function $\pi : X \to X/\sim$ that sends $x \mapsto [x]$. Note that $x \sim y$ if and only if $[x] = [y]$ if and only if $\pi(x) = \pi(y)$.

In particular, if $x \sim y$, then $\pi(x) = \pi(y)$. We shall now explain a sense in which $\pi : X \to X/\sim$ is universal with this property.

Example 5.2. Let X be a set and $\sim$ be an equivalence relation on X. The set $X/\sim$, together with the canonical projection $\pi : X \to X/\sim$ is a ”universal set equipped with a map that identifies $\sim$-equivalent elements of X”. By this, we mean the following: given any other set Y, together with a function $f : X \to Y$ such that $a \sim b$ implies $f(a) = f(b)$, there is a unique function $\bar{f} : X/\sim \to Y$ such that $f = \bar{f} \circ \pi$, namely $\bar{f}([a]) = f(a)$. Here is the diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\
X/\sim \arrow[ur, dashed, "\exists ! \bar{f}"'] &
\end{tikzcd}$$

Figure 5.1: Universal Property of Quotient Set

In this sense, every function $f : X \to Y$ that sends $\sim$-equivalent elements of X to equal elements of Y ”comes from the canonical projection $\pi : X \to X/\sim$, via composition with a (unique) map $\bar{f} : X/\sim \to Y$.

With these examples in mind, we now explain a close relationship between universality and hom functors. First, an observation:

Proposition 5.3. Let $F : X \to Y$ be a function between two sets. Then f is bijective if and only if

$(\star)$ for all $y \in Y$, there is a unique $x \in X$ such that $f(x) = y$.

Proof. Suppose that f is bijective, and let $y \in Y$. Then since f is surjective, there is $x \in X$ such that $y = f(x)$. Now, if $x' \in X$ is another element such that $f(x') = y$, then $f(x) = y = f(x')$, so $x = x'$ by injectivity. Thus, $(\star)$ is true.

Conversely, suppose that $(\star)$ is true. Given any $y \in Y$, there is a unique $x \in X$ such that $f(x) = y$, so f is surjective. Next, if $f(x) = f(x')$, then taking $y = f(x)$, we see that $x, x' \in X$ are both elements of X that map to y. By the uniqueness of $(\star)$, $x = x'$, so that f is injective. ☐

Now consider the previous examples.

Example 5.4 (Revisited).

1. For any set X, let $\eta_X : \mathbf{Set}(X, \{0,1\}) \to \mathcal{P}(X)$ send $(f : X \to \{0,1\}) \mapsto f^{-1}\{1\}$. This is a natural transformation, and in light of the previous proposition and the universal property of $\{1\} \subseteq \{0,1\}$, each η_X is a bijection. Thus, $\eta : \mathbf{Set}(-, \{0,1\}) \cong \mathcal{P}$, and the universality of $\{1\} \subseteq \{0,1\}$ has been repackaged in the natural isomorphism η.
2. For any set X, let $\eta_X : \mathbf{Set}(\{0,1\}, X) \to X \times X$ send $(f : \{0,1\} \to X) \mapsto (f(0), f(1))$. As in the previous example, the universality of the ordered pair $(0,1) \in \{0,1\} \times \{0,1\}$ translated into the fact that $\eta_X : \mathbf{Set}(\{0,1\}, X) \cong X \times X$ is a natural isomorphism.

3. Let X be a set and $\sim$ be an equivalence relation on X. Define a functor $F : \mathbf{Set} \to \mathbf{Set}$ by $FY = \{f : X \to Y \mid \text{if } a \sim b, \text{ then } f(a) = f(b)\}$ and $F(\varphi : Y \to Z) = (\varphi_* : FY \to FZ, f \mapsto \varphi \circ f)$. Then there is a natural transformation $\eta_Y : \mathbf{Set}(X/\sim, Y) \to FY$ that sends $(g : X/\sim \to Y) \mapsto (g \circ \pi : X \to Y)$ and the universality of $X \to X/\sim$ expresses the fact that $\eta : \mathbf{Set}(X/\sim, -) \cong F$ is a natural isomorphism.

Thus, we see that our examples' universal properties may be reinterpreted as natural isomorphisms between hom functors and another functor related to the universal property.

Our purpose going forward will be to study this correspondence systematically, but first some terminologies.

Definition 5.5 (Representable, Representation).

1. A covariant or contravariant functor F from a locally small category $\mathscr{C}$ to $\mathbf{Set}$ is representable if there is an object $C \in \mathscr{C}$ and a natural isomorphism between F and the hom functor of appropriate variance associated to C, in which case one says F is represented by C.

2. A representation of F is an object $C \in \mathscr{C}$ together with a specified natural isomorphism $\mathscr{C}(C, -) \cong F$ if F is covariant, or $\mathscr{C}(-, C) \cong F$ if F is contravariant.

So our examples show that universal properties give rise to representations.

What about the other direction? Does representations give rise to universal properties? To answer this, we will need to understand the data that goes into defining a natural isomorphism $\mathscr{C}(C, -) \cong F$ or $\mathscr{C}(-, C) \cong F$.

The key observation is that hom functors behave very much like one-dimensional vector spaces. We illustrate through the following analogy.

Consider $\mathbb{R}$ as a vector space over itself, and consider the vector $1 \in \mathbb{R}$. For any other vector $x \in \mathbb{R}$, there is a unique scalar $\lambda \in \mathbb{R}$ such that $x = \lambda \cdot 1$, namely $\lambda = x$. Thus, $\{1\}$ is a basis of $\mathbb{R}$. It follows that for any other vector space V and vector $v \in V$, there is a unique linear transformation $T_v : \mathbb{R} \to V$ such that $T_v(1) = v$, namely $T_v(x) = x \cdot v$. In other words, for any vector space V, there is a natural isomorphism $\mathbf{ev}_1 : \mathbf{Vect}_{\mathbb{R}}(\mathbb{R}, V) \xrightarrow{\cong} U(V)$ that sends $(T : \mathbb{R} \to V) \mapsto T(1)$, where:

- $\mathbf{Vect}_{\mathbb{R}}$ is the category of vector spaces over $\mathbb{R}$ and $\mathbb{R}$-linear transformations.
- $U : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Set}$ is the forgetful functor
- $\mathbf{ev}_1$ is the evaluation operation at $1 \in \mathbb{R}$.

The inverse to $\mathbf{ev}_1$ sends a vector $v \in U(V)$ to the unique linear transformation $T_v : \mathbb{R} \to V$ such that $T_v(1) = v$. Once we know $T_v(1) = v$, linearity forces $T_v(x) = T_v(x \cdot 1) = x \cdot T_v(1) = xv$.

Now suppose $\mathscr{C}$ is a locally small category, $C \in \mathscr{C}$ is an object, and consider the covariant hom functor $\mathscr{C}(C, -) : \mathscr{C} \to \mathbf{Set}$. There is a distinguished element $1_C \in \mathscr{C}(C, C)$ with the property that for all $D \in \mathscr{C}$ and $f \in \mathscr{C}(C, D)$, there is a unique $\lambda : C \to D$ such that $\lambda \circ 1_C = f$, namely $\lambda = f$.

In this sense, $\{1_C\}$ is a "basis" of $\mathscr{C}(C, -)$.

This has consequences similar to those above, which are the content of the Yoneda Lemma.

Lemma 5.6. (Yoneda, Version I) Let $\mathscr{C}$ be a locally small category and $C \in \mathscr{C}$. Then for any functor $F : \mathscr{C} \to \mathbf{Set}$, there is a bijection

$$\mathbf{ev}_1 : \mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C, -), F) \xrightarrow{\cong} FC,$$

which is natural in C and F. The function $\mathbf{ev}_1$ sends a natural transformation $\eta : \mathscr{C}(C, -) \Rightarrow F$ to $\eta_C(1_C) \in F$. The inverse to $\mathbf{ev}_1$ sends an element $x \in FC$ to the natural transformation $\eta(x) : \mathscr{C}(C, -) \Rightarrow F$, where D-component is $\mathscr{C}(C, D) \to FD$ that sends $f : C \to D$ to $Ff(x)$ (think of this as $f \cdot x$).

Proof. We first show that this is a bijection.

We begin by showing $\mathbf{ev}_1$ and $x \mapsto \eta(x)$ are inverse functions between $\mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C, -), F)$ and FC. To start, note that if $\eta : \mathscr{C}(C, -) \Rightarrow F$, then $\eta_C : \mathscr{C}(C, C) \to FC$, so that $\mathbf{ev}_1(\eta) = \eta_C(1_C) \in FC$. Thus, $\mathbf{ev}_1 : \mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C, -), F) \to FC$ is a function. Now suppose $x \in FC$. Then for any $f : C \to D$ in $\mathscr{C}$, $Ff : FC \to FD$, so $Ff(x) \in FD$. Thus, $\eta(x)_D : \mathscr{C}(C, D) \to FD$ that sends $f : C \to D$ to $Ff(x)$ is a function, which is natural in D by the functionality of F. Thus, we also have a function $FC \to \mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C, -), F)$ that sends x to $\eta(x)$.

Now suppose $\eta : \mathscr{C}(C, -) \Rightarrow F$. We must show that $\eta(\eta_C(1_C)) = \eta$. Consider D-components. The function $\eta(\eta_C(1_C))_D : \mathscr{C}(C, D) \to FD$ sends $f : C \to D$ to $Ff(\eta_C(1_C)) = \eta_D(f_*(1_C)) = \eta_D(f)$. Now by the naturality of η, the following diagram commutes:

$$\begin{tikzcd}
\mathscr{C}(C,C) \arrow[r, "\eta_C"] \arrow[d, "f_*"'] & FC \arrow[d, "Ff"] \\
\mathscr{C}(C,D) \arrow[r, "\eta_D"] & FD
\end{tikzcd}$$

Therefore, $\eta(\eta_C(1_C)) = \eta$. Next, suppose that $x \in FC$, then $\eta(x)_C(1_C) = F(1_C)(x) = 1_{FC}(x) = x$. This proves that $\mathbf{ev}_1$ and $x \mapsto \eta(x)$ are inverses.

We now show the naturality.

First we consider the naturality in C. Let $f : C \to D$ in $\mathscr{C}$. Then $Ff : FC \to FD$ in $\mathbf{Set}$, we have a natural transformation $f^* : \mathscr{C}(D,-) \Rightarrow \mathscr{C}(C,-)$, and we obtain the square

$$\begin{tikzcd}
\mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C,-),F) \arrow[r, "\mathbf{ev}_1"] \arrow[d, "(f^*)^*"'] & FC \arrow[d, "Ff"] \\
\mathbf{Set}^{\mathscr{C}}(\mathscr{C}(D,-),F) \arrow[r, "\mathbf{ev}_1"'] & FD
\end{tikzcd}$$

Let $\eta : \mathscr{C}(C,-) \Rightarrow F$. Note that $Ff(\eta_C(1_C))$ is equal to $\eta_D(f)$ by the naturality of η. On the other hand, $(\eta \circ f^*)_D(1_D) = \eta_D(f^*(1_D)) = \eta_D(f)$ as well. Therefore, $\mathbf{ev}_1$ is natural in C.

Now we consider the naturality in F. Let $\theta : F \Rightarrow G$, so that we have a square

$$\begin{tikzcd}
\mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C,-),F) \arrow[r, "\mathbf{ev}_1"] \arrow[d, "\theta_*"'] & FC \arrow[d, "\theta_C"] \\
\mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C,-),G) \arrow[r, "\mathbf{ev}_1"'] & GC
\end{tikzcd}$$

Let $\eta : \mathscr{C}(C,-) \Rightarrow F$. Note that $\theta_C(\eta_C(1_C)) = (\theta_C \circ \eta_C)(1_C) = (\theta \circ \eta)_C(1_C)$. Therefore, the diagram also commutes, so $\mathbf{ev}_1$ is natural in F. This concludes the proof. $\square$

Thus, natural transformation $\eta : \mathscr{C}(C,-) \Rightarrow F$ are in natural bijection with element $x \in FC$, but our original goal was to understand what natural isomorphisms $\mathscr{C}(C,-) \cong F$ correspond to.

Corollary 5.7. Let $\mathscr{C}$ be a locally small category, $C \in \mathscr{C}$ be an object, and $F : \mathscr{C} \to \mathbf{Set}$ be a functor. Then the bijections in the Yoneda Lemma restrict to bijections (correspondences)

$$\mathbf{ev}_1 : \mathbf{iso}(\mathscr{C}(C,-),F) \rightleftarrows \{x \in FC \mid (\star) \text{ for any } D \in \mathscr{C} \text{ and } y \in FD, \text{ there is a unique } f : C \to D \text{ such that } Ff(x) = y\}$$

Proof. Suppose $\eta : \mathscr{C}(C,-) \cong F$. Then $\eta_C(1_C) \in FC$, and we first verify that it has property $(\star)$. So suppose that $D \in \mathscr{C}$ and $y \in FD$, then since η is an isomorphism, we have a bijection $\eta_D : \mathscr{C}(C,D) \xrightarrow{\cong} FD$, so there is a unique $f : C \to D$ such that $\eta_D(f) = y$. However, $\eta_D(f) = \eta_D(f_*(1_C)) = Ff(\eta_C(1_C))$ by naturality, so $\eta_C(1_C)$ has property $(\star)$. Thus, $\mathbf{ev}_1$ is a function between the displayed sets.

Next, suppose $x \in FC$ has property $(\star)$, and define $\eta(x) : \mathscr{C}(C,-) \Rightarrow F$ by $\eta(x)_D(f) = Ff(x)$. Then $\eta(x)_D$ is bijective for all D by $(\star)$, and thus $\eta(x) : \mathscr{C}(C,-) \cong F$.

It follows that the inverse functions in the Yoneda Lemma restrict to a pair of inverse functions between the displayed structure listed in the corollary. $\square$

With this in mind, we make a definition.

Definition 5.8 (Universal Element)**.** Suppose $F : \mathscr{C} \to \mathbf{Set}$. A (covariant) universal element of F is a pair $(C \in \mathscr{C}, x \in FC)$ such that for any pair $(D \in \mathscr{C}, y \in FD)$, there is a unique $f : C \to D$ such that $Ff(x) = y$.

Corollary 5.9. Let $\mathscr{C}$ be a locally small category and $F : \mathscr{C} \to \mathbf{Set}$. Then there is a bijective correspondence between representations of F and universal elements of F given by

$$\mathbf{ev}_1 : (C \in \mathscr{C}, \eta : \mathscr{C}(C,-) \cong F) \mapsto (C \in \mathscr{C}, \eta_C(1_C) \in FC).$$

We have already seen a few universal elements.

Example 5.10.

1. The universal element associated to the representation

$$(\{0,1\}, \mathbf{Set}(\{0,1\}, X) \to X \times X : f \mapsto (f(0), f(1)))$$

   is $(\{0,1\}, (0,1) \in \{0,1\} \times \{0,1\})$, i.e. a universal set with an ordered pair.

2. If X is a set and $\sim$ is an equivalence relation on X with projection $\pi : X \to X/\sim$, then the universal element associated to the representation

$$(X/\sim, \mathbf{Set}(X/\sim, Y) \to \{f : X \to Y \mid a \sim b \Rightarrow f(a) = f(b)\} : g \mapsto g \circ \pi)$$

   is $(X/\sim, \pi : X \to X/\sim)$, i.e. a universal set with a map that identifies $\sim$-equivalent elements of X.

With these examples in mind, we define what it means for an object of a category to have a universal property.

Definition 5.11 (Universal Property). Let $\mathscr{C}$ be a locally small category. A (covariant) universal property of an object $C \in \mathscr{C}$ is a representable functor $F : \mathscr{C} \to \mathbf{Set}$, together with a universal element (C, x) that corresponds to a natural isomorphism $\mathscr{C}(C, -) \cong F$. In such a case, we say that "C is a universal object of $\mathscr{C}$ equipped with an element of F".

We have seen that representations and universal elements are essentially the same thing.

We now give another interpretation of universal elements that is ultimately more concise. We require a few more definitions.

Definition 5.12 (Category of Elements). Let $F : \mathscr{C} \to \mathbf{Set}$ be a functor. The category of elements of F, denoted $\int F$, is the category with

- objects as pairs $(C \in \mathscr{C}, x \in FC)$ and
- morphisms as $f : (C, x) \to (D, y)$ which is a morphism $f : C \to D$ in $\mathscr{C}$ such that $Ff(x) = y$.

Observe that (C, x) is a universal element of F if and only if

$$(\star) \text{ for any } (D, y) \in \textstyle\int F, \text{ there is a unique } (C, x) \to (D, y) \text{ in } \textstyle\int F.$$

This motivates the following definition.

Definition 5.13 (Initial). Let $\mathscr{C}$ be a category. An object $C \in \mathscr{C}$ is initial if, for any $D \in \mathscr{C}$, there is a unique morphism $C \to D$ in $\mathscr{C}$.

Proposition 5.14. Let $F : \mathscr{C} \to \mathbf{Set}$ be a functor.

1. (Covariant) universal elements of F are precisely initial objects of $\int F$.
2. F has a (covariant) universal element if and only if $\int F$ has an initial object.

Thus, we have three perspectives on universality. If $\mathscr{C}$ is a locally small category and $C \in \mathscr{C}$, then a (covariant) universal property of C is equivalently:

1. a functor $F : \mathscr{C} \to \mathbf{Set}$ together with a universal element $(C \in \mathscr{C}, x \in FC)$.
2. a functor $F : \mathscr{C} \to \mathbf{Set}$ together with a representation $\eta : \mathscr{C}(C, -) \cong F$.
3. a functor $F : \mathscr{C} \to \mathbf{Set}$ together with an initial object $(C, x) \in \int F$.

We conclude by recording the dual of the preceding discussions.

Lemma 5.15 (Yoneda, Version I). Let $\mathscr{C}$ be a locally small category and $C \in \mathscr{C}$. Then for any functor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$, there is a bijection

$$\mathbf{ev}_1 : \mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}(\mathscr{C}(-, C), F) \xrightarrow{\cong} FC,$$

which is natural in C and F. The function $\mathbf{ev}_1$ sends a natural transformation $\eta : \mathscr{C}(-, C) \Rightarrow F$ to $\eta_C(1_C) \in F$. The inverse to $\mathbf{ev}_1$ sends an element $x \in FC$ to the natural transformation $\eta(x) : \mathscr{C}(-, C) \Rightarrow F$, where D-component is $\mathscr{C}(D, C) \to FD$ that sends $f : D \to C$ to $Ff(x)$.

Corollary 5.16. Let $\mathscr{C}$ be a locally small category, $C \in \mathscr{C}$ be an object, and $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor. Then the bijections in the Yoneda Lemma restrict to bijections (correspondences)

$$\mathbf{ev}_1 : \mathbf{iso}(\mathscr{C}(-, C), F) \rightleftarrows \{x \in FC \mid (\star) \text{ for any } D \in \mathscr{C} \text{ and } y \in FD, \text{ there is a unique } f : D \to C \text{ such that } Ff(x) = y\}$$

Definition 5.17 (Universal Element)**.** Suppose $F : \mathscr{C} \to \mathbf{Set}$. A (contravariant) universal element of F is a pair $(C \in \mathscr{C}, x \in FC)$ such that for any pair $(D \in \mathscr{C}, y \in FD)$, there is a unique $f : D \to C$ such that $Ff(x) = y$.

Corollary 5.18. Let $\mathscr{C}$ be a locally small category and $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$. Then there is a bijective correspondence between representations of F and universal elements of F given by

$$\mathbf{ev}_1 : (C \in \mathscr{C}, \eta : \mathscr{C}(-, C) \cong F) \mapsto (C \in \mathscr{C}, \eta_C(1_C) \in FC).$$

Definition 5.19 (Universal Property)**.** Let $\mathscr{C}$ be a locally small category. A (covariant) universal property of an object $C \in \mathscr{C}$ is a representable functor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$, together with a universal element (C, x) that corresponds to a natural isomorphism $\mathscr{C}(-, C) \cong F$. In such a case, we say that "C is a universal object of $\mathscr{C}$ equipped with an element of F".

Definition 5.20 (Category of Elements)**.** Let $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor. The category of elements of F, denoted $\int F$, is the category with

- objects as pairs $(C \in \mathscr{C}, x \in FC)$ and
- morphisms as $f : (C, x) \to (D, y)$ which is a morphism $f : C \to D$ in $\mathscr{C}$ such that $Ff(y) = x$.

Note well that the category of elements $\int F$ of a contravariant functor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is the opposite category of the category of elements of F, regarded as a covariant functor.

Definition 5.21 (Terminal)**.** Let $\mathscr{C}$ be a category. An object $C \in \mathscr{C}$ is terminal if, for any $D \in \mathscr{C}$, there is a unique morphism $D \to C$ in $\mathscr{C}$.

Proposition 5.22. Let $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ be a functor.

1. (Contravariant) universal elements of F are precisely terminal objects of $\int F$.
2. F has a (contravariant) universal element if and only if $\int F$ has an terminal object.

Thus, we have three perspectives on universality. If $\mathscr{C}$ is a locally small category and $C \in \mathscr{C}$, then a (contravariant) universal property of C is equivalently:

1. a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ together with a universal element $(C \in \mathscr{C}, x \in FC)$.
2. a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ together with a representation $\eta : \mathscr{C}(-, C) \cong F$.
3. a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ together with a terminal object $(C, x) \in \int F$.

6 UNIVERSALITY AND THE YONEDA EMBEDDING

In this note, we explain how objects of a category are uniquely determined, up to isomorphisms, by universal properties. This makes it possible to define objects (categorically) by specifying their universal property.

Let us start with an example to see how this works.

Example 6.1. Consider $\{0, 1\} \in \mathbf{Set}$, together with the subset $\{1\} \subseteq \{0, 1\}$. This is a "universal set equipped with a subset" in the sense that if $X \in \mathbf{Set}$ and $A \subseteq X$, then there is a unique function $f : X \to \{0, 1\}$ such that $f^{-1}\{1\} = A$. Now suppose that $U \in \mathbf{Set}$ and $S \subseteq U$ is another universal set equipped with a subset. We shall show that there is a unique isomorphism $\{0, 1\} \cong U$ that is compatible with the universal properties. Here is how:

1. Since $\{1\} \subseteq \{0,1\}$ such that $f^{-1}\{1\} = S$.
2. Since $S \subseteq U$ is universal, there is a unique function $g : \{0,1\} \to U$ such that $g^{-1}S = \{1\}$.
3. Consider $g \circ f : U \to U$. This is a function such that $(g \circ f)^{-1}S = f^{-1}g^{-1}S = f^{-1}\{1\} = S$. However, $1_U : U \to U$ also has this property. Since $S \subseteq U$ is universal, there is only one function with this property, so $g \circ f = 1_U$.
4. A similar argument shows that $f \circ g = 1_{\{0,1\}}$.

It follows that the unique composition map $f : U \to \{0,1\}$ relating the subsets $S \subseteq U$ and $\{1\} \subseteq \{0,1\}$ si an isomorphism. Thus, $\{1\} \subseteq \{0,1\}$ is the unique (up to isomorphism) universal set with a subset.

This is a categorical definition of the set $\{0,1\}$.

We can generalize this argument to universal elements of set-valued functions.

Proposition 6.2. Suppose that $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is a functor and that (C,x) and (D,y) are universal elements of F. Then there is a unique morphism $f : C \to D$ such that $Ff(y) = x$, and it is an isomorphism.

Proof. By the universality of (D,y), there is a unique $f : C \to D$ such that $Ff(y) = x$. We must show f is an isomorphism. By the universality of (C,x), there is a unique $g : D \to C$ such that $Fg(x) = y$. Then $g \circ f : C \to C$ and $F(g \circ f)(x) = Ff(Fg(x)) = Ff(y) = x$, but also $1_C : C \to C$ and $F1_C(x) = 1_{FC}(x) = x$. By the universality of (C,x), it follows that $g \circ f = 1_C$, and $f \circ g = 1_D$ by a similar reasoning. Therefore, f is an isomorphism. ☐

Thus, objects that have the same universal property are isomorphic, so if $(C \in \mathscr{C}, x \in FC)$ is a universal element, then C is the unique (up to isomorphism) object of $\mathscr{C}$ equipped with an element of F. This is a categorical definition of $C \in \mathscr{C}$ up to isomorphism).

Now, we had two other perspectives on universality, one in terms of categories of elements, and another in term of hom functors. Let us see how this looks from these perspectives.

We begin by considering categories of elements. Recall that if $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$, then a universal element (C,x) of F is precisely the same thing as a terminal element of $\int F$. Thus, our study of universal elements reduces to the study of terminal objects in a category (which, in this case, encodes a universal property).

Just as there are unique comparison isomorphisms (which is just a term for the unique isomorphism) between universal elements, so too are there such isomorphisms between terminal objects (and by essentially the same argument).

Lemma 6.3. Suppose that $\mathscr{C}$ is a category and that $C, D \in \mathscr{C}$ are both terminal objects. Then there is a unique morphism $C \to D$ and it is an isomorphism.

Proof. Since D is terminal, then there is a unique morphism $f : C \to D$. We must show it is an isomorphism. Since C is terminal, there is a unique $g : D \to C$. Then $g \circ f : C \to C$ nad $1_C : C \to C$ are both morphisms $C \to C$, and since C is terminal, we conclude $g \circ f = 1_C$. Similarly, $f \circ g = 1_D$. ☐

As before, this says that terminal objects are unique (up to isomorphism). However, thinking in terms of terminal objects allows us to express this uniqueness more conceptually.

Let $\mathbb{1}$ be the category with the single object $*$ and its identity morphism 1_*. Then $\mathbb{1}$ is a terminal category, and we have the following:

Proposition 6.4. Since $\mathscr{C}$ is a category and $J \subseteq \mathscr{C}$ is the full subcategory of $\mathscr{C}$ whose objects are the terminal objects of $\mathscr{C}$. Then either $J = \varnothing$ or the unique functor $F : J \to \mathbb{1}$ is an equivalence.

Proof. Suppose $J \neq \varnothing$. For any $C, D \in J$, we have $J(C,D) \cong *$, so $F : J \to \mathbb{1}$ is fully faithful. It is also essentially surjective on objects, which implies $F : J \to \mathbb{1}$ is (part of) an equivalence of categories. ☐

Aside: we call a category that is isomorphic to $\mathbb{1}$ a contractible groupoid. Such a category is necessarily a groupoid because equivalences reflect isomorphisms, and it is "contractible" because it is equivalent to the "point" $\mathbb{1}$.

If the full subcategory J were isomorphic to $\mathbb{1}$, then $\mathscr{C}$ would have a unique terminal object. Saying $J \simeq *$ means that $\mathscr{C}$ has a "categorically unique" terminal object.

We conclude by examining how the essentially uniqueness of universal objects appears on the level of representations.

Suppose that $\mathscr{C}$ is a locally small category and that $C, D \in \mathscr{C}$ have the same universal property, in the sense that there is a single functor $F : \mathscr{C}^{\text{op}} \to \mathbf{Set}$ and a pair of representations $\eta : \mathscr{C}(-, C) \cong F$ and $\theta : \mathscr{C}(-, D) \cong F$. Then $\mathscr{C}(-, C) \cong \mathscr{C}(-, D)$, and in light of the previous discussion, we would like to conclude that $C \cong D$. This is an important consequence of the properties of the Yoneda Embedding, which we now introduce.

Definition 6.5 (Yoneda Embedding)**.** Let $\mathscr{C}$ be a locally small category. The covariant Yoneda Embedding is $Y : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}^{\text{op}}}$ that sends $C \mapsto \mathscr{C}(-, C)$ and $(f : C \to D) \mapsto (f_* : \mathscr{C}(-, C) \Rightarrow \mathscr{C}(-, D))$, and the contravariant Yoneda Embedding is $Y : \mathscr{C}^{\text{op}} \to \mathbf{Set}^{\mathscr{C}}$ that sends $C \mapsto \mathscr{C}(C, -)$ and $(f : C \to D) \mapsto (f^* : \mathscr{C}(D, -) \Rightarrow \mathscr{C}(C, -))$

The next result is also called the Yoneda Lemma.

Lemma 6.6 (Yoneda, Version II)**.** The Yoneda Embeddings are fully faithful.

Proof. We consider the covariant embedding. By the other Yoneda Lemma, $\mathbf{ev}_1 : \mathbf{Set}^{\mathscr{C}^{\text{op}}}(\mathscr{C}(-, C), \mathscr{C}(-, D)) \xrightarrow{\cong} \mathscr{C}(C, D)$ is a bijection. Consider the inverse bijection $\mathbf{ev}_1^{-1}$. For any $f : C \to D$ in $\mathscr{C}$, we have a natural transformation $f_* : \mathscr{C}(-, C) \Rightarrow \mathscr{C}(-, D)$ such that $\mathbf{ev}_1(f_*) = f_*(1_C) = f$. Then $\mathbf{ev}_1^{-1}(f) = f_*$, i.e.

$$y = \mathbf{ev}_1^{-1} : \mathscr{C}(C, D) \xrightarrow{\cong} \mathbf{Set}^{\mathscr{C}^{\text{op}}}(\mathscr{C}(-, C), \mathscr{C}(-, D)).$$

Thus, y is fully faithful. ☐

Corollary 6.7. Let $\mathscr{C}$ be a locally small category and $C, D \in \mathscr{C}$. The following are equivalent:

1. $C \cong D$ in $\mathscr{C}$
2. $\mathscr{C}(-, C) \cong \mathscr{C}(, -D)$ in $\mathbf{Set}^{\mathscr{C}^{\text{op}}}$
3. $\mathscr{C}(C, -) \cong \mathscr{C}(D, -)$ in $\mathbf{Set}^{\mathscr{C}}$

Proof. If $F : \mathscr{C} \to \mathscr{D}$ is a fully faithful functor and $C, D \in \mathscr{C}$, then $C \cong D$ if and only if $FC \cong FD$ in $\mathscr{D}$. ☐

Remark 6.8. This is an important principle. Roughly speaking, it says that objects of $\mathscr{C}$ are determined, up to isomorphism, by how they stand in relation to other objects of $\mathscr{C}$.

Returning to our earlier discussion, if $C, D \in \mathscr{C}$ have the same universal property, i.e. $\mathscr{C}(-, C) \cong F \cong \mathscr{C}(-, D)$ for some $F : \mathscr{C}^{\text{op}} \to \mathbf{Set}$, then the Yoneda Lemma implies that $C \cong D$.

If we keep track of the representations $\eta : \mathscr{C}(-, C) \cong F$ and $\theta : \mathscr{C}(-, D) \cong F$, then $\theta^{-1} \circ \eta : \mathscr{C}(-, C) \cong \mathscr{C}(-, D)$, and the unique $f : C \to D$ such that $\theta^{-1} \circ \eta = f_*$ is $\mathbf{ev}_1(\theta^{-1} \circ \eta) = \theta_C^{-1}(\eta_C(1_C)) \in \mathscr{C}(C, D)$. Moreover, $f = \theta_C^{-1}(\eta_C(1_C)) : C \to D$ is an isomorphism because $y(f) = f_* = \theta^{-1} \circ \eta$ is an isomorphism and y reflects isomorphisms.

Finally, note that $f = \theta_C^{-1}(\eta_C(1_C))$ is the unique $f : C \to D$ such that $\theta_C(f) = \eta_C(1_C)$. Now $\theta : \mathscr{C}(-, D) \cong F$, so $\theta_C(f) = \theta_C(f^*(1_D)) = Ff(\theta_D(1_D))$:

$$\begin{tikzcd}
\mathscr{C}(D, D) \arrow[r, "\theta_D"] \arrow[d, "f^*"'] & FD \arrow[d, "Ff"] \\
\mathscr{C}(C, D) \arrow[r, "\theta_C"'] & FC
\end{tikzcd}$$

So putting things together, the unique morphism $f : C \to D$ corresponding to $\theta^{-1} \circ \eta : \mathscr{C}(-, C) \cong \mathscr{C}(-, D)$ is $f = \theta_C^{-1}(\eta_C(1_C))$, and this is the unique morphism $f : C \to D$ such that $Ff(\theta_D(1_D)) = \theta_C(1_C)$.

This is the unique comparison map between the universal elements of F corresponding to the representations $\eta : \mathscr{C}(-, C) \cong F$ and $\theta : \mathscr{C}(-, D) \cong F$.

7 LIMITS

Recall that the Yoneda Lemma implies that two objects $C, D \in \mathscr{C}$ of a (locally small) category are isomorphic if and only if their corresponding covariant or contravariant hom functor are naturally isomorphic.

Informally, this means that objects of a category are determined by how they "relate" to other objects of the category. One consequence is that objects with the same universal property are isomorphic. This means it is possible to define an object in a category (up to isomorphism) by specifying a universal property that it satisfies.

In what follows, we shall take these ideas seriously, and define objects in categories by specifying their "position" relative to given diagrams – such objects will be examples of categorical limits and colimits.

When specialized to the concrete categories we have been considering, we will see that a number of familiar and important constructions can be described as limits and colimits.

Thus, limits and colimits unify formally analogous constructions in different settings, and given that they can be defined in any category, they can be used to export familiar concepts to new or unfamiliar contexts.

We begin by describing some special cases of limits, before introducing the general concept. Let us start with an example.

Example 7.1. Suppose that X and Y are sets. The Cartesian product of X and Y is the set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ of all ordered pairs whose first coordinate is in X and whose second coordinate is in Y.

Note that there are projection functions $\pi_1 : X \times Y \to X$ that sends $(x, y) \mapsto x$ and $\pi_2 : X \times Y \to Y$ that sends $(x, y) \mapsto y$ from $X \times Y$ to the factors, and they are universal in the following sense: given any other set A, together with a pair of functions $X \xleftarrow{s} A \xrightarrow{t} Y$, there is a unique function $u : A \to X \times Y$ such that $\pi_1 \circ u = s$ and $\pi_2 \circ u = t$, namely $u(a) = (s(a), t(a))$. Here is the diagram:

$$\begin{tikzcd}
 & A \arrow[dl, "s"'] \arrow[d, dashed, "\exists! u"] \arrow[dr, "t"] & \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}$$

Thus, $X \times Y$ is the universal set equipped with a pair of functions to X and Y, and this universal property defines $X \times Y$ up to a unique isomorphism that is compatible with the projection maps.

The upshot is that $X \times Y$ can be defined (up to isomorphism) by a universal property that relates it diagrammatically to X and Y.

In what follows, we will be looking at similar characterizations of objects in categories. We start by defining arbitrary products in categories.

Definition 7.2 (Product)**.** Suppose $\mathscr{C}$ is a category and $\{A_i \mid i \in I\}$ is an indexed collection of objects of $\mathscr{C}$. A product of these objects, if it exists, is an object $\prod\limits_{i \in I} A_i \in \mathscr{C}$, equipped with morphisms $\pi_i : \prod\limits_{i \in I} A_i \to A_i$ (called projections) for each $i \in I$, which is universal with this property. In this case, universality means that for any object $T \in \mathscr{C}$ together with morphisms $f_i : T \to A_i$, there is a unique morphism $\langle f_i \rangle_{i \in I} : T \to \prod\limits_{i \in I} A_i$ such that $\pi_i \circ \langle f_i \rangle_{i \in I} = f_i$ for all $i \in I$:

$$\begin{tikzcd}
T \arrow[d, dashed, "\langle f_i \rangle_{i \in I}"', "\exists!"] \arrow[dr, "f_i"] & \\
\prod\limits_{i \in I} A_i \arrow[r, "\pi_i"'] & A_i
\end{tikzcd}$$

Figure 7.1: Universal Property of Product

Remark 7.3. A terminal object is a product of no objects.

Example 7.4. Suppose I is an indexing set.

1. If $\{X_i \mid i \in I\} \subseteq \mathbf{Set}$, then $\prod\limits_{i \in I} X_i = \{(x_i)_{i \in I} \mid x_i \in X_i \text{ for all } i \in I\}$, together with the projections $\pi_i : \prod\limits_{i \in I} X_i \to X_i$ by mapping $(x_i)_{i \in I} \mapsto x_i$ is a product of $\{X_i \mid i \in I\}$ in $\mathbf{Set}$.

2. If $\{(M_i, \circ_i, e_i \mid i \in I\} \subseteq \mathbf{Mon}$, then $\prod\limits_{i \in I} M_i$ equipped with the operation $(x_i)_{i \in I} \circ (y_i)_{i \in I} = (x_i \circ_i y_i)_{i \in I}$, the element $(e_i)_{i \in I} \in \prod\limits_{i \in I} M_i$, and the coordinate projections $\pi_i : \prod\limits_{i \in I} M_i \to M_i$ above is a product of $\{M_i \mid i \in I\}$ in $\mathbf{Mon}$.
3. If $\{(P_i, \leqslant_i) \mid i \in I\} \subseteq \mathbf{Preord}$, the $\prod\limits_{i \in I} P_i$, equipped with the relation $(x_i)_{i \in I} \leqslant (y_i)_{i \in I}$ if and only if $x_i \leqslant_i y_i$ for all $i \in I$, together with the projections $\pi_i : \prod\limits_{i \in I} P_i \to P_i$ is a product of $\{P_i \mid i \in I\}$ in $\mathbf{Preord}$.

Definition 7.5 (Equalizer)**.** Suppose $\mathscr{C}$ is a category and $f, g : A \to B$ is a pair of morphisms in $\mathscr{C}$. An equalizer of this pair, if it exists, is an object $E \in \mathscr{C}$, together with a morphism $e : E \to A$ such that e is a universal morphism with the property that $fe = ge$. In other words, $fe = ge$, and given any other $T \in \mathscr{C}$ and $h : T \to A$ such that $fh = gh$, there is a unique $\tilde{h} : T \to E$ such that $e \circ \tilde{h} = h$:

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
T \arrow[u, dashed, "\exists! \tilde{h}"] \arrow[ru, "h"'] & &
\end{tikzcd}$$

Figure 7.2: Universal Property of Equalizer

Example 7.6.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ sends $(x, y) \mapsto x^2 + y^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ sends $(x, y) \mapsto 1$ in $\mathbf{Set}$. Then $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, together with the inclusion $i : S^1 \hookrightarrow \mathbb{R}^2$ is an equalizer of f and g in $\mathbf{Set}$:

$$\begin{tikzcd}
S^1 \arrow[r, hook, "i"] & \mathbb{R}^2 \arrow[r, shift left, "x^2+y^2"] \arrow[r, shift right, "1"'] & \mathbb{R}
\end{tikzcd}$$

2. Let $X, Y \in \mathbf{Set}$ and $f, g : X \to Y$ be functions. Then

$$\begin{tikzcd}
\{x \in X \mid f(x) = g(x)\} \arrow[r, hook, "inc"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "f"'] & Y
\end{tikzcd}$$

is an equalizer diagram in $\mathbf{Set}$, i.e. $i : \{x \in X \mid f(x) = g(x)\} \hookrightarrow X$ is an equalizer of f and g.

3. Let $M, N \in \mathbf{Mon}$ and $f, g : M \to N$ be monoid homomorphisms. Let $E = \{m \in M \mid f(m) = g(m)\}$. Then $e_M \in E$ and E is closed under $\cdot_M$. Thus, the monoid structure on M restricts to a monoid structure on E, with $e_E = e_M$ and $x \cdot_E y = x \cdot_M y$ for all $x, y \in E$, and

$$\begin{tikzcd}
E \arrow[r, hook, "inc"] & M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N
\end{tikzcd}$$

is an equalizer diagram in $\mathbf{Mon}$.

4. Let $P, Q \in \mathbf{Preord}$ and $f, g : P \to Q$ be monotone maps. Let $E = \{o \in P \mid f(p) = g(p)\}$, equipped with the order relation: for all $p, p' \in E$, $p \leqslant_E p'$ if and only if $p \leqslant_P p'$. Then E is a preorder, and

$$\begin{tikzcd}
E \arrow[r, hook, "inc"] & P \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Q
\end{tikzcd}$$

is an equalizer diagram in $\mathbf{Preord}$.

Definition 7.7 (Pullback/Fiber Product)**.** Let $\mathscr{C}$ be a category, and consider the following diagram in $\mathscr{C}$:

$$\begin{tikzcd}
 & B \arrow[d, "g"] \\
A \arrow[r, "f"] & C
\end{tikzcd}$$

A pullback or fiber product of f and g, if one exists, is an object $A \times_C B \in \mathscr{C}$, together with morphisms $\pi_A : A \times_C B \to A$ and $\pi_B : A \times_C B \to B$, which are universal with the property that $f \circ \pi_A = g \circ \pi_B$. In other words, $f\pi_A = g\pi_B$, and given any other object $T \in \mathscr{C}$ with morphisms $p : T \to A$ and $q : T \to B$ such that $fp = qg$, there is a unique $\langle p, q\rangle : T \to A \times_C B$ such that $\pi_A \circ \langle p, q\rangle = p$ and $\pi_B \circ \langle p, q\rangle = q$:

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists!\langle p,q\rangle"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
 & A \times_C B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"] & B \arrow[d, "g"] \\
 & A \arrow[r, "f"] & C
\end{tikzcd}$$

Figure 7.3: Universal Property of Pullback

Example 7.8. 1. Suppose $f : X \to Y$ in $\mathbf{Set}$ and $B \subseteq Y$. Let

(a) $\tilde{f} : f^{-1}B \to B$ be the restriction of $f : X \to Y$,

(b) $i : B \hookrightarrow Y$ be the inclusion, and

(c) $j : f^{-1}B \hookrightarrow X$ be the inclusion.

Then the square

$$\begin{tikzcd}
f^{-1}B \arrow[r, "\tilde{f}"] \arrow[d, "j"'] & B \arrow[d, "i"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

is a pullback, i.e. $f^{-1}B$, together with j and $\tilde{f}$, is a pullback of f and i in $\mathbf{Set}$.

2. In general, suppose $X \xrightarrow{f} Z \xleftarrow{g} Y$ are functions in $\mathbf{Set}$. Then the set $X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$, together with the projections $\pi_X : X \times_Z Y \to X$ that sends $(x, y) \mapsto x$ and $\pi_Y : X \times_Z Y \to Y$ that sends $(x, y) \mapsto y$, is a pullback of f and g in $\mathbf{Set}$:

$$\begin{tikzcd}
\{(x,y) \in (X,Y) \mid f(x) = g(y)\} \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

Similarly, if $X \to Z \leftarrow Y$ is in $\mathbf{Mon}$, then the set $X \times_Z Y$ above becomes a monoid with identity (e_X, e_Y) and componentwise multiplication and the square above is a pullback in $\mathbf{Mon}$.

If $X \to Z \leftarrow Y$ is in $\mathbf{Preord}$, then the set $X \times_Z Y$ above becomes a preorder with relation $(x, y) \leqslant (x', y')$ if and only if $x \leqslant x'$ and $y \leqslant y'$, and the square above is a pullback in $\mathbf{Preord}$.

Definition 7.9 (Inverse Limit)**.** Let $\mathscr{C}$ be a category, and let

$$\cdots \qquad A_3 \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

be an infinite sequence of composable morphisms in $\mathscr{C}$. An inverse limit of this sequence, if one exists, is an object $\varprojlim A_n \in \mathscr{C}$, together with morphisms $\varprojlim A_n \xrightarrow{\lambda_n} A_n$ for all $n \geqslant 0$, which are universal with the property that $f_n \circ \lambda_n = \lambda_{n-1}$ for all $n > 0$.

$$\begin{tikzcd}
 & \varprojlim A_n \arrow[d, "\lambda_3"'] \arrow[dr, "\lambda_2"'] \arrow[drr, "\lambda_1"] \arrow[drrr, "\lambda_0"] & & & \\
\cdots \arrow[r] & A_3 \arrow[r, "f_3"'] & A_2 \arrow[r, "f_2"'] & A_1 \arrow[r, "f_1"'] & A_0
\end{tikzcd}$$

Thus, for any object $T \in \mathscr{C}$, together with morphisms $t_n : T \to A_n$ such that $f_n \circ t_n = t_{n-1}$ for all $n > 0$, there is a unique morphism $t : T \to \varprojlim A_n$ such that $\lambda_n \circ t = t_n$ for all $n \geqslant 0$:

$$\begin{tikzcd}
T \arrow[rr, dashed, "\exists! t"] \arrow[dr, "t_n"] & & \varprojlim A_n \arrow[dl, "\lambda_n"] \\
 & A_n &
\end{tikzcd}$$

Figure 7.4: Universal Property of Inverse Limit

Example 7.10. 1. Let $X_0, X_1, \cdots \in \mathbf{Set}$ be a sequence of sets such that $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$, and let $f_i : X_i \hookrightarrow X_{i-1}$ be the inclusion map. Then $\bigcap\limits_{i=0}^{\infty} X_i$, together with the inclusions $i_n : \bigcap\limits_{i=0}^{\infty} X_i \hookrightarrow X_n$ is an inverse limit of $\cdots \to X_2 \to X_1 \to X_0$.

2. More generally, suppose $\cdots \to X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$ is an infinite sequence of composable morphisms in $\mathbf{Set}$. Then

$$L = \{(x_i)_{i=0}^{\infty} \in \prod_{i=0}^{\infty} X_i \mid f_i(x_i) = x_{i-1} \text{ for all } i > 0\},$$

together with the coordinate projections $\pi_n : L \to X_n$ by sending $(x_i)_{i=0}^{\infty} \mapsto x_n$ is an inverse limit of $\cdots \to X_2 \to X_1 \to X_0$ in $\mathbf{Set}$.

Given an infinite sequence $\cdots \to M_2 \to M_1 \to M_0$ in $\mathbf{Mon}$, the same construction equipped with componentwise operations is an inverse limit.

Given an infinite sequence $\cdots \to P_2 \to P_1 \to P_0$ in $\mathbf{Preord}$, the same construction, equipped with the componentwise order operation, is an inverse limit.

We now give the general definition of a limit of a diagram, which unifies the constructions we just presented.

Definition 7.11 (Diagram of Shape, Constant Diagram, Cone, Limit)**.** Suppose J and $\mathscr{C}$ are categories.

1. A diagram of shape J in $\mathscr{C}$ is a functor $F : J \to \mathscr{C}$.

2. For any $C \in \mathscr{C}$, the constant diagram $C : J \to \mathscr{C}$ is the functor that sends all objects $D \in \mathscr{C}$ to C, and all morphisms $f \in \mathscr{C}$ to 1_C.

3. A cone over the diagram $F : J \to \mathscr{C}$ with vertex $C \in \mathscr{C}$ is a natural transformation $\lambda : C \Rightarrow F$. The components of λ are called the legs of the cone.

   Spelled out, a cone $\lambda : C \Rightarrow F$ over a diagram $F : J \to \mathscr{C}$ consists of morphisms $\lambda_k : C \to F_j$ indexed by $\mathbf{Ob}(J)$ such that for any morphisms $f : i \to j$ in J, the triangle

$$\begin{tikzcd}
 & C \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
Fi \arrow[rr, "Ff"'] & & Fj
\end{tikzcd}$$

Figure 7.5: Cone Triangle

commutes, i,e, $Ff \circ \lambda_i = \lambda_j$.

4. A limit of $F : J \to \mathscr{C}$ is a terminal cone over F, i.e. an object $\lim_J F \in \mathscr{C}$ together with a cone $\lambda : \lim_J F \Rightarrow F$ such that for any other object $T \in \mathscr{C}$, together with a cone $\tau : T \Rightarrow F$, there is a unique morphism $t : T \to \lim_J F$ such that $\lambda_j \circ t = \tau_j$ for all $j \in \mathbf{Ob}(J)$:

$$\begin{tikzcd}
T \arrow[rr, dashed, "\exists! t"] \arrow[dr, "\tau_j"'] & & \lim_J F \arrow[dl, "\lambda_j"] \\
& Fj &
\end{tikzcd}$$

Figure 7.6: Universal Property of Limit

If J is small and $\mathscr{C}$ is locally small, then we can describe limits more conceptually. In this case, there is a functor $\mathbf{Cone}(-, F) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ that sends $C \in \mathscr{C}$ to the set $\mathbf{Cone}(C, F)$ of all cones over F with vertex C, and a morphism $f : C \to D$ to pre-composition $f^* : \mathbf{Cone}(D, F) \to \mathbf{Cone}(C, F)$ with f. A limit $(\lim_J F, \lambda)$ is then a universal element of $\mathbf{Cone}(-, F)$. Alternatively, a limit is a representation $\mathscr{C}(-, \lim_J F) \cong \mathbf{Cone}(-, F)$ or a terminal object in $\int \mathbf{Cone}(-, F)$.

Now, unpacking the definition and simplifying, we can see that:

1. A product is a limit of a diagram $F : J \to \mathscr{C}$ where J is discrete, i.e. has no non-identity morphisms.

2. An equalizer is a limit of a diagram $F : \{ \begin{tikzcd} \bullet \arrow[r, shift left, bend left] \arrow[r, shift right, bend right] & \bullet \end{tikzcd} \} \to \mathscr{C}$.

3. A pullback is a limit of a diagram $F : \left\{ \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \right\} \to \mathscr{C}$.

4. An inverse limit of a sequence of morphisms is a limit of a diagram

$$F : \{ \begin{tikzcd} \cdots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \} \to \mathscr{C}.$$

Example 7.12. Suppose J is a small category and $F : J \to \mathscr{C}$ is a diagram. Then the set

$$L = \{(x_j)_{j \in J} \in \prod_{j \in J} Fj \mid \text{ for all } f : i \to j \text{ in } J : Ff(x_i) = x_j\},$$

together with the projections $\pi_j : L \to Fj$ sending $(x_j)_{j \in J} \mapsto x_j$ is a limit of F in $\mathbf{Set}$. Similarly for $\mathbf{Mon}$ and $\mathbf{Preord}$.

Our formulas for pullbacks, inverse limits, and general limits in $\mathbf{Set}$ suggest that limits of diagram, which are indexed by small categories, can be constructed as subobjects of products. This is true.

Theorem 7.13. Let J be a small category and $F : J \to \mathscr{C}$ be a J-shaped diagram in $\mathscr{C}$. Suppose that $\mathscr{C}$ has equalizers of all parallel pairs of morphisms and products of all indexed sets of objects. Then F has a limit. More precisely, consider the diagram

$$\begin{tikzcd}
L \arrow[r, "e"] & \prod\limits_{j \in \mathbf{Ob}(J)} Fj \arrow[r, bend left, "\langle Ff \circ \pi_{\mathbf{domain}(f)} \rangle"] \arrow[r, bend right, "\langle \pi_{\mathbf{codomain}(f)} \rangle"'] & \prod\limits_{f \in \mathbf{Mor}(J)} F(\mathbf{codomain}(f))
\end{tikzcd}$$

Figure 7.7: Construction of Small Limits

where:

1. $\left\langle Ff \circ \pi_{\mathbf{domain}(f)} \right\rangle$ is the unique morphism such that $\pi_f \circ \left\langle Ff \circ \pi_{\mathbf{domain}(f)} \right\rangle = Ff \circ \pi_{\mathbf{domain}(f)}$ for all $f \in \mathbf{Mor}(J)$,
2. $\left\langle \pi_{\mathbf{codomain}(f)} \right\rangle$ is the unique morphism such that $\pi_f \circ \left\langle \pi_{\mathbf{codomain}(f)} \right\rangle = \pi_{\mathbf{codomain}(f)}$ for all $f \in \mathbf{Mor}(J)$, and
3. e is an equalizer of $\left\langle Ff \circ \pi_{\mathbf{domain}(f)} \right\rangle$ and $\left\langle \pi_{\mathbf{codomain}(f)} \right\rangle$.

Then L, equipped with the morphisms $(\lambda_j = \pi_j \circ e : L \to Fj)_{j \in J}$ is a limit of $F : J \to \mathscr{C}$ in $\mathscr{C}$.

The simplest proof is a verification that (L, λ) has the correct universal property. The next lemma is helpful.

Lemma 7.14.

1. Suppose

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$$

   is an equalizer diagram. Then e is monic.
2. Suppose $\prod\limits_{i \in I} A_i$, together with the projections $\pi_i : \prod\limits_{i \in I} A_i \to A_i$ is a product of $\{A_i \mid i \in I\}$. Then the π_i's are "jointly monic" in the following sense: for any object T and morphisms $h, k : T \to \prod\limits_{i \in I} A_i$, if $\pi_i \circ h = \pi_i \circ k$ for all $i \in I$, then $h = k$.

Proof.

1. Suppose T is an object and $h, k : T \to E$ are morphisms such that $eh = ek$. Write $t = eh : T \to A$. Then $ft = feh = geh = gt$, so by the universal property of $e : E \to A$, there is a unique $\tilde{t} : T \to E$ such that $e\tilde{t} = t = eh$. both $\tilde{t} = h$ and $\tilde{t} = k$ work, so $h = k$ by uniqueness.
2. Suppose T is an object and $h, k : T \to \prod\limits_{i \in I} A_i$ are morphisms such that $\pi_i h = \pi_i k$ for all $i \in I$. Write $t_i = \pi_i h = \pi_i k : T \to A_i$ for all i. By the universal property of $\prod\limits_{i \in I} A_i$, there is a unique $t : T \to \prod\limits_{i \in I} A_i$ such that $\pi_i t = t_i$ for all i. But $t = h$ and $t = k$ both work. Therefore, $h = k$ by uniqueness.

□

We can now prove the theorem.

Proof. Suppose that $(T, (\tau_j : T \to F_j)_{j \in J})$ is a cone over $F : J \to \mathscr{C}$, so that for any $f : i \to j$ in J, $Ff \circ \tau_i = \tau_j$. By the universal property of a product, there is a unique map $\langle \tau_j \rangle : T \to \prod\limits_{j \in \mathbf{Ob}(J)} Fj$ such that $\pi_j \circ \langle \tau_j \rangle = \tau_j$ for all $j \in \mathbf{Ob}(J)$. Next, consider the morphisms $\left\langle Ff \circ \pi_{\mathbf{domain}(f)} \right\rangle \circ \langle \tau_j \rangle$,$\left\langle \pi_{\mathbf{codomain}(f)} \right\rangle \circ \langle \tau_j \rangle : T \to \prod\limits_{f \in \mathbf{Mor}(J)} F(\mathbf{codomain}(f))$. For any $f : i \to j$ in J, we have

$$\begin{aligned} \pi_f \circ \left\langle Ff \circ \pi_{\mathbf{domain}(f)} \right\rangle \circ \langle \tau_j \rangle &= Ff \circ \pi_i \circ \langle \tau_j \rangle \\ &= Ff \circ \tau_i \\ &= \tau_j \\ &= \pi_j \circ \langle \tau_j \rangle \\ &= \pi_f \circ \left\langle \pi_{\mathbf{codomain}(f)} \right\rangle \circ \langle \tau_j \rangle \end{aligned}$$

and therefore $\left\langle Ff \circ \pi_{\mathbf{domain}(f)} \right\rangle \circ \langle \tau_j \rangle = \left\langle \pi_{\mathbf{codomain}(f)} \right\rangle \circ \langle \tau_j \rangle$ by the second part of the previous lemma. Thus, there is a unique $t : T \to L$ such that $et = \langle \tau_j \rangle$. Thus, $\lambda_j t = \pi_j et = \pi_j \langle \tau_j \rangle = \tau_j$, and we have shown that there is a morphism $t : T \to L$ such that $\lambda_j t = \tau_j$ for all $j \in J$.

It remains to show $t : T \to L$ is unique. Suppose $t' : T \to L$ is also such that $\lambda_j t' = \tau_j$ for all j. Then for any $j \in \mathbf{Ob}(J)$, we have $\pi_j et' = \lambda_j t' = \tau_j = \lambda_j t = \pi_j et$, and therefore by the first part of the lemma, $t' = t$. Then $t : T \to L$ is the unique morphism such taht $\lambda_j t = \tau_j$ for all $j \in J$. □

We end with some terminology that summarizes our work.

Definition 7.15 (Small, Complete)**.** A diagram is small if its indexing category is a small category.

A category $\mathscr{C}$ is complete if it "has all small limits", i.e. if it admits limits of all small diagrams valued in $\mathscr{C}$.

Most concrete categories encountered in practice are complete. Our work shows explicitly that $\mathbf{Set}, \mathbf{Mon}$ and $\mathbf{Preord}$ are complete.

We can also restate the most recent theorem.

Theorem 7.16. Suppose $\mathscr{C}$ is a category. Then $\mathscr{C}$ is complete if and only if it has equalizers and products of all indexed sets of objects in $\mathscr{C}$.

8 Colimits

Colimits are the dual notion of limits.

In this note, we introduce the general notion of a colimit, along with a number of special cases, and we illustrate these constructions with examples.

Definition 8.1 (Cocone, Colimit)**.** Suppose J and $\mathscr{C}$ are categories and $F : J \to \mathscr{C}$ is a diagram.

1. A cocone under F with vertex $C \in \mathscr{C}$ is an object $C \in \mathscr{C}$ together with a natural transformation $\lambda : F \Rightarrow C$, where the target of λ is the constant diagram valued at $C \in \mathscr{C}$. The components of λ are called the legs of the cocone.

   Spelled out, a cocone $\lambda : F \Rightarrow C$ consists of morphisms $\lambda_j : Fj \to C$, indexed by the objects $j \in J$, such that for any $f : i \to j$ in J, the triangle

$$\begin{tikzcd}
Fi \arrow[rr, "Ff"] \arrow[dr, "\lambda_i"'] & & Fj \arrow[dl, "\lambda j"] \\
& C &
\end{tikzcd}$$

Figure 8.1: Cocone Triangle

   commutes, i.e. $\lambda_j \circ Ff = \lambda_i$.

2. A colimit of F is an initial cocone under F. That is, it is an object $\mathbf{colim}_J F \in \mathscr{C}$, together with a cocone $\lambda : F \Rightarrow \mathbf{colim}_J F$ such that for any other object $T \in \mathscr{C}$, together with a cocone $\tau : F \Rightarrow T$, there is a unique $t : \mathbf{colim}_J F \to T$ such that for all $j \in J$, the triangle

$$\begin{tikzcd}
& Fj \arrow[dl, "\lambda_j"'] \arrow[dr, "\tau_j"] & \\
\mathbf{colim}_J F \arrow[rr, dashed, "\exists! t"'] & & T
\end{tikzcd}$$

Figure 8.2: Universal Property of Colimit

   commutes, i.e. $t \circ \lambda_j = \tau_j$.

As with limits, this definition can be recast if $\mathscr{C}$ is locally small and J is small. In this case, there is a functor $\mathbf{Cocone}(F, -) : \mathscr{C} \to \mathbf{Set}$ that sends an object $C \in \mathscr{C}$ to the set of cocones under F with vertex C, and a morphims $f : C \to D$ to post-composition $f_* : \mathbf{Cocone}(F, C) \to \mathbf{Cocone}(F, D)$. A colimit is then a universal element $(\mathbf{colim}_J F, \lambda)$ of $\mathbf{Cocone}(F, 0)$, or equivalently, a representation $\mathscr{C}(\mathbf{colim}_J F, -) \cong \mathbf{Cocone}(F, -)$ or an initial element of $\int \mathbf{Cocone}(F, -)$.

As with limits, colimits with respect to certain diagram shapes get special names.

Definition 8.2 (Coproduct)**.** A coproduct is a colimit of a diagram indexed by a discrete category.

If J is discrete, then a diagram $F : J \to \mathscr{C}$ is equivalent to an indexed collection $\{Fj \mid j \in J\}$ of objects of $\mathscr{C}$. A coproduct of this collection, if it exists, consists of an object $\coprod\limits_{j\in J} Fj \in \mathscr{C}$, together with morphisms $i_j : Fj \to \coprod\limits_{j\in J} Fj$ indexed by $\mathbf{Ob}(J)$, such that for any $T \in \mathscr{C}$, together with morphisms $t_j : Fj \to T$, there is a unique $[t_j]_{j\in J} : \coprod\limits_{j\in J} Fj \to T$ such that $[t_j]_{j\in J} \circ i_j = t_j$ for all $j \in J$:

$$\begin{tikzcd}
Fj \arrow[r, "i_j"] \arrow[dr, "t_j"'] & \coprod\limits_{j\in J} Fj \arrow[d, dashed, "\exists! [t_j]_{j\in J}"] \\
& T
\end{tikzcd}$$

Figure 8.3: Universal Property of Coproduct

Remark 8.3. An initial object is a coproduct of no objects.

Example 8.4. If J is an indexing set and $\{A_j \mid j \in J\} \subseteq \mathbf{Set}$, then the disjoint union $\coprod\limits_{j\in J} A_j = \{(j, x) \mid j \in J, x \in A_j\}$ together with the inclusion $i_j : A_j \to \coprod\limits_{j\in J} A_j$ that sends $x \mapsto (j, x)$ is a coproduct of $\{A_j \mid j \in J\}$ in $\mathbf{Set}$.

Definition 8.5 (Coequalizer)**.** A coequalizer is a colimit of a diagram of the form $F : \{ \bullet \rightrightarrows \bullet \} \to \mathscr{C}$.

Unwinding the definition and simplifying, we find that a $\{ \bullet \rightrightarrows \bullet \}$-shaped diagram is a parallel pair $f, g : A \to B$ of morphisms, and a coequalizer of such a pair is an object Q, together with a morphism $q : B \to Q$ that is initial with the property that $qf = qg$. Then, given any object T and morphism $t : B \to T$ such that $tf = tg$, there is a unique $\bar{t} : Q \to T$ such that $t = \bar{t} \circ g$:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "t"'] & Q \arrow[d, dashed, "\exists! \bar{t}"] \\
& & T
\end{tikzcd}$$

Figure 8.4: Universal Property of Coequalizer

Example 8.6. Consider the sets $A = \{*\}$ and $B = [0, 2\pi] \subseteq \mathbb{R}$, then the diagram

$$\begin{tikzcd}
\{*\} \arrow[r, shift left, "x\mapsto 0"] \arrow[r, shift right, "x\mapsto 2\pi"'] & {[0, 2\pi]} \arrow[r, "\theta\mapsto(\cos(\theta), \sin(\theta))"] & \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}
\end{tikzcd}$$

is a coequalizer in $\mathbf{Set}$. In this case, coequalizing the two left maps ”glues” the two endpoints of $[0, 2\pi]$ together.

General coequalizers in $\mathbf{Set}$ are somewhat complicated. We shall describe them momentarily, but first, some preliminaries.

Suppose X is a set and R is a binary relation on X. The equivalence relation generated by R, denoted $\sim_R$, is defined as follows: for any $x, y \in X$, $x \sim_R y$ if and only if there is $n \geqslant 0$ and $x_0, x_1, \cdots, x_n \in X$ such that

1. $x = x_0$ and $y = x_n$, and
2. for all $0 \leqslant i < n$, either $x_i R x_{i+1}$ or $x_{i+1} R x_i$.

The relation $\sim_R$ has the following properties:

1. $\sim_R$ is an equivalence relation on X.
2. if xRy, then $x \sim_R y$, and

3. if $\approx$ is an equivalence relation on X such that xRy implies $x \approx y$, then for any $x, y \in X$, if $x \sim_R y$, then $x \approx y$.

In other words, $\sim_R$ is the smallest equivalence relation that contains R.

Example 8.7. Suppose $f, g : X \to Y$ in **Set**. We construct a coequalizer of f and g. Let R be the relation on Y defined by $y_1 R y_2$ if and only if there is $x \in X$ such that $y_1 = f(x)$ and $y_2 = g(x)$ and consider the quotient $\pi : Y \to Y/\sim_R$. Then

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[r, "\pi"] & Y/\sim_R
\end{tikzcd}$$

is a coequalizer diagram.

Definition 8.8 (Pushout)**.** A pushout is a colmit of a diagram of the form $F : \left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \right\} \to \mathscr{C}$.

Unwinding and simplifying, we find that a $\left\{ \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \right\}$-shaped diagram is just a corner of morphisms

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \\
C &
\end{tikzcd}$$

with a common domain, and a pushout of this corner is an object P, together with morphisms $h : C \to P$ and $K : B \to P$, which are initial with the property that $kf = hg$. Thus, given any object T, together with morphisms $s : C \to T$ and $t : B \to T$ such that $tf = sg$, there is a unique morphism $u : P \to T$ such that $uh = s$ and $uk = t$. The diagram is the following:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \arrow[ddr, bend left, "t"] & \\
C \arrow[r, "h"'] \arrow[drr, bend right, "s"'] & P \arrow[dr, dashed, "\exists! u"] & \\
& & T
\end{tikzcd}$$

Figure 8.5: Universal Property of Pushout

Example 8.9. Consider the following sets:

$$A = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z \geqslant 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z \leqslant 0\}$$

Then the square

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, hook] & B \arrow[d, hook] \\
C \arrow[r, hook] & \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}
\end{tikzcd}$$

is a pushout in **Set**, where all maps are inclusions. Pushing out glues the two hemispheres together along their common boundary in this case.

Definition 8.10 (Sequential Colimit/Direct Limit)**.** A sequential colimit or direct limit is a colimit of a diagram of the form $F : \mathbb{N} \to \mathscr{C}$, where we regard $\mathbb{N}$, equipped with its usual ordering, as a poset category. Direct limits are frequently denoteda as $\varinjlim F_n$.

Unwinding and simplifying, a $\mathbb{N}$-shaped diagram is an infinite sequence $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \to \cdots$ of composable morphisms, and a direct limit of such a sequence is an object $\varinjlim A_n$, together with morphisms $\lambda_n : A_n \to \varinjlim A_n$, which are initial with the property that $\lambda_n \circ f_n = \lambda_{n-1}$ for all $n > 0$.

$$\begin{tikzcd}
A_0 \arrow[r, "f_1"] \arrow[rrrd, "\lambda_0"'] & A_1 \arrow[r, "f_2"] \arrow[rrd, "\lambda_1"] & A_2 \arrow[r] \arrow[rd, "\lambda_2"] & \cdots \\
& & & \varinjlim A_n
\end{tikzcd}$$

Thus, given any object T and morphisms $\tau_n : A_n \to T$ such that $\tau_n \circ f_n = \tau_{n-1}$ for all $n > 0$, there is a unique $t : \varinjlim \to T$ such that $t \circ \lambda_n = \tau_n$ for all $n \geqslant 0$:

$$\begin{tikzcd}
& A_n \arrow[ld, "\lambda_n"'] \arrow[rd, "\tau_n"] & \\
\varinjlim A_n \arrow[rr, dashed, "\exists! t"'] & & T
\end{tikzcd}$$

Figure 8.6: Universal Property of Direct Limit

Example 8.11. Let $A_0, A_1, \cdots$ be sets such that $A_0 \subseteq A_1 \subseteq \cdots$ and let $f_n : A_{n-1} \hookrightarrow A_n$ be the inclusion. Then $\bigcup\limits_{n=0}^{\infty} A_n$, together wtih the inclusions $i_n : A_n \hookrightarrow \bigcup\limits_{n=0}^{\infty} A_n$ is a direct limit of

$$\begin{tikzcd}
A_0 \arrow[r, hook, "f_1"] & A_1 \arrow[r, hook, "f_2"] & A_2 \arrow[r, hook] & \cdots
\end{tikzcd}$$

Thus, there are a wide variety of colimits that we encounters in practice, but they tend to "feel" like gluing constructions.

Just as general limits can be built from products and equalizers, so too can colimits be built from coproducts and coequalizers.

Theorem 8.12. Let $F : J \to \mathscr{C}$ be a small diagram. Suppose that $\mathscr{C}$ has coequalizers of all parallel pairs of morphisms and coproducts of all indexed sets of objects. Then F has a colimit.

This is the dual to the analogous theorem for limits, so we omit the proof.

As with limits, we end with some terminology.

Definition 8.13 (Cocomplete)**.** A category $\mathscr{C}$ is cocomplete if it admits colimits of all small diagrams valued in $\mathscr{C}$.

Our previous work, combined with the theorem above shows that **Set** is complete. Most of the large categories one encounters in practice are cocomplete.

We may rephrase the theorem above:

Theorem 8.14. A category $\mathscr{C}$ is cocomplete if and only if it has coequalizers and colimits of all indexed set of objects in $\mathscr{C}$.

9 Limit Functors, Limit of Functors, Iterated Limits

By forming limits and colimits, we can construct objects out of diagrams. We now explain how to extend this procedure to construct "limit morphisms" or "colimit morphisms" out of natural transformation. We need them to formulate a number of important results. We shall focus on limits as the situation for colimits is dual.

First, an observation.

Lemma 9.1. Suppose $F, G : J \to \mathscr{C}$ are two J-shaped diagrams, $\eta : F \Rightarrow G$ is a natural transformation, and $(\lim_J F, (\lambda_j : \lim_J F \to Fj)_{j \in J})$ and $(\lim_J G, (\mu_j : \lim_J G \to Gj)_{j \in J})$ are limits of F and G respectively. Then there is a unique morphism $\lim_J \eta : \lim_J F \to \lim_J G$ such that the square

$$\begin{tikzcd}
\lim_J F \arrow[r, "\lim_J \eta"] \arrow[d, "\lambda_j"'] & \lim_J G \arrow[d, "\mu_j"] \\
Fj \arrow[r, "\eta_j"'] & G_j
\end{tikzcd}$$

commutes for all $j \in J$.

Proof. The vertical composite $\eta \circ \lambda : \lim_J F \Rightarrow F \Rightarrow G$ is a cone over G, so by the universal property of $\lim_J G$, there is a unique $\lim_J \eta : \lim_J F \to \lim_J G$ such that $\mu_j \circ \lim_J \eta = (\eta \circ \lambda)_j = \eta_j \circ \lambda_j$ for all $j \in J$. ☐

By choosing limits of J-shaped diagrams, this lemma allows us to extend $\lim_J$ to a functor.

Let J be an indexing category, $\mathscr{C}$ be another category, and write $\mathscr{C}^J_{\mathbf{lim}}$ for the full subcategory of $\mathscr{C}^J$ whose objects are those diagrams $F : J \to \mathscr{C}$ that have a limit.

Proposition 9.2. For each $F \in \mathscr{C}^J_{\mathbf{lim}}$, choose a limit $(\lim_J F, \lambda_F : \lim_J F \Rightarrow F)$ of F. Then this choise extends ot a functor $\lim_J : \mathscr{C}^J_{\mathbf{lim}} \to \mathscr{C}$ that sends a natural transformation $\eta : F \Rightarrow G$ in $\mathscr{C}^J_{\mathbf{lim}}$ to the unique morphism $\lim_J \eta : \lim_J F \to \lim_J G$ of the previous lemma.

Proof. The mapping $\lim_J$ preserves domains and codomains by definition, so it remains to check that it preserves identity and compositions. Suppose $F \in \mathscr{C}^J_{\mathbf{lim}}$ and $1_F : F \Rightarrow F$. Then $\lim_J 1_F : \lim_J F \to \lim_J F$ is the unique morphism such that the square

$$\begin{tikzcd}
\lim_J F \arrow[r, "\lim_J 1_F"] \arrow[d, "(\lambda_F)_j"'] & \lim_J F \arrow[d, "(\lambda_F)_j"] \\
Fj \arrow[r, "1_{Fj}"'] & Fj
\end{tikzcd}$$

commutes for all $j \in J$. Since $1_{\lim_J F}$ has this property, it follows that $\lim_J 1_F = 1_{\lim_J F}$. Next, suppose that $\eta : F \Rightarrow G$ and $\theta : G \Rightarrow H$ in $\mathscr{C}^J_{\mathbf{lim}}$. Then the diagram

$$\begin{tikzcd}
\lim F \arrow[r, "\lim_J \eta"] \arrow[d, "(\lambda_F)_j"'] & \lim G \arrow[r, "\lim_J \theta"] \arrow[d, "(\lambda_G)_j"] & \lim H \arrow[d, "(\lambda_H)_j"] \\
F_j \arrow[r, "\eta_i"'] \arrow[rr, bend right, "(\theta \circ \eta)_j"'] & G_j \arrow[r, "\theta_j"'] & K_j
\end{tikzcd}$$

commutes for all $j \in J$. Thus, $(\lambda_H)_j \circ (\lim_J \theta \circ \lim_J \eta) = (\theta \circ \eta)_j \circ (\lambda_F)_j$ for all $j \in J$, but $\lim_J(\theta \circ \eta) : \lim_J F \to \lim_J H$ is the unique morphism with this property. Thus, $\lim_J(\theta \circ \eta) = \lim_J \theta \circ \lim_J \eta$. Thus, $\lim_J$, as defined in this property, is a functor. ☐

Corollary 9.3. If $F, G : J \to \mathscr{C}$ have limits and $F \cong G$ naturally, then $\lim_J F \cong \lim_J G$ in $\mathscr{C}$.

Proof. Choose an isomorphism $\eta : F \cong G$. This is a morphism in $\mathscr{C}^J_{\mathbf{lim}}$ and hence any choice of a functor $\lim_J : \mathscr{C}^J_{\mathbf{lim}} \to \mathscr{C}$ carries η to an isomorphism $\lim_J \eta : \lim_J F \xrightarrow{\cong} \lim_J G$ in $\mathscr{C}$. ☐

We can also use the functoriality of $\lim_J$ in natural transformations to construct limits "pointwise" in functor categories.

To state the result, note first that if J is an indexing category and $\mathscr{C}$ is another category, then for any $j \in J$, thee is an evaluation functor $\mathbf{ev}_j : \mathscr{C}^J \to \mathscr{C}$ that sends $F \mapsto Fj$ and $(\eta : F \Rightarrow G) \mapsto \eta_j : Fj \to Gj$.

Theorem 9.4. Let I and J be indexing categories and $\mathscr{C}$ be another category. Suppose $F : I \to \mathscr{C}^J$ is a diagram such that for all $j \in J$, the diagram

$$F(-)(j) := \mathbf{ev}_j \circ F : I \to \mathscr{C}$$

has a limit. Then F has a limit. Moreover, we can construct $\lim_I F$ in such a way that

$$\mathbf{ev}_j(\lim_I F) = \lim_I(\mathbf{ev}_j \circ F) = \lim_I F(-)(j).$$

Proof. We construct a limit of $F : I \to \mathscr{C}^J$ from limits of the diagrams $F(-)(j) : I \to \mathscr{C}^J \to \mathscr{C}$. For each $j \in J$, choose a limit cone

$$(\lim_I F(-)(j), (\lambda(j)_i : \lim_I F(-)(j) \to F(i)(j)_{i \in I})$$

We begin by extending these choices to a functor $\lim_I F(-) : J \to \mathscr{C}$. Let $f : j \to j'$ in J. Then for each $e : i \to i'$ in I, the diagram

$$\begin{tikzcd}
& \lim_I F(-)(j) \arrow[dl, "\lambda(j)_i"'] \arrow[dr, "\lambda(j)_{i'}"] & \\
F(i)(j) \arrow[rr, "F(e)_j"] \arrow[d, "F(i)(f)"'] & & F(i')(j) \arrow[d, "F(i')(f)"] \\
F(i)(j') \arrow[rr, "F(e)_{j'}"'] & & F(i')(j')
\end{tikzcd}$$

commutes, because $\lambda(j) : \lim_I F(-)(j) \Rightarrow F(-)(j)$ is a cone over $F(-)(j)$ and $F(e) : F(i) \Rightarrow F(i')$ is natural. By the universal property of $\lim_I F(-)(j')$, there is a unique induced morphism $\lim_I F(-)(f) : \lim_I F(-)(j) \to \lim_I F(-)(j')$ such that the square

$$\begin{tikzcd}
\lim_I F(-)(j) \arrow[r, "\lim_I F(-)(f)"] \arrow[d, "\lambda(j)_i"'] & \lim_I F(-)(j') \arrow[d, "\lambda(j')_i"] \\
F(i)(j) \arrow[r, "F(i)(f)"'] & F(i)(j')
\end{tikzcd}$$

commutes for all $i \in I$. We obtain a functor $\lim_I F(-) : J \to \mathscr{C}$. Indeed, $\lim_I F(-)$ preserves domains and codomains by construction, and the uniqueness of the maps $\lim_I F(-)(f)$ induced by morphisms $f : j \to j'$ in J ensures that $\lim_I F(-)$ also preserves identities and composition.

Now define morphisms $\Lambda(i)_j$ by

$$\Lambda(i)_j := \lambda(j)_i : \lim_I F(-)(j) \to F(i)(j).$$

Then, by the definition of the morphisms $\lim_I F(-)(f)$, the morphisms $\Lambda(i)_j$ are natural in $j \in J$, so that we obtain natural transformations $\Lambda(i) : \lim_I F(-) \Rightarrow F(i)$ for all $i \in I$. Moreover, for each $e : i \to i'$ in I, the triangle

$$\begin{tikzcd}
& \lim_I F(-) \arrow[dl, "\Lambda(i)"'] \arrow[dr, "\Lambda(i')"] & \\
F(i) \arrow[rr, "F(e)"'] & & F(i')
\end{tikzcd}$$

of natural transformations commutes in $\mathscr{C}^J$ because evaluating at each $j \in J$ recovers a commuting triangle for the (limit) cone $(\lim_I F(-)(j), (\lambda(j)_i : \lim_I F(-)(j) \to F(i)(j))_i)$. Thus $(\lim_I F(-), (\Lambda(i) : \lim_I F(-) \Rightarrow F(i))_i)$ is a cone over $F : I \to \mathscr{C}$. To see that it is a limit cone, note that if $(T, (\tau(i) : T \Rightarrow F(i))_{i \in I})$ is any other cone over $F : I \to \mathscr{C}^J$, then any comparison map $\sigma : T \Rightarrow \lim_I F(-)$ such that $\Lambda(i) \circ \sigma = \tau(i)$ for all $i \in I$ must have j-th component satisfying $\lambda(j)_i \circ \sigma_j = \tau(i)_j$ for all $i \in I$, so by the universal property of $\lim_I F(-)(j)$, it must be the unique morphism induced by the cone $(Tj, (\tau(i)_j : T_j \to F(i)(j))_{i \in I})$. Thus, σ is unique if it exists, and taking these induced maps as the definition of σ gives the desired comparison natural transformation $\sigma : T \Rightarrow \lim_I F(-)$. ☐

Corollary 9.5. If $\mathscr{C}$ is complete, then so is every functor category $\mathscr{C}^J$.

Finally, we consider iterated limits.

Here is the setup: let I and J be indexing categories and $F : I \times J \to \mathscr{C}$ be a diagram in the category $\mathscr{C}$. For each $i \in I$, there is a diagram $F(i,-) : J \to \mathscr{C}$ that sends $j \mapsto F(i,j)$ and $(f : j \to j') \mapsto (F(i,f) := F(1_i, f) : F(i,j) \to F(i,j'))$.

Suppose that for all $i \in I$, the diagram $F(i,-)$ has a limit, and choose a limit cone $(\lim_{j\in J} F(i,j), \lambda(i) : \lim_{j\in J} F(i,j) \Rightarrow F(i,-))$ for each $i \in I$. We extend these choices to a functor $\lim_{j\in J} F(i,j) : I \to \mathscr{C}$ as follows: note first that for any $e : i \to i'$ in I and $f : j \to j'$ in J, the diagram

$$\begin{tikzcd}
& \lim\limits_{j\in J} F(i,j) \arrow[dl, "\lambda(i)_j"'] \arrow[dr, "\lambda(i)_{j'}"] & \\
F(i,j) \arrow[rr, "F(1_i,f)"] \arrow[d, "F(e,1_j)"'] & & F(i,j') \arrow[d, "F(e,1_{j'})"] \\
F(i',j) \arrow[rr, "F(1_{i'},f)"'] & & F(i',j')
\end{tikzcd}$$

commutes. By the universal property of $\lim_{j\in J} F(i',j)$, there is a unique morphism $\lim_{j\in J} F(e,j) : \lim_{j\in I} F(i,j) \to \lim_{j\in J} F(i',j)$ such that the square

$$\begin{tikzcd}
\lim\limits_{j\in J} F(i,j) \arrow[r, "\lim\limits_{j\in J} F(e,j)"] \arrow[d, "\lambda(i)_j"'] & \lim\limits_{j\in J} F(i',j) \arrow[d, "\lambda(i')_j"] \\
F(i,j) \arrow[r, "F(e,1_j)"'] & F(i',j)
\end{tikzcd}$$

commutes for all $j \in J$. The uniqueness of the morphisms $\lim_{j\in J} F(e,j)$ ensures that we obtain a functor $\lim_{j\in J} F(-,j) : I \to \mathscr{C}$.

Suppose further that $\lim_{j\in J} F(i,j) : I \to \mathscr{C}$ has a limit, and choose a limit cone

$$(\lim_{i\in I} \lim_{j\in J} F(i,j), \mu : \lim_{i\in I} \lim_{j\in J} F(i,j) \Rightarrow \lim_{j\in J} F(i,j)).$$

Theorem 9.6. Keep the setup above. Then the object $\lim_{i\in I} \lim_{j\in J} F(i,j)$, together with the morphisms

$$(\lambda(i)_j \circ \mu_i : \lim_{i\in I} \lim_{j\in J} F(i,j) \to \lim_{j\in J} F(i,j) \to F(i,j))_{(i,j)\in I\times J}$$

form a limit cone over $F : I \times J \to \mathscr{C}$. Consequently $\lim_{i\in I} \lim_{j\in J} F(i,j) \cong \lim_{(i,j)\in I\times J} F(i,j)$.

Proof. A diagram chase shows that $(\lambda(i)_j \circ \mu_i)_{i,j}$ is a con. Now suppose $T \in \mathscr{C}$ and $(\tau_{i,j} : T \to F(i,j))_{(i,j)\in I\times J}$ is a cone over F. Fix $i \in I$. Then for any $f : j \to j'$ in J, the diagram

$$\begin{tikzcd}
& T \arrow[dl, "\tau_{i,j}"'] \arrow[dr, "\tau_{i,j'}"] & \\
F(i,j) \arrow[rr, "F(1_i,f)"'] & & F(i,j')
\end{tikzcd}$$

commutes, so by the universal property of $\lim_{j\in J} F(i,j)$ there is a unique $\tau_i : T \to \lim_{j\in J} F(i,j)$ such that $\lambda(i)_j \circ \tau_i = \tau_{i,j}$ for $j \in J$. Now, as $i \in I$ varies, we obtain a cone $(T, (\tau_i : T \to \lim_{j\in J} F(i,j))_{i\in I})$ over $\lim_{j\in J} F(i,j)$, so by the universal property of $\lim_{i\in I} \lim_{j\in J} F(i,j)$, there is a unique morphism $\tau : T \to \lim_{i\in I} \lim_{j\in J} F(i,j)$ such that $\mu_i \circ \tau = \tau_i$ for all $i \in I$. Then $\lambda(i)_j \circ \mu_i \circ \tau = \lambda(i)_j \circ \tau_i = \tau_{i,j}$ for all $(i,j) \in I \times J$, so a comparison morphism $\tau : T \to \lim_{i\in I} \lim_{j\in J} F(i,j)$ exists. If τ'

is another such comparison morphism, then the uniqueness portion of the universal property of $\lim\limits_{i\in I}\lim\limits_{j\in J} F(i,j)$ and the $\lim\limits_{j\in J} F(i,j)$'s implies that $\tau = \tau'$, so the comparison map is unique. □

Similar considerations apply to the iterated limit $\lim\limits_{j\in J}\lim\limits_{i\in I} F(i,j)$ (provided all the necessary limits exist), and we arrive at the following conclusion:

Corollary 9.7. If the limits $\lim\limits_{i\in I}\lim\limits_{j\in J} F(i,j)$ and $\lim\limits_{j\in J}\lim\limits_{i\in I} F(i,j)$ associated to a diagram $F : I \times J \to \mathscr{C}$ exists in $\mathscr{C}$, then they are isomorphic and define the limit $\lim\limits_{(i,j)\in I\times J} F(i,j)$.

This can be thought of as a kind of Fubini Theorem for categorical limits.

10 LIMITS AND HOM FUNCTORS

We now describe an important interaction between (co)limits and hom functors.

First, a definition.

Definition 10.1 (Preserve, Continuous/Cocontinuous)**.** For any class of diagrams $K : J \to \mathscr{C}$ valued in a category $\mathscr{C}$, a functor $F : \mathscr{C} \to \mathscr{D}$ preserves those (co)limits if, for any diagram $K : J \to \mathscr{C}$ in the class and (co)limit (co)cone over K, the image of this (co)cone defines a (co)limit (co)cone over (under) the composite diagram $FK : J \to \mathscr{D}$.

We say that a functor $F : \mathscr{C} \to \mathscr{D}$ is (co)continuous if it preserves all small (co)limits.

Covariant hom functors are an important class of limit-preserving functors.

Theorem 10.2. Let $\mathscr{C}$ be a locally small category and $C \in \mathscr{C}$. Then the covariant hom functor $\mathscr{C}(C,-) : \mathscr{C} \to \mathbf{Set}$ preserves any limit that exists in $\mathscr{C}$, so if $F : J \to \mathscr{C}$ is a diagram and $\lim\limits_{j\in J} F_j$ exists, then $\lim\limits_{j\in J}\mathscr{C}(C,Fj)$ exists and $\mathscr{C}(C,\lim\limits_{j\in J} Fj) \cong \lim\limits_{j\in J}\mathscr{C}(C,Fj)$.

Proof. Suppose $F : J \to \mathscr{C}$ is a diagram and $(\lim_J F, \lambda : \lim_J F \Rightarrow F)$ is a limit cone over F. Apply $\mathscr{C}(C,-)$. We obtain a cone with vertex $\mathscr{C}(C,\lim_J F)$ and lets $\mathscr{C}(C,\lambda_j) = (\lambda_j)_* : \mathscr{C}(C,\lim_J F) \to \mathscr{C}(C,Fj)$, which we must show is a limit cone in $\mathbf{Set}$. To that end, suppose $T \in \mathbf{Set}$ and that $\tau_j : T \to \mathscr{C}(C,Fj)$ are the legs of a cone over $\mathscr{C}(C,F-) : J \to \mathbf{Set}$. Then for each element $x \in T$, $\tau_j(x) : C \to Fj$ and for all $f : j \to j'$ in J, the triangle below commutes:

$$\begin{tikzcd} & T \arrow[dl, "\tau_j"'] \arrow[dr, "\tau_{j'}"] & \\ \mathscr{C}(C,Fj) \arrow[rr, "Ff_*"'] & & \mathscr{C}(C,Fj') \end{tikzcd}$$

Thus, $Ff \circ \tau_j(x) = Ff_*(\tau_j(x)) = \tau_{j'}(x)$, meaning

$$\begin{tikzcd} & C \arrow[dl, "\tau_j(x)"'] \arrow[dr, "\tau_{j'}(x)"] & \\ Fj \arrow[rr, "Ff"'] & & Fj' \end{tikzcd}$$

commutes for all $f : j \to j'$ in J. This means $(C, (\tau_j(x))_{j\in J})$ is a cone over $F : J \to \mathscr{C}$ in $\mathscr{C}$ for all $x \in T$.

Now suppose $\tau : T \to \mathscr{C}(C,\lim_J F)$ is a function such that $(\lambda_j)_* \circ \tau = \tau_j$ for all $j \in J$. Then for all $x \in T$, $\tau(x) : C \to \lim_J F$ and $\tau_j(x) = (\lambda_j)_*(\tau(x)) = \lambda_j \circ \tau(x)$. Thus, $\tau(x)$ is the unique morphism into $\lim_J F$ induced by the cone $(C, (\tau_j(x))_{j\in J})$. This shows that the values of τ are completely determined, so that τ is unique if it exists.

On the other hand, since $(C, (\tau_j(x))_{j\in J})$ is a cone over F for all $x \in T$, we can define $\tau(x) : C \to \lim_J F$ to be the unique comparison map. This gives a functor $\tau : T \to \mathscr{C}(C,\lim_J F)$, and for all $j \in J$, we have $((\lambda_j)_* \circ \tau)(x) = \lambda_j(\tau(x)) = \tau_j(x)$, so that $(\lambda_j)_* \circ \tau = \tau_j$. Thus τ is a comparison map between $(T, (\tau_j)_{j\in J})$ and $(\mathscr{C}(C,\lim_J F), ((\lambda_j)_*)_{j\in J})$, which shows that $(\mathscr{C}(C,\lim_J F), ((\lambda_j)_* : \mathscr{C}(C,\lim_J F), ((\lambda_j)_*)_{j\in J})$ is a limit cone. □

Corollary 10.3. Let $\mathscr{C}$ be a locally small category. Then the covariant Yoneda embedding $Y : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}^{\text{op}}}$ preserves limits, so if $F : J \to \mathscr{C}$ is a diagram and $\lim\limits_{j\in J} Fj$ exists , then $\lim\limits_{j\in J} \mathscr{C}(-, Fj)$ exists and $\mathscr{C}(-, \lim\limits_{j\in J} Fj) \cong \lim\limits_{j\in J} \mathscr{C}(-, Fj)$.

Proof. Suppose $F : J \to \mathscr{C}$ is a diagram and $(\lim_J F, \lambda : \lim_J F \Rightarrow F)$ is a limit cone over F. Consider the composite diagram $Y \circ F : J \to \mathbf{Set}^{\mathscr{C}^{op}}$. For any $C \in \mathscr{C}$, the diagram

$$\begin{gathered}\mathbf{ev}_C \circ Y \circ F : J \to \mathbf{Set}\\ j \mapsto \mathscr{C}(C, Fj)\\ (f : j \to j') \mapsto (Ff_* : \mathscr{C}(C, Fj) \to \mathscr{C}(C, Fj'))\end{gathered}$$

equals $\mathscr{C}(C, F-) : J \to \mathbf{Set}$, and by the previous theorem, it has a limit given by $(\mathscr{C}(C, \lim_J F), ((\lambda_j)_* : \mathscr{C}(C, \lim_J F) \to \mathscr{C}(C, Fj))_{j\in J})$. We can construct a limit of $YF : J \to \mathbf{Set}^{\mathscr{C}^{\text{op}}}$ pointwise from these limits. Unwinding these construction, the result is precisely the cone

$$(\mathscr{C}(-, \lim_J F), ((\lambda_j)_* : \mathscr{C}(-, \lim_J F) \Rightarrow \mathscr{C}(-, Fj))_{j\in J})$$

which is just $(Y \lim_J F, Y\lambda_j : Y \lim_J F \Rightarrow YFj)$.

Thus, Y preserves the limit $(\lim_J F, \lambda : \lim_J F \Rightarrow F)$. ☐

Just as functors can reflect isomorphisms, so too can they reflect (co)limits.

Definition 10.4 (Reflect)**.** For any class of diagrams $K : J \to \mathscr{C}$ valued in a category $\mathscr{C}$, a functor $F : \mathscr{C} \to \mathscr{D}$ reflects these limits if any cone over a diagram $K : J \to \mathscr{C}$, whose image upon applying F is a limit cone for the diagram $FK : J \to \mathscr{D}$, is a limit cone for $K : J \to \mathscr{C}$.

Reflection of colimits is defined similarly.

Proposition 10.5. If $F : \mathscr{C} \to \mathscr{D}$ is fully faithful, then F reflects limits and colimits.

Proof. We treat the case for limits. The argument for colimits is dual.

Suppose $K : J \to \mathscr{C}$ is a diagram in $\mathscr{C}$ and $(C, \lambda : C \Rightarrow K)$ is a cone over K such that $(FC, F\lambda : FC \Rightarrow FK)$ is a limit over FK. We must prove that $(C, \lambda : C \Rightarrow K)$ is a limit cone as well.

Suppose $(T, (\tau_j : T \to Kj)_{j\in J})$ is a cone over K. Applying F, we obtain a cone $(FT, (F\tau_j : FT \to FK_j)_{j\in J})$ and a limit cone $(FC, (F\lambda_j : FC \to FK_j)_{j\in J})$ over FK. Thus, there is a unique $t : FT \to FC$ such that $F\lambda_j \circ t = F_{\tau_j}$ for all $j \in J$. Since F is full, there is some $\tilde{t} : T \to C$ such that $F\tilde{t} = t$, and then $F(\lambda_j \circ \tilde{t}) = F\lambda_j \circ t = F\tau_j$, and then since F is faithful, it follows $\lambda_j \circ \tilde{t} = \tau_j$ for all $j \in J$. Thus, there is a comparison map between $(T, \tau : T \Rightarrow K)$ and $(C, \lambda : C \Rightarrow K)$.

It remains the show that $\tilde{t} : T \to C$ is unique. Let $s : T \to C$ be another morphism such that $\lambda_j \circ s = \tau_j$ for all $j \in J$, and apply F. Then $F\lambda_j \circ Fj = F\tau_j = F\lambda_j \circ F\tilde{t}$ for all $j \in J$, and since $(FC, (F\lambda_j : FC \to FKj)_{j\in J})$ is a limit cone, it follows that $Fs = F\tilde{t}$, then $s = \tilde{t}$ because F is faithful, which proves that $\tilde{t}$ is unique. This proves that $(C, \lambda : C \Rightarrow F)$ is a limit over F. ☐

Corollary 10.6. Let $\mathscr{C}$ be a locally small category. Then the covariant Yoneda embedding $Y : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}^{op}}$ reflects limits.

In summary:

Theorem 10.7. Let $\mathscr{C}$ be a locally small set.

1. For any $C \in \mathscr{C}$, the functor $\mathscr{C}(C, -) : \mathscr{C} \to \mathbf{Set}$ preserves limits.
2. The Yoneda embedding $Y : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}^{op}}$ preserves and reflects limits.

These facts allow us to study limits in locally small categories in terms of limits in $\mathbf{Set}$. Indeed, if $F : J \to \mathscr{C}$ is a small diagram and $(\lim_J F, \lambda : \lim_J F \Rightarrow F)$ is a limit of F in $\mathscr{C}$, then by (2) $(Y(\lim_J F), Y\lambda : Y(\lim_J F) \Rightarrow YF)$ is a limit of YF in $\mathbf{Set}^{\mathscr{C}^{op}}$, i.e. $\mathscr{C}(-, \lim\limits_{j\in J} Fj) \cong \lim\limits_{j\in J} \mathscr{C}(-, Fj)$, and this limit is constructed pointwise in $\mathbf{Set}$.

In fact, we can do a bit better. Suppose J is a small category, $\mathscr{C}$ is a locally small category, and $F : J \to \mathscr{C}$ is a diagram. Then the diagram $Y \circ F : J \to \mathbf{Set}^{\mathscr{C}^{op}}$ always has a limit, because for all $C \in \mathscr{C}^{\text{op}}$, $\mathbf{ev}_C \circ Y \circ F : J \to \mathbf{Set}$ is a small diagram and $\mathbf{Set}$ is complete. Thus, YF has a limit, constructed pointwise, and unwinding the construction reveals that $\lim_J YF \cong \mathbf{Cone}(-, F)$ naturally. Thus, $\lim\limits_{j\in J} Fj$, if it exists, is a representation of $\lim\limits_{j\in J} \mathscr{C}(-, Fj) \cong \lim_J YF$.

Definition 10.8 (Representable Definition of Limits)**.** Suppose $F : J \to \mathscr{C}$ is a small diagram in a locally small category. A limit of F, if it exists, is a representation of $\lim\limits_{j \in J} \mathscr{C}(-, Fj) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$.

Thus, once we know what limits in $\mathbf{Set}$ are, we can define limits in $\mathbf{Set}^{\mathscr{C}^{op}}$ pointwise, and then define general limits representably in terms of these limits in $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$. Again, this allows us to study arbitrary limits in terms of limits in $\mathbf{Set}$.

We conclude by recording the dual results for colimits, using the fact that limits in $\mathscr{C}^{\mathrm{op}}$ are colimits in $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}(C, -) = \mathscr{C}(-, C) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$.

Theorem 10.9. Let $\mathscr{C}$ be any locally small category.

1. For any $C \in \mathscr{C}$, the functor $\mathscr{C}(-, C) : \mathscr{C}^{\mathrm{op}} : \mathbf{Set}$ carries colimits in $\mathscr{C}$ to limits in $\mathbf{Set}$.
2. The contrvariant Yoneda Embedding $Y : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}^{\mathscr{C}}$ both preserves and reflects limits in $\mathscr{C}^{\mathrm{op}}$, i,e, a cocone under a diagram in $\mathscr{C}$ is a colimit cocone if and only if its image under y defines a limit cone in $\mathbf{Set}^{\mathscr{C}}$.

Definition 10.10 (Representable Definition of Colimits)**.** Suppose $F : J \to \mathscr{C}$ is a small diagram in a locally small category. A colimit of F, if it exists, is a representation of $\lim\limits_{j \in J^{\mathrm{op}}} \mathscr{C}(F^{\mathrm{op}} j, -) : \mathscr{C} \to \mathbf{Set}$.

11 Cardinality and Limits

Many of the large categories that one encounters in practice are complete and cocomplete. On the other hand, small, complete/cocomplete categories are far less common.

The difference ultimately stems from Cantor's diagonal argument, which we now discuss.

Definition 11.1 (Equipotent)**.** Two sets A and B are equipotent or have the sam cardinality if there is a bijection $f : A \to B$. In such a case, we write $|A| = |B|$.

Note that equipotence is an equivalence relation:

- the identity $1_A : A \to A$ is bijective.
- if $f : A \to B$ is bijective, then so if $f^{-1} : B \to A$.
- the composite of two bijections is a bijection.

Definition 11.2. Let A and B be sets. The cardinality of A is less than or equal to the cardinality of B if there is an injective function $f : A \to B$. In such a case, we write $|A| \leqslant |B|$.

Note that the relation $|A| \leqslant |B|$ is reflexive and transitive:

- The identity $1_A : A \to A$ is injective.
- The composite of two injections is an injection.

As suggested by the notion and terminology, the relation $|\cdot| \leqslant |\cdot|$ is also antisymmetric, but this is a theorem.

Theorem 11.3 (Cantor-Schröder-Berstein)**.** Let A and B be sets. If $|A| \leqslant |B|$ and $|B| \leqslant |A|$, then $|A| = |B|$.

Proof. Let $f : A \to B$ and $g : B \to A$ be injections. We must construct a bijection between A and B. The strategy is to partition A and B into subsets in a way that makes the behavior of f and g transparent, and then to build a bijection compatible with the partitions.

Let $a \in A$. By taking preimages repeatedly, we can construct a finite or infinite sequence

$$(*)\; a, g^{-1}a, f^{-1}g^{-1}a, g^{-1}f^{-1}g^{-1}a, \cdots$$

of elements that alternate between A and B. Define the length of a to be the number of terms in $(*)$ if the sequence is finite and ∞ if the sequence is infinite. Since each $a \in A$ has a unique length, we can partition A be length:

$$A = A_1 \coprod A_2 \coprod \cdots \coprod A_\infty,$$

where $A_n = \{a \in A \mid a \text{ has length } n\}$. Similarly, we can construct a partition $B = B_1 \bigsqcup B_2 \bigsqcup \cdots \bigsqcup B_\infty$ of B by length.

Next, observe that for any $0 < n < \infty$, $f : A \to B$ restricts to a bijection $f_n : A_n \to B_{n+1}$, and that $f : A \to B$ also restricts to a bijection $f_\infty : A_\infty \to B_\infty$. Similar considerations apply to $g : B \to A$.

Finally, define a bijection $h : A \to B$ by

$$h(a) = \begin{cases} f_n(a) \text{ if the length of } a \text{ is an odd number } n \\ g_{n-1}^{-1}(a) \text{ if the length of } a \text{ is an even number } n \\ f_\infty(a) \text{ if the length of } a \text{ is } \infty \end{cases}$$

□

Thus, $\leqslant$ behaves like a partial order. One can also prove that any two sets A and B have comparable cardinalities using the well-ordering principle and transfinite recursion, but we shall not need this.

Definition 11.4. Let A and B be sets. We write $|A| < |B|$ if $|A| \leqslant |B|$ and $|A| \neq |B|$, i.e. if there is an injection $A \to B$ but no bijections.

Here is a classic example of sets A and B such that $|A| < |B|$.

Theorem 11.5 (Cantor's Diagonal Argument)**.** For any set A, we have $|A| < |2^A|$, where $2^A = \mathbf{Set}(A, \{0, 1\})$.

Proof. For any $a \in A$, let $\delta_a(x) = \begin{cases} 1 \text{ if } x = a \\ 0 \text{ if } x \neq a \end{cases}$.

Then the function $\delta : A \to 2^A$ sending $a \mapsto \delta_a$ is injective, so that $|A| \leqslant |2^A|$.

Now suppose for contradiction that $|A| = |2^A|$, so that there is a bijection $f : A \xrightarrow{\cong} 2^A$. Write $f(a) = f_a$ for all $a \in A$, then we have the following table:

$$\begin{array}{ccccc} & f_{a_0} & f_{a_1} & f_{a_2} & \cdots \\ a_0 & f_{a_0}(a_0) & f_{a_1}(a_0) & f_{a_2}(a_0) & \cdots \\ a_1 & f_{a_1}(a_1) & f_{a_1}(a_1) & f_{a_2}(a_1) & \cdots \\ a_2 & f_{a_0}(a_2) & f_{a_1}(a_2) & f_{a_2}(a_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

and define $g(a) = 1 - f_a(a) : A \to \{0, 1\}$. Then $g \neq f_a$ for any $a \in A$ because if $g = f_a$, then $1 - f_a(a) = g(a) = f_a(a)$, then $f : A \to 2^A$ is not surjective, contradiction. Therefore, $|A| \neq |2^A|$. □

Corollary 11.6. For any set A, there is no injection $2^A \to A$.

Proof. If there were an injection $2^A \to A$, then $|2^A| \leqslant |A|$ and $|A| \leqslant |2^A|$. Then $|A| = |2^A|$ by Cantor-Schröder-Bernstein theorem, a contradiction to Cantor's diagonal argument. □

With these preliminaries on cardinality finished, we return to category theory.

Theorem 11.7 (Freyd)**.** If $\mathscr{C}$ is a small category and $\mathscr{C}$ is complete, then $\mathscr{C}$ is a preorder category.

Proof. Let $\mathscr{C}$ be a small, complete category, and suppose for contradiction that $\mathscr{C}$ is not a preorder category. Then there are objects $A, B \in \mathscr{C}$ such that there is more than one morphism $A \to B$. Choose two distinct morphisms $f, g : A \to B$, and consider the morphisms $A \to \prod\limits_{\mathbf{Mor}(\mathscr{C})} B$. Note that the right-hand product exists because $\mathbf{Mor}(\mathscr{C})$ is a set and $\mathscr{C}$ is complete. Then

$$\begin{aligned} 2^{\mathbf{Mor}(\mathscr{C})} &\cong \{f, g\}^{\mathbf{Mor}(\mathscr{C})} \\ &\cong \prod_{\mathbf{Mor}(\mathscr{C})} \{f, g\} \end{aligned}$$

$$\begin{aligned} &\subseteq \prod_{\mathbf{Mor}(\mathscr{C})} \mathscr{C}(A,B) \\ &\cong \mathscr{C}(A, \prod_{\mathbf{Mor}(\mathscr{C})} B) \\ &\subseteq \mathbf{Mor}(\mathscr{C}) \end{aligned}$$

which gives an injection $2^{\mathbf{Mor}}(\mathscr{C}) \to \mathbf{Mor}(\mathscr{C})$. This contradicts the previous corollary, so $\mathscr{C}$ is a preorder category. □

So, just as there are plenty of small categories that are not preorder categories, so too are there plenty of small categories that are not complete.

12 ADJUNCTIONS

Suppose $\mathscr{C}$ and $\mathscr{D}$ are categories and $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ are a pair of functors.

There are a number of ways F and G might be related. Here are two that we have considered: F and G might be

1. inverse - i.e. $G \circ F = 1_{\mathscr{C}}$ and $F \circ G = 1_{\mathscr{D}}$. In this case, F and G "undo" each other.
2. pseudoinverse - i,e, there are natural isomorphisms $\eta : 1_{\mathscr{C}} \cong GF$ and $\varepsilon : FG \cong 1_{\mathscr{D}}$. In this case, F and G "undo" each other up to natural isomorphism.

In both of these cases, the functors F and G are dual in the sense that they are opposite to each other.

We now consider a more general form of duality that a pair of functors $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ might exhibit. By way of example, suppose $\mathscr{C} = \mathbf{Set}$ and $\mathscr{D} = \mathbf{Mon}$ are the categories of sets and monoids, respectively. There is a "forgetful" functor $U : \mathbf{Mon} \to \mathbf{Set}$ that sends a monoid to its underlying set and a monoid homomorphism to its underlying function. Intuitively, this is the "most efficient" way of turning a monoid into a set. But what about the other direction? What is the "most efficient" way of turning a set into a monoid?

Given a set X, one could try to choose a unit element $e \in X$ and define a binary operator $\cdot : X \times X \to X$ that makes $(X, \cdot, e)$ into a monoid, but these choices are in no way canonical.

A more natural thing to do is to start multiplying the elements of X together to generate a monoid. This construction is called the free monoid on X.

Remark 12.1 (Construction)**.** Let X be a set. The free monoid on X, denoted MX, is defined as follow:

1. The underlying set of MX is the set of all finite tuples $(x_1, \cdots, x_m)$ of elements in X. The empty tuple is allowed, and is denoted ( ).
2. The multiplication on MX is concatenation, i.e. $(x_1, \cdots, x_m) \cdot (y_1, \cdots, y_n) = (x_1, \cdots, x_m, y_1, \cdots, y_n)$.
3. The unit of MX is the empty tuple ( ).

We think of an element $x \in X$ as the length 1 tuple (x), so that an arbitrary element $(x_1, \cdots, x_m) \in MX$ is uniquely expressed as a product $(x_1, \cdots, x_m) = (x_1) \cdot \cdots \cdot (x_m) \approx x_1 \cdot \cdots \cdot x_m$.

In this sense, MX is the set of all finite, formal products of elements in X, and the elements of X form a "basis" of MX.

Now, we can extend M to a functor $M : \mathbf{Set} \to \mathbf{Mon}$. Given a set map $f : X \to Y$, we define a monoid homomorphism $Mf : MX \to MY$ by sending each generator (x) to $(f(x))$ and then extending multiplicatively. In other words, $Mf(x_1, \cdots, x_m) = (f(x_1), \cdots, f(x_m))$.

In summary, there is a free monoid functor $M : \mathbf{Set} \to \mathbf{Mon}$ that builds a monoid out of a set in the "most efficient" way possible. Intuitively, one would like to say that $M : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U$ are "dual:. They are certainly not inverse, but they are doing the same sort of thing, but in opposite directions.

To formalize this kind of duality, let us analyze the situation a bit further.

We have said repeatedly that MX is the monoid built from X in the "most efficient" way possible. Category theory gives us the tools to make this idea precise.

Observe that MX is a monoid, and that there is a function $\eta_X : X \to UMX$ that sends $x \mapsto (x)$, i.e. maps x in as a "basis". This expresses the fact that MX is a monoid that is built from X.

Next, note that if N is any monoid and $f : X \to UN$ is a function, then there is a unique monoid homomorphism $\bar{f} : MX \to N$ such that $U\bar{f} \circ \eta_X = f$, i.e. such that $\bar{f}(x) = f(x)$. Here is the diagram:

$$\begin{tikzcd}
MX \arrow[r, dashed, "\exists ! \bar{f}"] & N
\end{tikzcd}$$

such that

$$\begin{tikzcd}
UMX \arrow[r, "U\bar{f}"] & UN \\
X \arrow[u, "\eta_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Indeed, the set X behaves very much like a basis. Given $f : X \to UN$, we can define a monoid homomorphism $\bar{f} : MX \to N$ by sending $(x) \mapsto f(x)$ and then extending multiplicatively, i.e. $\bar{f}(x_1, \cdots, x_m) = f(x_1) \cdot_N \cdots \cdot_N f(x_m)$.

By design, $*U\bar{f} \circ \eta_X)(x) = \bar{f}(x) = f(x)$, and if $\varphi : MX \to N$ is any monoid homomorphism such taht $U\varphi \circ \eta_X = f$, then we have $\varphi(x) = (U\varphi \circ \eta_X)(x) = f(x)$, so that

$$\begin{aligned}
\varphi(x_1, \cdots, x_m) &= \varphi((x_1) \cdot \cdots (x_m)) \\
&= \varphi(x_1) \cdot_N \cdots \cdot_N \varphi(x_m) \\
&= f(x_1) \cdot_N \cdots \cdot_N f(x_m) \\
&= \bar{f}(x_1, \cdots, x_m)
\end{aligned}$$

Thus, $\bar{f} : MX \to N$ is the unique monoid homomorphism such that $U\bar{f} \circ \eta_X = f$.

This shows that MX, together with the function $\eta_X : X \to UMX$ is the initial monoid, together with a function from X to its underlying set. This formalizes the idea that MX is built from X in the "most efficient" way possible.

Said differently, $(MX, \eta_X : X \to UMX)$ is a universal element of the functor $\mathbf{Set}(X, U-) : \mathbf{Mon} \to \mathbf{Set}$, which by Yoneda Lemma is eqiuvalent to a representation

$$\begin{aligned}
\varphi_{X,N} : \mathbf{Mon}(MX, N) &\cong \mathbf{Set}(X, UN) \\
\varphi &\mapsto U\varphi \circ \eta_X
\end{aligned}$$

natural in $N \in \mathbf{Mon}$.

However, even more is true. The map $\eta_X : X \to UMX$ is natural in X, and this implies $\varphi_{X,N}$ is also natural in X.

We arrive at the following definition:

Definition 12.2 (Adjunction, Adjoint)**.** An adjunction consists of a pair of functors $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, together with isomorphisms

$$\varphi_{C,D} : \mathscr{D}(FC, D) \cong \mathscr{C}(C, GD)$$

for all $C \in \mathscr{C}$ and $D \in \mathscr{D}$ that are natural in both variables. Here F is left adjoint to G and G is right adjoint to F. We write $F \dashv G$ to indicate this relation between F and G.

Naturality in $D \in \mathscr{D}$ is equivalent to saying that for any $f : FC \to D$ and $k : D \to D'$ in $\mathscr{D}$, we have

$$\begin{tikzcd}
C \arrow[r, "\varphi(f)"] \arrow[dr, "\varphi(k \circ f)"'] & GD \arrow[d, "Gk"] \\
& GD'
\end{tikzcd}$$

as a commutative diagram.

Naturality in $C \in \mathscr{C}$ is equivalent to saying that for any $f : FC \to D$ and $h : C' \to C$, we have

$$\begin{tikzcd}
C \arrow[r, "\varphi(f)"] & GD \\
C' \arrow[u, "h"] \arrow[ur, "\varphi(f\circ Fh)"'] &
\end{tikzcd}$$

commutes.

Here is another formulation of naturality, which can be useful.

Lemma 12.3. Suppose $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ are functors and that $\varphi_{C,D} : \mathscr{D}(FC, D) \cong \mathscr{C}(C, GD)$ is an isomorphism for all $C \in \mathscr{C}$ and all $D \in \mathscr{D}$. Then φ is natural in C and D simultaneously if and only if $(*)$ for any $f : FC \to D$ and $g : FC' \to D'$, $h : C \to C'$ and $k : D \to D'$:

$$\begin{tikzcd}
FC \arrow[r, "f"] \arrow[d, "Fh"'] & D \arrow[d, "k"] \\
FC' \arrow[r, "g"'] & D'
\end{tikzcd}$$

commutes if and only if

$$\begin{tikzcd}
C \arrow[r, "\varphi(f)"] \arrow[d, "h"'] & GD \arrow[d, "Gk"] \\
C' \arrow[r, "\varphi(g)"'] & GD'
\end{tikzcd}$$

commutes.

Proof. We first prove the $(\Rightarrow)$ direction.

Suppose φ is natural in C and D. Given any morphisms f, g, h, k as in $(*)$, notice that the left square commutes if and only if $k \circ f = g \circ Fh$ if and only if $\varphi(k \circ f)\varphi(g \circ Fh)$ (because φ is a bijection) if and only if $Gk \circ \varphi(f) = \varphi(g) \circ h$ (since φ is natural) if and only if the right square commutes.

Therefore, $(*)$ is true.

We now prove the $(\Leftarrow)$ direction.

Conversely, suppose that $(*)$ is true. Then, given any $f : FC \to D$ and $K : D \to D'$, we know since the following square commutes:

$$\begin{tikzcd}
FC \arrow[r, "f"] \arrow[d, "F1_C"'] & D \arrow[d, "k"] \\
FC \arrow[r, "k\circ f"'] & D'
\end{tikzcd}$$

then $Gk \circ \varphi(f) = \varphi(k \circ f)$, i.e. the following square commutes:

$$\begin{tikzcd}
C \arrow[r, "\varphi(f)"] \arrow[d, "1_C"'] & GD \arrow[d, "Gk"] \\
C \arrow[r, "\varphi(k\circ f)"'] & GD'
\end{tikzcd}$$

This shows that φ is natural in D. On the other hand, given any $f : FC \to D$ and $h : C' \to C$, we have

$$\begin{tikzcd}
FC' \arrow[r, "f\circ Fh"] \arrow[d, "Fh"'] & D \arrow[d, "1_D"] \\
FC \arrow[r, "f"'] & D
\end{tikzcd}$$

commutes, which means $\varphi(f) \circ h = \varphi(f \circ Fh)$, i.e. the following diagram also commutes:

$$\begin{tikzcd}
C' \arrow[r, "\varphi(f\circ Fh)"] \arrow[d, "h"'] & GD \arrow[d, "G1_D"] \\
C \arrow[r, "\varphi(f)"'] & GD
\end{tikzcd}$$

Therefore, φ is natural in C. This concludes the proof.

□

With this definition in too, we now consider some other examples of adjunctions.

Example 12.4. 1. $(\ )_+ : \mathbf{Set} \rightleftarrows \mathbf{Set}_* : U$

Let $\mathbf{Set}$ and $\mathbf{Set}_*$ be the categories of sets and pointed sets, respectively. There is a forgetful functor $U : \mathbf{Set}_* \to \mathbf{Set}$ that sends a pointed set (X, x) to its underlying set X and a basepoint-preserving function $f : (X, x) \to (Y, y)$ ot its underlying function $f : X \to Y$. In the other direction, there is a functor $(-)_+ : \mathbf{Set} \to \mathbf{Set}_*$ that sends a set X to ”the set X with a new basepoint adjoined”, i.e, $X_+ := X \sqcup \{*\} = \{(x, 0) \mid x \in X\} \cup \{(*, 1)\}$, where $(*, 1)$ is regarded as the basepoint of X_+.

The functor $(-)_+$ sends a function $f : X \to Y$ to the basepoint-preserving function $f_+ : X_+ \to Y_+$ that sends $(x, 0) \mapsto (f(x), 0)$ and $(*, 1) \mapsto (*, 1)$.

For any set X and pointed set (Y, y), there is a set bijection $\mathbf{Set}_*(X_+, (Y, y)) \cong \mathbf{Set}(X, U(Y, y))$ that sends $(f : X_+, (Y, y)) \mapsto (X \to U(y, y), x \mapsto f(x, 0))$, and $(g : X \to U(Y, y)) \mapsto (X_+ \to (Y, y), (x, 0) \mapsto g(x), (*, 1) \mapsto y)$ that is natural in $X \in \mathbf{Set}$ and $(Y, y) \in \mathbf{Set}_*$.

Thus, there is an adjunction $(-)_+ \dashv U$, i.e. adjoining a new basepoint is left adjoint to forgetting an existing basepoint.

2. Let X and Y be sets and $f : X \to Y$ be a function. Then there are inclusion-preserving functions $f : \mathcal{P}(X) \rightleftarrows \mathcal{P}(Y) : f^{-1}$ where $A \mapsto f(A) = \{f(a) \mid a \in A\}$, and $B \mapsto f^{-1}(B) = \{x \in X \mid f(x) \in B\}$.

Moreover, we have that for any $A \in \mathcal{P}(X)$ and $B \in \mathcal{P}(Y)$, $f(A) \subseteq B$ if and only if $A \subseteq f^{-1}(B)$. Now regard $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ as poset categories and $f : \mathcal{P}(X) \rightleftarrows \mathcal{P}(Y) : f^{-1}$ as functors. Then for any $A \in \mathcal{P}(X)$ and $B \in \mathcal{P}(Y)$, there is a bijection $\mathcal{P}(Y)(f(A), B) \cong \mathcal{P}(X)(A, f^{-1}(B))$ because both sides are either singletons or empty, and these bijections are natural in A and B. Therefore, there is an adjunction $f \dashv f^{-1}$, i.e. forming images is left adjoint to forming inverse images.

In fact, f^{-1} also has a right adjoint. For any set $A \subseteq X$, let $f_*(A) = \{y \in Y \mid f^{-1}(y) \subseteq A\}$. Then f_* is an inclusion-preserving function $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$ and for any $B \in \mathcal{P}(Y)$ and $A \in \mathcal{P}(X)$, we have $f^{-1}(B) \subseteq A$ if and only if $B \subseteq f_*(A)$.

Now regard f_* as a functor $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$ between poset categories. Then as above, there are bijections $\mathcal{P}(X)(f^{-1}(B), A) \cong \mathcal{P}(Y)(B, f_*(A))$, natural in $B \in \mathcal{P}(Y)$ and $A \in \mathcal{P}(X)$, so that $f^{-1} \dashv f_*$. In summary, we have a chain of adjunctions

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r, bend left, "f"] \arrow[r, bend right, "f_*"'] & \mathcal{P}(Y) \arrow[l, "f^{-1}"]
\end{tikzcd}$$

where $f \dashv f^{-1}$ and $f^{-1} \dashv f_*$, associated to any set map $f : X \to Y$.

3. $(-) \times B : \mathbf{Set} \rightleftarrows \mathbf{Set} : (-)^B$

Let B be a set. Then B determines a covariant functor $(-) \times B : \mathbf{Set} \to \mathbf{Set}$ that sends $A \mapsto A \times B$ and $(f : A \to A') \mapsto (f \times 1_B : A \times B \to A' \times B, (a, b) \mapsto (f(a), b))$ and a covariant functor $(-)^B : \mathbf{Set} \to \mathbf{Set}$ that sends $C \mapsto C^B$ and $(f : C \to C') \mapsto (f_* : C^B \to (C')^B, t \mapsto f \circ t)$.

For any sets A and C, there is a bijection $\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, C^B)$ where $(f : A \times B \to C) \mapsto (A \to C^B, a \mapsto f(a, -))$ and $(g : A \to C^B) \mapsto (A \times B \to C, (a, b) \mapsto g(a)(b))$, which is natural in A and C. Thus, there is an adjunction $(-) \times B \dashv (-)^B$ for every set $B \in \mathbf{Set}$.

We conclude by reinterpreting limit and colimit functors in terms of adjunctions.

Let $\mathscr{C}$ be a category and J be an indexing category. There is a constant diagram functor $\Delta : \mathscr{C} \to \mathscr{C}^J$ that sends C to the constant functor valued at C, and morphism $f : C \to C'$ to the constant natural transformation with components $f : C \to C'$.

Example 12.5. Suppose that $\mathscr{C}$ has a limit for each J-shaped diagram $F : J \to \mathscr{C}$, and choose a limit functor $\lim_J : \mathscr{C}^J \to \mathscr{C}$. In other words, choose a limit $(\lim_J F, \lambda^F : \lim_J F \Rightarrow F)$ for each diagram $F : J \to \mathscr{C}$, and for any natural transformation $\eta : F \Rightarrow F'$ in $\mathscr{C}^J$, let $\lim_J \eta : \lim_J F \to \lim_J F'$ be the unique morphism such that the square

$$\begin{tikzcd}
\lim_J F \arrow[r, "\lim_J \eta"] \arrow[d, "\lambda_j^F"'] & \lim_J F' \arrow[d, "\lambda_j^{F'}"] \\
Fj \arrow[r, "\eta_j"'] & Fj'
\end{tikzcd}$$

commutes for all $j \in J$. Observe that for any diagram $F : J \to \mathscr{C}$ and $C \in \mathscr{C}$, we have $\mathscr{C}^J(\Delta C, F) = \mathbf{Cone}(C, F)$. By the universal property of limits, we obtain a bijection $\mathscr{C}^J(\Delta C, F) \cong \mathscr{C}(C, \lim_J F)$ where $f : C \to \lim_J F$ is send back to $\lambda^F \circ \Delta f : \Delta C \Rightarrow F$, that is natural in C and F. Therefore, this is an adjunction $\Delta \dashv \lim_J$.

Dually, if $\mathscr{C}$ has a colimit for every J-shaped diagram $F : J \to \mathscr{C}$, and we choose a colimit $(\mathbf{colim}_J(F), \iota^F : F \Rightarrow \mathbf{colim}_J F)$ for each diagram $F : J \to \mathscr{C}$ and if, for any natural transformation $\eta : F \Rightarrow F'$, we set $\mathbf{colim}_J \eta : \mathbf{colim}_J F \to \mathbf{colim}_J F'$ to be the unique morphism such that the square

$$\begin{tikzcd}
Fj \arrow[r, "\eta_j"] \arrow[d, "\iota_j^F"'] & Fj' \arrow[d, "\iota_j^{F'}"] \\
\mathbf{colim}_J F \arrow[r, "\mathbf{colim}_J \eta"'] & \mathbf{colim}_J F'
\end{tikzcd}$$

commutes for all $j \in J$, then the universal property of colimits gives a bijection $\mathscr{C}(\mathbf{colim}_J F, C) \cong \mathscr{C}^J(F, \Delta C)$ that sends $f : \mathbf{colim}_J F \to C$ to $(\Delta f \circ \iota^F : F \Rightarrow \Delta C$, which is natural in F and in C. Thus, we obtain an adjunction $\mathbf{colim}_J \dashv \Delta$.

13 THE UNIT AND COUNIT OF AN ADJUNCTION

Given any adjunction $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, together with natural isomorphisms $\mathscr{D}(FC, D) \cong \mathscr{C}(C, GD)$, there are two natural transformations $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$ that we define, called the unit and counit of adjunction $F \dashv G$.

These natural transformations are important for several reasons. Among other things:

- They have universal properties.
- They can be used to give an alternate, 2-categorical definition of an adjunction, and
- They can be used to construct adjunctions.

In this note, we shall define the unit and counit of an adjunction and explore their properties.

To define unit, note that for fixed $C \in \mathscr{C}$, we have an isomorphism $\varphi : \mathscr{D}(FC, -) \cong \mathscr{C}(C, G-)$, which is natural in $D \in \mathscr{D}$.

We define the C-component of $\eta : 1_{\mathscr{C}} \Rightarrow GF$ by $\eta_C := \varphi(1_F C) : C \to GFC$.

Example 13.1. Consider the free-forgetful adjunction $M : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U$, with natural transformations $\varphi : \mathbf{Mon}(MX, N) \cong \mathbf{Set}(X, UN)$ defined by $\alpha \mapsto U_\alpha \circ \eta_X$, where $\eta_X : X \to UMX$ sends $x \in X$ to $(x) \in UMX$. Then the unit of $M \dashv U$ is just this η, which inserts X as the free generator of MX.

By Yoneda, η_C is a universal element of $\mathscr{C}(C, G-)$, i.e. for any $D \in \mathscr{D}$ and $f : C \to GD$, there is a unique $\bar{f} : FC \to D$ such that $G\bar{f} \circ \eta_C = \varphi(\bar{f}) = f$.

In the case of $M \dashv U$ above, the universal property of η given by abstract category is the familiar universal property of $\eta_X : X \to UMX$ that encodes the fact that X is a basis of UMX.

We also have that η is natural in the $M \dashv U$ example. This is true on general grounds.

Lemma 13.2. Suppose $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, $\varphi : \mathscr{D}(FC, D) \cong \mathscr{C}(C, GD)$ is an adjunction and define $\eta_C = \varphi(1_{FC}) : C \to GFC$. Then η is a natural transformation $1_{\mathscr{C}} \Rightarrow GF$, called the unit of the adjunction $F \dashv G$.

Proof. Since φ is natural in C and D and

$$\begin{tikzcd}
FC \arrow[r, "1_{FC}"] \arrow[d, "Ff"'] & FC \arrow[d, "Ff"] \\
FC' \arrow[r, "1_{FC'}"'] & FC'
\end{tikzcd}$$

commutes for all $f : C \to C'$ in $\mathscr{C}$, it follows that the transposed square

$$\begin{tikzcd}
C \arrow[r, "\varphi(1_{FC})=\eta_C"] \arrow[d, "f"'] & GFC \arrow[d, "GFf"] \\
C' \arrow[r, "\varphi(1_{FC'})=\eta_{C'}"'] & GFC'
\end{tikzcd}$$

also commutes by an earlier lemma. ☐

As with all categorical construction, the unit has a dual called the counit of the adjunction.

To define the counit, note that for fixed $D \in \mathscr{D}$ in an adjunction $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, $\varphi : \mathscr{D}(FC, D) \cong \mathscr{C}(C, GD)$, there is an isomorphism $\varphi^{-1} : \mathscr{C}(-, GD) \cong \mathscr{D}(F-, D)$, which is natural in $C \in \mathscr{C}$.

We define the D-component of the counit $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$ by $\varepsilon_D := \varphi^{-1}(1_{GD}) : FGD \to D$.

Example 13.3. Consider the free-forgetful adjunction $M : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U$. Then for any monoid N, the counit $\varepsilon_N : MUN \to N$ is the unique monoid homomorphism such that $U_{\varepsilon N} \circ \eta_{UN} = 1_{UN}$. In other words, $\varepsilon_N(x) = x$ for all $x \in N$, so that

$$\begin{aligned}
\varepsilon_N(x_1, \cdots, x_m) &= \varepsilon_N((x_1) \cdot \cdots \cdot (x_m)) \\
&= \varepsilon_N(x_1) \cdot \cdots \varepsilon_N(x_m) \\
&= x_1 \cdot \cdots \cdot x_m
\end{aligned}$$

i.e. ε_N sends a formal composite of elements of N to their actual composite in N.

By Yoneda, $\varepsilon_D : FGD \to D$ is a universal element of $\mathscr{D}(F-, D)$, i.e. for any $C \in \mathscr{C}$, together with a morphism $f : FC \to D$, there is a unique morphism $\bar{f} : C \to GD$ such that $\varepsilon_D \circ F\bar{f} = \varphi^{-1}(\bar{f}) = f$.

As with the unit $\eta : 1_{\mathscr{C}} \Rightarrow GF$, the counit $\varepsilon_D : FGD \to D$ is natural in D.

Lemma 13.4. Suppose $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, $\varphi : \mathscr{D}(FC, D) \cong \mathscr{C}(C, GD)$ is an adjunction and define $\varepsilon_D = \varphi^{-1}(1_{GD}) : FGD \to D$. Then ε is a natural transformation $FG \Rightarrow 1_{\mathscr{D}}$, called the counit of the adjunction $F \dashv G$.

Proof. Dual to the previous lemma. ☐

In the most recent example, where we determined the counit of the free-forgetful adjunction $M : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U$, we saw that $U\varepsilon_N \circ \eta_{UN} = 1_{UN}$, i.e. $U\varepsilon \circ \eta U = 1_U$.

This equation, and a dual equation hold for every adjunction on general grounds.

For, suppose that $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, $\varphi : \mathscr{D}(FC, D) \cong \mathscr{C}(C, GD)$ is an adjunction. Then naturality in C and D, together with the commutativity of

$$\begin{tikzcd}
FGD \arrow[r, "1_{FGD}"] \arrow[d, "F(1_{GD})=1_{FGD}"'] & FGD \arrow[d, "\varepsilon_D"] \\
FGD \arrow[r, "\varepsilon_D"'] & D
\end{tikzcd}$$

imply that the transposed square

$$\begin{tikzcd}
GD \arrow[r, "\varphi(1_{FGD}"] \arrow[d, "1_{GD}"'] & GFGD \arrow[d, "G\varepsilon_D"] \\
GD \arrow[r, "\varphi(\varepsilon_D)=1_{GD}"'] & GD
\end{tikzcd}$$

commutes. Thus, $G\varepsilon_D \circ \eta_{GD} = 1_{GD}$ for all $D \in \mathscr{D}$, i.e. $G\varepsilon \circ \eta G = 1_G$.

Dually, the commutativity of

$$\begin{tikzcd}
C \arrow[r, "\varphi(1_{FC})=\eta_C"] \arrow[d, "\eta_C"] & GFC \arrow[d, "1_{GFC}=G1_{FC}"] \\
GFC \arrow[r, "\varphi(\varepsilon_{FC})=1_{GFC}"] & GFC
\end{tikzcd}$$

implies that the square

$$\begin{tikzcd}
FC \arrow[r, "1_{FC}"] \arrow[d, "F\eta_C"'] & FC \arrow[d, "1_{FC}"] \\
FGFC \arrow[r, "\varepsilon_{FC}"'] & FC
\end{tikzcd}$$

commutes for all $C \in \mathscr{C}$. Thus, $\varepsilon F \circ F\eta = 1_F$.

The equations $\varepsilon F \circ F\eta = 1_F$ and $G\varepsilon \circ \eta G = 1_G$ are called the triangle identities. In terms of pasting diagrams, we have the following diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[dr, "1_{\mathscr{C}}"'] & \mathscr{D} \arrow[d, "G"] \arrow[dr, "1_{\mathscr{D}}"] & \\
 & \mathscr{C} \arrow[r, "F"'] & \mathscr{D}
\end{tikzcd}$$

with $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$, equivalent to

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "1_F \Uparrow"] & \mathscr{D}
\end{tikzcd}$$

and similarly we have

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "G"] \arrow[dr, "1_{\mathscr{D}}"'] & \mathscr{D} \arrow[d, "F"] \arrow[dr, "1_{\mathscr{C}}"] & \\
 & \mathscr{D} \arrow[r, "G"'] & \mathscr{C}
\end{tikzcd}$$

with $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$, equivalent to

$$\begin{tikzcd}
\mathscr{D} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "1_G \Downarrow"] & \mathscr{C}
\end{tikzcd}$$

The triangle identities assert that ”the counit if the left inverse of the unit modulo translation”.

Now suppose that $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ is a pair of functors. The surprise is that a pair of natural transformations $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$, satisfying the triangle identities, defines an adjunction.

Indeed, given such natural transformations η and ε, we can define bijections $\Phi_{C,D} : \mathscr{D}(FC, D) \xrightarrow{\cong} \mathscr{C}(C, GD)$ by mapping $(f : FC \to D) \mapsto (Gf \circ \eta_C : C \to GFC \to GD)$, with inverse assignment $(g : C \to GD) \mapsto (\varepsilon_D \circ Fg : FC \to FGD \to D)$, where the bijection is natural in C and D.

All told, we have the following result.

Theorem 13.5. Let $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ be functors. Then there is a bijective correspondence between tuples $(\varphi_{CmD} : \mathscr{D}(FC, D) \cong \mathscr{C}(C, GD))_{C,D}$ that are natural in C and D and the natural transformations $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$ satisfying triangular identities.

Proof. One can check that the two ways of going between these data are inverse. □

Definition 13.6 (Adjunction, in terms of unit and counit)**.** An adjunction consists of functors $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, together with natural transformations $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$ that satisfy the triangle identities $\varepsilon F \circ F\eta = 1_F$ and $G\varepsilon \circ \eta G = 1_G$.

By the considerations above, this definition of an adjoint is equivalent to our previous "hom-set definition".

The previous definition specifies all of the data in an adjunction, but one can get away with less.

We conclude by describing a useful method of constructing an adjunction $F \dashv G$ from the functor G, together with maps $\eta_C : C \to GFC$ that have the universal property of the unit.

Proposition 13.7. Suppose that $G : \mathscr{D} \to \mathscr{C}$ is a functor and that for all $C \in \mathscr{C}$, we have a chosen pair $(FC \in \mathscr{D}, \eta_C : C \to GFC)$ such that for any $D \in \mathscr{D}$ and $f : C \to GD$, there is a unique $\bar{f} : FC \to D$ such that $G\bar{f} \circ \eta_C = f$. Then there is a unique extension of $C \mapsto FC$ to a functor such that $\eta : 1_{\mathscr{C}} \Rightarrow GF$ is a natural transformation, and moreover, we can extend $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ to an adjunction such that η is the unit of $F \dashv G$.

Proof. If such an extension of $C \mapsto FC$ exists, then η must be a natural transformation $1_{\mathscr{C}} \Rightarrow GF$. Then, for any $f : C \to C'$ in $\mathscr{C}$, we must have $Ff : FC \to FC'$ such that

$$\begin{tikzcd}
GFC \arrow[r, "GFf"] & GFC' \\
C \arrow[u, "\eta_C"] \arrow[r, "f"] & C' \arrow[u, "\eta_{C'}"]
\end{tikzcd}$$

so that $Ff : FC \to FC'$ must be the unique morphism induced by $C \xrightarrow{f} C' \xrightarrow{\eta_{C'}} GFC'$. Thus, F is completely determined by the naturality of η, and is unique if it exists. ON the other hand, it is straightforward to check that this definition of F on morphisms does define a functor such that $\eta : 1_{\mathscr{C}} \Rightarrow GF$ is natural.

It remains to check that $F \dashv G$ with unit η. Define functions $\varphi_{C,D} : \mathscr{D}(FC, D) \to \mathscr{C}(C, GD)$

by the formula $\varphi_{C,D}(f) = Gf \circ \eta_C$. Then by the universal property of η, $\varphi_{C,D}$ is bijective for all C and D. It is immediate that φ is natural in D, and the naturality of η in C implies that φ is also natural in C. Thus, we have an adjunction $F \dashv G$, and $\varphi(1_{FC}) = G1_{FC} \circ \eta_C = \eta_C$, so that η is the unit of $F \dashv G$. □

14 THEORETICAL PROPERTIES OF ADJUNCTIONS

In this note, we collect some fundamental theoretical properties of adjunctions.

We start by proving that adjoints are unique up to isomorphisms.

Proposition 14.1. Suppose $G : \mathscr{D} \to \mathscr{C}$ is a functor and that $F, F' : \mathscr{C} \to \mathscr{D}$ are both left adjoints to G. Then $F \cong F'$ naturally.

Proof. For any fixed $C \in \mathscr{C}$, we have isomorphisms $\mathscr{D}(FC, D) \cong \varphi_{C,D}\mathscr{C}(C, GD) \cong \psi^{-1}_{C,D}\mathscr{D}(F'C, D)$ that are natural in D, where the first one named $\varphi_{C,D}$ and the second one named $\psi^{-1}_{C,D}$, i.e. $\psi^{-1}_{C,-} \circ \varphi_{C,-} : \mathscr{D}(FC, -) \cong \mathscr{D}(F'C, -)$. Since the contravariant Yoneda Embedding is fully faithful, there is a unique $\theta_C : F'C \to FC$ such that $\psi^{-1}_{C,-} \circ \varphi_{C,-} = \theta^*_C$, and moreover, $\theta)C^*$ is an isomorphism because full and faithful functors reflect isomorphisms. We can recover θ_C by evaluating $\theta^*_C = \psi^{-1}_{C,-} \circ \varphi_{C,-}$ at the identity, i.e. $\theta_C = \psi^{-1}(\varphi(1_{FC}))$, and by the naturality of φ and ψ^{-1}, it follows that for any $f : C \to C'$ in $\mathscr{C}$, we have $Ff \circ \theta_C = \psi^{-1}(\varphi(Ff)) = \theta_{C'} \circ F'f$.

Thus, $\theta : F' \cong F$ is a natural isomorphism. □

Thus, it is possible to define functors, up to isomorphisms by requiring them to be adjoints to a given functor.

Next, we show that adjoints compose.

Proposition 14.2. Suppose $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ and $F' : \mathscr{D} \rightleftarrows \mathscr{E} : G'$ are functors, and that $F \dashv G$ and $F' \dashv G'$. Then $F'f \dashv GG'$.

Proof. For any $C \in \mathscr{C}$ and $E \in \mathscr{E}$, there is an isomorphismm $\varepsilon(F'FC, E) \cong \mathscr{D}(FC, G'E) \cong \mathscr{C}(C, GG'E)$ that is natural in C and E. Thus $F'F \dashv GG'$. ☐

We now explain how adjointness is a form of duality that generalize that duality exhibited by psudoinverse functors.

Proposition 14.3. Any equivalence of categories $F : \mathscr{C} \rightleftarrows \mathscr{D} : G, \eta : 1_{\mathscr{C}} \cong GF, \varepsilon : FG \cong 1_{\mathscr{D}}$ can be converted into an adjoint equivalence, in which the natural isomorphisms η and ε satisfy the triangle identities, by replacing one of η or ε with a new natural isomorphism.

Proof. Suppose we are given an equivalence as above. We shall show how to replace ε with a new ε' in such a way that $(F, G, \eta, \varepsilon')$ is an adjunction and an equivalence. To start, note that G is fully faithful and η is isomorphism. Then, for any $C \in \mathscr{C}$ and $D \in \mathscr{D}$, there is an isomorphism

$$\varphi_{C,D} : \mathscr{D}(FC, D) \xrightarrow[\cong]{G} \mathscr{C}(GFC, GD) \xrightarrow[\cong]{\eta_C^*} \mathscr{C}(C, GD)$$

that maps $(f : FC \to D) \mapsto Gf \circ \eta_C$, that is natural in C and D. Thus, we obtain an adjunction. The unit of this adjunction is $(\varphi(1_{FC}) = \eta_C)_{C \in \mathscr{C}} = \eta$. Denote the counit by ε'. The transformation ε' may or may not be equal to the original ε, but we know that $(F, G, \eta, \varepsilon')$ is an adjunction. It remains to show that ε' is a natural isomorphism. By the triangle identities, we know $G\varepsilon' \circ \eta G = 1_G$, i.e. $G_{\varepsilon'_D} \circ \eta_{GD} = 1_{GD}$ for all $D \in \mathscr{D}$. Since η is an isomorphism, it follows $G\varepsilon'_D = \eta_{GD}^{-1} = 1_{GD}$ and since G is fully faithful, it follows that ε'_D is an isomorphism. Therefore $(F, G, \eta, \varepsilon')$ is an adjunction and an equivalence. ☐

Corollary 14.4. If $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ are pseudoinverse, i.e. there are natural isomorphisms $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$, then $F \dashv G \dashv F$.

Proof. By the previous proposition, we can convert an equivalence $(F, G, \eta, \varepsilon)$ into an adjoint equivalence $(F, G, \eta, \varepsilon')$, so that $F \dashv G$. However, $(G, F, \varepsilon^{-1}, \eta^{-1})$ is also an equivalence, so $G \dashv F$ as well. ☐

Our next proposition shows that an adjunction $F \dashv G$ induces pre-composition and post-composition adjunctions.

First, some preliminary knowledge. Suppose $F : \mathscr{C} \to \mathscr{D}$ is a functor and $\mathscr{E}$ is another category. There is a pre-composition functor $F^* : \mathscr{E}^{\mathscr{D}} \to \mathscr{E}^{\mathscr{C}}$ that takes $G \mapsto GF$ and $(\eta : G \Rightarrow G') \mapsto (\eta F : GF \Rightarrow G'F)$ and also a post-composition functor $F_* : \mathscr{C}^{\mathscr{E}}$ that takes $G \mapsto FG$ and $(\eta : G \Rightarrow G') \mapsto (F\eta : FG \Rightarrow FG')$.

Now for the result.

Proposition 14.5. Suppose $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ and $F \dashv G$ with unit η and counit ε. Then:

1. For any category J, there is $F_* : \mathscr{C}^J \rightleftarrows \mathscr{D}^J : G_*$ gives an adjunction $F_* \dashv G_*$.
2. For any category $\mathscr{E}$, $G^* : \mathscr{E}^{\mathscr{C}} \rightleftarrows \mathscr{E}^{\mathscr{D}} : F_*$ gives an adjunction $G^* \dashv F^*$.

Proof.

1. Let $H \in \mathscr{C}^J$ and $K \in \mathscr{D}^J$. Then there is a bijection $\Phi_{H,K} : \mathscr{D}^J(FH, ZK) \xrightarrow{\cong} \mathscr{C}^J(H, GK)$ given by $(\theta : FH \Rightarrow K) \mapsto (G\theta \circ \eta H : H \Rightarrow GFH \Rightarrow GK)$ and inverse $(\zeta : H \Rightarrow GK) \mapsto (\varepsilon K : F\zeta : FH \Rightarrow FGK \Rightarrow K$, that is natural in H and K. Thus, $F_* \dashv G_*$.
2. Let $H \in \mathscr{E}^{\mathscr{C}}$ and $K \in \mathscr{E}^{\mathscr{C}}(H, KF)$ be given by $(\theta : HG \Rightarrow K) \mapsto (\theta F \circ H\eta : H \Rightarrow HGF \Rightarrow KF)$ and an inverse $(\zeta : H \Rightarrow KF) \mapsto (K\varepsilon \circ \zeta G : HG \Rightarrow KFG \Rightarrow K)$ that is natural in H and K. Thus, $G^* \dashv F^*$.

☐

We arrive at an important result.

Theorem 14.6 (RAPL)**.** Right adjoints preserve limits.

Proof. Let $F : \mathscr{C} \rightleftarrows \mathscr{D} : G, \eta : 1_{\mathscr{C}} \Rightarrow GF, \varepsilon : FG \Rightarrow 1_{\mathscr{D}}$ be an adjunction, $K : J \to \mathscr{D}$ be a diagram, and $(L, \lambda : L \Rightarrow K)$ be a limit cone over K. We must prove that $(GL, G\lambda : GL \Rightarrow GK)$ is a limit cone over GK. There is a string of isomorphisms:

$$\mathscr{C}(C, GL) \cong \mathscr{D}(FC, L)$$

$$\begin{aligned}&\cong \mathscr{D}^J(\Delta FC, K)\\&= \mathscr{D}^J(F\Delta C, K)\\&\cong \mathscr{C}^J(\Delta C, GK)\end{aligned}$$

that sends $f \mapsto \varepsilon_L \circ Ff \mapsto (\lambda_j \circ \varepsilon_L \circ Ff)_{j\in J} = \mu \mapsto G\mu \circ \eta(\Delta C) = (G\lambda_j \circ G\varepsilon_L \circ GFf \circ \eta_C)_{j\in J} = (G\lambda_j \circ f)_{j\in J}$, which is natural in $C \in \mathscr{C}$.

The first equivalence is given by $F \dashv G$, the second equivalence is given by the universal property of L, the third equivalence is given by $\Delta \circ F = F_* \circ \Delta$, and the last equivalence is given by $F_* \dashv G_*$ constructed in the previous proposition from $F \dashv G$. The bijectivity of $\mathscr{C}(C, GK) \cong \mathscr{C}^J(\Delta C, GK)$ given by $(f : C \to G :) \mapsto (G\lambda_j \circ f)_{j\in J}$ means precisely that $(GL, G\lambda : GL \Rightarrow GK)$ is a limit cone. ☐

Dually, we have the following result:

Theorem 14.7 (LAPC)**.** Left adjoints preserve colimits.

The contrapositives of the last two results are worth noting:

- If $F : \mathscr{C} \to \mathscr{D}$ does not preserves limits, then F does not have a left adjoint.
- If $F : \mathscr{C} \to \mathscr{D}$ does not preserve colimits, then F does not have a right adjoint.

Here are two nice applications of RAPL and LAPC.

Example 14.8. Suppose $f : X \to Y$ is a set map. Then $f : \mathcal{P}(X) \to \mathcal{P}(Y)$ preserves unions and $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ preserves unions and intersections.

Proof. Regarding $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ as poset categories, we have adjunctions $f \dashv f^{-1} \dashv f_*$, where $f_* : \mathcal{P}(X) \to \mathcal{P}(Y)$ sends A to $f_*(A) = \{y \in Y \mid f^{-1}\{y\} \subseteq A\}$. Since coproducts in $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are just unions, and $f : \mathcal{P}(X) \rightleftarrows \mathcal{P}(Y) : f^{-1}$ are both left adjoints, it follows that f and f^{-1} preserves unions. Since products in $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are just intersections, and f^{-1} is a right adjoint, then it follows that f^{-1} also preserves intersections. ☐

Example 14.9.

1. For any sets A, A', B, $(A \sqcup A') \times B \cong (A \times B) \sqcup (A' \times B)$.
2. For any sets B, C, C', $(C \times C')^B \cong C^B \times (C')^B$.
3. For any sets B, B', C, $C^{B \sqcup B'} \cong C^B \times C^{B'}$.

Proof. For part 1 and part 2, notice that for any set B, there is an adjunction $(-) \times B : \mathbf{Set} \rightleftarrows \mathbf{Set} : (-)^B$. Therefore, $(-) \times B$ preserves coproducts (as disjoint unions) and $(-)^B$ preserves products.

For part 3, let C be a set and consider the functor $C^{(-)} = \mathbf{Set}(-, C) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$. We can also think of $C^{(-)}$ as a functor $\mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$, and for any sets $A \in \mathbf{Set}$ and $B \in \mathbf{Set}^{\mathrm{op}}$, we have

$$\begin{aligned}\mathbf{Set}^{\mathrm{op}}(C^A, B) &= \mathbf{Set}(B, C^A)\\&\cong \mathbf{Set}(B \times A, C)\\&\cong \mathbf{Set}(A \times B, C)\\&\cong \mathbf{Set}(A, C^B)\end{aligned}$$

naturally in A and in B. Therefore, there is an adjunction between functors $C^{(-)} : \mathbf{Set} \rightleftarrows \mathbf{Set}^{\mathrm{op}} : C^{(-)}$ so that $C^{(-)}$ sends colimits in $\mathbf{Set}$ to limits in $\mathbf{Set}$. Therefore, $C^{(-)}$ sends coproducts to products. ☐

Thus, the distributive law and the laws of exponents in arithmetic are consequences of certain adjointness relations.

15 FREYD'S ADJOINT FUNCTOR THEOREM

Right adjoints preserve limits, but what about the converse? Is every limit-preserving functor a right adjoint?

No. We illustrate by example.

Example 15.1. Let $\mathbf{FinSet}$ be the category of finite sets and let $\mathbf{Set}$ be the category of sets. Then the covariant hom functor $\mathbf{FinSet}(*,-) : \mathbf{FinSet} \to \mathbf{Set}$ preserves all limits, but it does not have a left adjoint. Indeed, suppose for contradiction that there were some functor $L : \mathbf{Set} \to \mathbf{FinSet}$ such that $L \dashv G$. Then for every set X and finite set Y, we would have a natural bijection $\mathbf{FinSet}(LX, Y) \cong \mathbf{Set}(X, GY)$. Let $X = \mathbb{N}$ and $Y = \{0,1\}$. Then $L\mathbb{N}$ is finite, so that $|\mathbf{FinSet}(L\mathbb{N}, \{0,1\}| = 2^{|L\mathbb{N}|}$, which is a finite natural number. On the other hand, since $GY \cong Y$ for all finite sets Y, we have $|\mathbf{Set}(\mathbb{N}, G\{0,1\}| = |\mathbf{Set}(\mathbb{N}, \{0,1\}| = |2^{\mathbb{N}}| > |\mathbb{N}|$, which is infinite. Since $\mathbf{FinSet}(L, \mathbb{N}, \{0,1\})$ and $\mathbf{Set}(\mathbb{N}, G\{0,1\})$ are in bijection, they must be the same size, so we have arrived at a contradiction.

Thus, it is natural to ask what conditions beyond continuity ensure that a functor has a left adjoint. Freyd's Adjoint Functor Theorem identifies sufficient extra conditions that ensure we can construct a left adjoint. Here is the statement:

Theorem 15.2 (Freyd's Adjoint Functor Theorem). Let $U : \mathcal{A} \to \mathcal{S}$ be a continuous functor whose domain is locally small and complete. Suppose that U satisfies the following **solution set condition**:

($*$) For every $S \in \mathcal{S}$, there exists a set I_S and an I_S-indexed set $\Phi_S = \{(A_i \in \mathcal{A}, \eta_{s,i} : S \to UA_i \in \mathcal{S}) \mid i \in I_S\}$ such that for any $(A, f : S \to UA)$, there exists an $i \in I_S$ and a morphism $\bar{f} : A_i \to A$ such that $f = U\bar{f} \circ \eta_{s,i}$.

Then U has a left adjoint.

This theorem is also known as the General Adjoint Functor Theorem. There is a Special Adjoint Functor Theorem, but we shall not consider it.

As mentioned above, the hypotheses in the General Adjoint Functor Theorem are conditions that ensure we can construct a left adjoint. Implicit is the following result, which shows how to construct a left adjoint from a collection of arrows that have the universal property of the unit of an adjunction.

Proposition 15.3. Suppose that $U : \mathcal{A} \to \mathcal{S}$ is a functor and that for all $S \in \mathcal{S}$, we have a chosen pair $(FS, \mathcal{A}, \eta_S : S \to UFS \in \mathcal{S})$ such that

($*_S$) For any $A \in \mathcal{A}$ and $f : S \to UA$, there is a unique $\bar{f} : FS \to A$ such that $f = U\bar{f} \circ \eta_S$, such that

$$\begin{tikzcd} UFS \arrow[r, "U\bar{f}"] & UA \\ S \arrow[u, "\eta_S"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

Then U has a left adjoint.

Proof. We extend the choice $S \mapsto FS$ to a functor $F : \mathcal{S} \to \mathcal{A}$ that is left adjoint to U. Suppose $f : S \to S'$ is a morphism in $\mathcal{S}$. Then $\eta_{S'} \circ f : S \to S' \to UFS'$, so there is a unique morphism $Ff : FS \to FS'$ such that $UFf \circ \eta_S = \eta_{S'} \circ f$. Here is the diagram:

$$\begin{tikzcd} UFS \arrow[r, "UFf"] & UFS' \\ S \arrow[u, "\eta_S"] \arrow[r, "f"] & S' \arrow[u, "\eta_{S'}"'] \end{tikzcd}$$

The uniqueness of Ff and the functorality of U ensures that $F : \mathcal{S} \to \mathcal{A}$ is a functor, and by definition, $\eta : 1_{\mathcal{S}} \Rightarrow UF$ is a natural transformation. Now, for any $S \in \mathcal{S}$ and $A \in \mathcal{A}$, ($*_S$) implies that $\varphi_{S,A} : \mathcal{A}(FS, A) \xrightarrow{\cong} \mathcal{S}(S, UA)$, which sends $\bar{f} : FS \to A$ to $U\bar{f} \circ \eta_S : S \to UA$, is a bijection, It is straightforward to check that φ is natural in A, and the naturality of η implies that φ is natural in S. Then $F : \mathcal{S} \rightleftarrows \mathcal{A} : U$, together with φ is an adjunction with unit η, so that U has a left adjoint. ☐

Reframing this property in terms of the General Adjoint Functor Theorem, we see that the universal arrow $(FS, \eta : S \to UFS)$ determines a single-element solution set $\{(FS, \eta_S : S \to UFS)\}$, with the added condition that the comparison map to any other $(A, f : S \to UA)$ is unique.

The point of the General Adjoint Functor Theorem is to construct a single universal arrow (FS, η_S) from an approximating solution set Φ_S, for all objects S.

To understand the construction of (FS, η_S) from Φ_S, it is helpful to recast the problem. Here is a definition.

Definition 15.4 (Comma Category)**.** For any $S \in \mathcal{S}$, the comma category $S \downarrow U$ has objects as pairs $(A \in \mathcal{A}, f : S \to UA \in S)$ and the morphisms of the category are $\varphi : (A, f) \to (B, g)$, considered as $\varphi : A \to B$ in $\mathcal{A}$ such that $U\varphi \circ f = g$:

$$\begin{tikzcd}
UA \arrow[rr, "U\varphi"] & & UB \\
& S \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

In these terms, a universal arrow $(FS, \eta_S : S \to UFS)$ is precisely an initial object in $S \downarrow U$. Similarly, a solution set Φ_S is a jointly weakly initial (indexed) set of objects in $S \downarrow U$, in the following sense:

Definition 15.5 (Jointly Weakly Initial Category)**.** An indexed set of objects $\Phi = \{C_i \mid i \in I\}$ in a category $\mathscr{C}$ is jointly weakly initial if, for any object $D \in \mathscr{C}$, there is an index $i \in I$ and a morphism $f : C_i \to D$. (We do not require i or f to be unique.)

Thus, the problem is to construct an initial object from a jointly weakly initial set of objects. The following example gives the idea:

Example 15.6. Suppose $\mathscr{C} = [0, 1]$, regarded as a poset category. Then the initial object of $[0, 1]$ is 0, the least element. On the other hand, if $\Phi = \{X_i \mid i \in I\}$ is jointly (weakly) initial in $[0, 1]$, then it must contain points arbitrarily close to 0 (and 0 itself). We recover 0 as $\min \Phi = \inf \Phi$.

By analogy, an initial object in a category can be thought of as a "least element" (in a sense we shall momentarily make precise), while a jointly weakly initial set can be thought of as a set of objects that includes objects that are "arbitrary close" to an initial object. We recover an initial object by taking a limit.

We now turn these ideas into mathematics.

First, we give an equivalent condition for an object of a category to be initial. It makes precise a sense in which initial objects are "least".

Lemma 15.7. Suppose $\mathscr{C}$ is a category and $C \in \mathscr{C}$, then C is initial if and only if there is a cone $K : C \Rightarrow 1_{\mathscr{C}}$ such that $K_C = 1_C$.

Proof. ($\Rightarrow$) Suppose C is initial. For each $D \in \mathscr{C}$ let $K_D : C \to D$ be the unique morphism. Then the uniqueness of the K_D's ensures that $K : C \Rightarrow 1_{\mathscr{C}}$ is a cone and that $K_C = 1_C$.

($\Leftarrow$) Conversely, suppose there is a cone $K : C \Rightarrow 1_{\mathscr{C}}$ such that $K_C = 1_C$. We claim that for any $D \in \mathscr{C}$, $K_D : C \to D$ is the unique morphism from C to D. Indeed, K_D is a morphism $C \to D$, and if $f : C \to D$ is any such morphism, then

$$\begin{tikzcd}
& C \arrow[dl, "1_C=K_C"'] \arrow[dr, "K_D"] & \\
C \arrow[rr, "f"] & & D
\end{tikzcd}$$

must commute because $K : C \Rightarrow 1_{\mathscr{C}}$ is a cone. Thus, $f = f \circ 1_C = f \circ K_C = K_D$. Thus, for any $D \in \mathscr{C}$, there is a unique morphism $C \to D$, which proves that C is initial.

□

Next, we consider the problem of constructing an initial object as a limit (as opposed to an empty colimit).

First, a warm up: suppose $\mathscr{C}$ is a category and that L is a limit of $1_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$. We shall prove that L is an initial object of $\mathscr{C}$.

Proof. Let $\lambda : L \Rightarrow 1_{\mathscr{C}}$ be a limit cone. By the previous lemma, it will be enough to show that $\lambda_L = 1_L$. First, note that since $\lambda : L \Rightarrow 1_{\mathscr{C}}$ is a cone over $1_{\mathscr{C}}$, there is a unique $f : L \to L$ such that

$$\begin{tikzcd}
L \arrow[rr, "f"] \arrow[dr, "\lambda_C"] & & L \arrow[dl, "\lambda_C"] \\
& C &
\end{tikzcd}$$

commutes for all $C \in \mathscr{C}$, namely $f = 1_L$. We claim that λ_L also has this property. Indeed, suppose $C \in \mathscr{C}$ and consider $\lambda_C : L \to C$. Since λ is a cone over $1_{\mathscr{C}}$, the triangle

$$\begin{tikzcd}
& L \arrow[dl, "\lambda_L"'] \arrow[dr, "\lambda_C"] & \\
L \arrow[rr, "\lambda_C"] & & C
\end{tikzcd}$$

commutes, which is precisely what we need. Thus, $\lambda_L = 1_L$, and L is initial. ☐

Now we consider the situation for a jointly weakly initial set of objects.

Proposition 15.8. Let $\mathscr{C}$ be a category of pullbacks, $\Phi = \{C_i \mid i \in I\}$ be a jointly weakly initial set of objects in $\mathscr{C}$, and $\mathscr{D}$ be the full subcategory of $\mathscr{C}$ whose objects are the objects in Φ. Suppose that L is a limit of the inclusion functor $i : \mathscr{D} \hookrightarrow \mathscr{C}$. Then L is an initial object of $\mathscr{C}$.

Proof. We construct a cone $\lambda : L \Rightarrow 1_{\mathscr{C}}$ with the property that $\lambda_L = 1_L$. Let $K : L \Rightarrow i$ be a limit cone. We define λ as follows: given any $C \in \mathscr{C}$,

1. if $C \notin \Phi$, choose an index $i \in I$ and a morphism $h_C : C_i \to C$, and let $\lambda_C = h_C \circ K_{C_i} : L \to C_i \to C$.
2. if $C \in \Phi$, choose an index $i \in I$ such that $C = C_i$, let $h_C = 1_C : C_i \to C$, and let $\lambda_C = h_C \circ K_{C_i} = K_C : L \to C$.

We claim that $(\lambda_C : L \to C)_{C \in \mathscr{C}}$ is a cone over $1_{\mathscr{C}}$. To that end, suppose that $f : C \to D$ is a morphism in $\mathscr{C}$. Let $P \in \mathscr{C}$ be a pullback of $h_D : C_j \to D$ along $f \circ h_C : C_i \to C \to D$, and then choose an index $k \in I$ and a morphism $g : C_k \to P$. We obtain the following commutative diagram:

$$\begin{tikzcd}
& L \arrow[d, "K_{C_k}"] \arrow[ddl, bend right, "K_{C_i}"'] \arrow[ddr, bend left, "K_{C_j}"] & \\
& C_k \arrow[d, "g"] \arrow[dr, "\beta_g"] \arrow[ddl, "\alpha_g"'] & \\
& P \arrow[r, "\beta"] \arrow[dl, "\alpha"] & C_j \arrow[d, "h_D"] \\
C_i \arrow[r, "h_C"'] & C \arrow[r, "f"'] & D
\end{tikzcd}$$

Therefore, $f \circ \lambda_C = f \circ h_C \circ K_{C_i} = h_D \circ K_{C_j} = \lambda_D$, which shows $\lambda : L \Rightarrow 1_{\mathscr{C}}$ is a cone.

To show $\lambda_L = 1_L$, we argue as before. Suppose $C \in \Phi$ and consider $K_C : L \to C$. Then, since λ is a cone, the triangle

$$\begin{tikzcd}
& L \arrow[dl, "\lambda_L"'] \arrow[dr, "\lambda_C = K_C"] & \\
L \arrow[rr, "K_C"'] & & C
\end{tikzcd}$$

commutes. Thus, λ_L factors the cone K through itself, but 1_L is the only morphism that does this. Therefore, $\lambda_L = 1_L$. ☐

Corollary 15.9. Suppose $\mathscr{C}$ is locally small, complete, and has a jointly weakly initial set of objects. Then $\mathscr{C}$ has an initial object.

Proof. The category $\mathscr{C}$ has pullback because it is a composition. Now, let $\Phi, \mathscr{D}$ and $i : \mathscr{D} \hookrightarrow \mathscr{C}$ by as in the previous proposition. Then $\mathscr{D}$ is a small category because $\mathscr{C}$ is locally small, and hence $i : \mathscr{D} \hookrightarrow \mathscr{C}$ has a limit L because $\mathscr{C}$ is complete. By the previous proposition, L is initial. ☐

We now return to the General Adjoint Functor Theorem. Let $U : \mathcal{A} \to \mathcal{S}$ be a continuous functor. As discussed earlier, our goal in the General Adjoint Functor Theorem is to construct an initial object of $S \downarrow U$ from a weak initial set $\Phi_S \subseteq S \downarrow U$, for all objects $S \in \mathcal{S}$.

We would like to apply the previous corollary, so we need to know that $S \downarrow U$ is locally small and complete. One easy way to ensure that $S \downarrow U$ is locally small is to require $\mathcal{A}$ ot be locally small. Indeed, $(S \downarrow U)((A, f), (B, g)) = \{\varphi : A \to B \mid U\varphi \circ f = g\} \subseteq \mathcal{A}(A, B)$, so $S \downarrow U$ has hom sets whenever $\mathcal{A}$ does.

The less obvious part is how to ensure that $S \downarrow U$ is complete. The next proposition does the trick.

Proposition 15.10. Let $U : \mathcal{A} \to \mathcal{S}$ be a continuous functor, and suppose that $\mathcal{A}$ is complete. Then for any object $S \in \mathcal{S}$, the category $S \downarrow U$ is also complete.

Proof. We show that $S \downarrow U$ has products of all indexed sets of objects and equalizers.

Suppose that $\{(A_j, f : S \to UA_j) \mid j \in J\} \subseteq S \downarrow U$. Since A is complete, there is a product $\prod_{j\in J} A)j \in \mathcal{A}$, with projections $\pi_J : \prod_{j\in J} A_j \to A_j$ for all $j \in J$. Since U is continuous, $U \prod_{j\in J} A_j \in \mathcal{S}$, together with $U\pi_j : U \prod_{j\in J} A_j \to UA_j$ is also a product, and hence the morphism $f : S \to UA_j$ induce a unique morphism $\langle f_j \rangle : S \to U \prod_{j\in J}$ such that $U\pi_j \circ \langle f_j \rangle = f_j$ for all $j \in J$. This defines an object $(\pi A_j, \langle f_j \rangle)$ of $S \downarrow U$.

Moreover, the equation $U\pi_j \circ \langle f_j \rangle = f_j$ implies that $\pi_j : (\prod_{j\in J} A_j, \langle f_j \rangle) \to (A_j, f_j)$ is a morphism in $S \downarrow U$ for all $j \in J$.

Now one can verify that $(\prod_{j\in J} A_j, \langle f_j \rangle)$, together with the maps $\pi_j : (\prod_{j\in J} A_j, \langle f_j \rangle) \to (A_j, f_j)$ is a product in $S \downarrow U$.

Now for equalizers. Suppose that $s, t : (A, f) \to (B, g)$ are a pair of parallel morphisms in $S \downarrow U$. Since $\mathcal{A}$ is complete, there is an equalizer

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & B \end{tikzcd}$$

in $\mathcal{A}$, and since U is continuous, we also has an equalizer as below:

$$\begin{tikzcd} UE \arrow[r, "Ue"] & UA \arrow[r, shift left, "Us"] \arrow[r, shift right, "Ut"'] & UB \\ & S \arrow[ul, dashed, "\exists!h"] \arrow[u, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

Since $s, t : (A, f) \to (B, g)$ in $S \downarrow U$, we know that $Us \circ f = g = Ut \circ f$, and then there is a unique $h : S \to UE$ such that $Ue \circ h = f$. Thus, $(E, h) \in S \downarrow U$ and $e : (E, h) \to (A, f)$, and one can check that it is an equalizer of $s, t : (A, f) \to (B, g)$.

Thus, $S \downarrow U$ is complete because it has all the products and equalizers. ☐

We now prove the General Adjoint Functor Theorem. It is just a matter of putting the pieces together.

Proof. Suppose $U : \mathcal{A} \to \mathcal{S}$ is a continuous functor, $\mathcal{A}$ is locally small and complete, and that U satisfies the solution set condition. Then for any object $S \in \mathcal{S}$:

1. The category $S \downarrow U$ is locally small because $\mathcal{A}$ is.
2. The category $S \downarrow U$ is complete because $\mathcal{A}$ is complete and U is continuous (see the previous proposition), and
3. The category $S \downarrow U$ has a jointly weakly initial set of objects because U satisfies the Solution Set Condition.

By an earlier corollary, it follows that all of the comma categories $S \downarrow U$ have initial objects, and then, we can construct a left adjoint $F \dashv U$ by the method explained in the proposition at the outset. ☐

Here is a standard application of the General Adjoint Functor Theorem: the construction of free groups. The standard method has fussy combinatorics, which this avoids.

Example 15.11 (Application)**.** Let $\mathbf{Grp}$ be the category of groups and let $U : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor. The category $\mathbf{Grp}$ is complete - if $F : J \to \mathbf{Grp}$ is a small diagram, then

$$L = \{(x_j)_{j \in J} \in \prod_{j \in J} Fj \mid \forall f : i \to j \in J : Ff(x_i) = x_j\}.$$

equipped with componentwise multiplication is a group, the coordinate projections $\pi_j : L \to Fj$ are homomorphisms, and L, together with these π_j's is a limit of F in $\mathbf{Grp}$. Applying U gives the standard construction of a limit of UF in $\mathbf{Set}$, so that U preserves these particular limits. However, all limits cones of a given diagram $F : J \to \mathbf{Grp}$ are isomorphic, so it follows that U preserves all small limits, i.e. U is continuous.

Next, observe that $\mathbf{Grp}$ is locally small, because a group homomorphism $\varphi : G \to H$ is a particular kind of set map from G to H.

Finally, we verify that U satisfies the solution set condition. First, some definitions. Let G be a group. A subgroup of G is a subset $H \subseteq G$ that is closed under multiplication, contains the identity and is closed under inversion. If $H \subseteq G$ is a subgroup, then the group structure on G restricts to a group structure on H such that the inclusion map $i : H \hookrightarrow G$ is a group homomorphism. Next, if $X \subseteq G$ is a subset, then the subgroup generated by X is the set

$$\langle X \rangle = \{e\} \cup \{x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \mid n \geqslant 1, x_1, \cdots, x_n \in S, \varepsilon_1, \cdots, \varepsilon_n \in \{\pm 1\}\}.$$

This is the smallest subgroup of G that contains X.

Back to the solution set condition. Let S be a fixed set. The key observation is that there is an upper bound on the size of the subgroups that image of S can generate in other groups. More precisely, note that if G is a group and $f : S \to UG$ is a set map, then there is a surjection

$$T = \coprod_{n \geqslant 0} (S \times \{\pm 1\})^n \to \langle \mathbf{im}(f) \rangle$$

that sends ( ) to e and sends $((s_1, \varepsilon_1), \cdots, (s_n, \varepsilon_n))$ to $f(s_1)_1^{\varepsilon} \cdots f(s_n)^{\varepsilon_n}$.

Choosing a section $s : \langle \mathbf{im}(f) \rangle \to T$, we see that $\langle \mathbf{im}(f) \rangle$ is in bijection with a subset of T.

Now to construct our solution set, let

$$I_S = \{(R, \gamma, \eta \mid R \subseteq T \text{ as a subset}, \gamma \text{ is a group structure on } R, \eta : S \to R \text{ as a set map }\}$$

and let $\Phi_S = \{((R, \gamma), \eta : S \to R) \mid (R, \gamma, \eta) \in I_S\}$. Then, given any group G and $f : S \to UG$, we say above that $\langle \mathbf{im}(f) \rangle \subseteq G$ is in bijection with a subset $R \subseteq T$ via some $s^{-1} : R \rightleftarrows \mathbf{im}(\langle f \rangle) : s$. Pushing the group structure on $\langle \mathbf{im}(f) \rangle$ over to a structure γ on R and letting $\eta = s \circ f : S \to R$, we obtain a group homomorphism

$$\bar{f} : i \circ s^{-1} : (R, \gamma) \xrightarrow{\cong} \langle \mathbf{im}(f) \rangle \hookrightarrow G$$

such that the diagram below commutes:

$$\begin{tikzcd}
R \arrow[r, "Us^{-1}", "\cong"'] & U \langle \mathbf{im}(f) \rangle \arrow[r, hook, "Ui"] & UG \\
S \arrow[u, "\eta"] \arrow[ur, "f"] \arrow[urr, "f"'] & &
\end{tikzcd}$$

This proves that Φ_S is a solution set, so by the General Adjoint Functor Theorem, it follows that $U : \mathbf{Grp} \to \mathbf{Set}$ has a left adjoint.

We conclude with another theorem that can be proven using the same techniques we developed for the General Adjoint Functor Theorem.

The following theorem gives a method of showing that a functor is representable.

Theorem 15.12 (Freyd's Representability Theorem)**.** Let $F : \mathscr{C} \to \mathbf{Set}$ be a continuous functor and suppose that $\mathscr{C}$ and locally small. If F satisfies the solution set condition below:

$(*)$ There exists a set Φ of objects of $\mathscr{C}$ such that for any $D \in \mathscr{C}$ and $y \in FD$, there is an object $C \in \Phi$, an element $x \in FC$ and a morphism $f : C \to D$ such that $Ff(x) = y$.

then F is representable.

Proof. Consider the comma category $* \downarrow F$. We have that

1. $* \downarrow F$ is locally small because $\mathscr{C}$ is, and
2. $* \downarrow F$ is complete because $\mathscr{C}$ is and F is continuous (see the earlier proposition).

Since $* \downarrow F \cong \int F$, it follows that $\int F$ has these same two properties. Now consider the set $\Psi = \{(C, x) \mid c \in \Phi, x \in FC\}$, then Ψ is jointly weakly initial in $\int F$ by the solution set condition. By an earlier corollary, it follows that $\int F$ has an initial object, which is the same thing as a universal element, which is the same thing as a representation of F by the Yoneda Lemma. ☐

16 Monads from Adjunctions

Suppose that $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ are adjoints with unit $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and counit $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$. This adjunction casts a ”shadow” in $\mathscr{C}$. In particular:

1. We can compote G and F to obtain an endofunctor $F = GF : \mathscr{C} \to \mathscr{C}$,
2. The unit $\eta : 1_{\mathscr{C}} \Rightarrow T$ is a map of endofunctors, and
3. The counit is not quite visible, but we can whisker it to obtain a natural map $G\varepsilon F : GFGF \Rightarrow G1_{\mathscr{D}}F = GF$. We denote this map $\mu : T^2 \Rightarrow T$.

The unit η and counit ε are natural and satisfy the triangle identities, and this implies that certain diagrams relating η and μ commute. All told, we obtain a monad.

Definition 16.1 (Monad)**.** Suppose $\mathscr{C}$ is a category. A monad on $\mathscr{C}$ is a triple (T, η, μ), where:

1. $T : \mathscr{C} \to \mathscr{C}$ is an endofunctor,
2. $\eta : 1_{\mathscr{C}} \Rightarrow T$ is a natural transformation, and
3. $\mu : T \circ T \Rightarrow T$ is a natural transformation.

such that $\mu \circ \eta T = 1_T = \mu \circ T\eta$ and $\mu \circ T\mu = \mu \circ \mu T$, i.e. the diagrams

$$\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "1_T"] \\
& T &
\end{tikzcd}$$

and

$$\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "\mu T"] \arrow[d, Rightarrow, "T\mu"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

commutes.

As mentioned earlier, every adjunction gives rise to a monad on the domain of the left adjoint.

Proposition 16.2. If $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, together with $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$ is an adjunction, then $(GF, \eta, G\varepsilon F)$ is a monad.

Proof. For any $C \in \mathscr{C}$, the diagram

$$\begin{tikzcd}
GFC \arrow[r, "\eta_{GFC}"] \arrow[dr, "1_{GFC}"'] & GFGFC \arrow[d, "G\varepsilon_{FC}"] & GFC \arrow[l, "GF\eta_C"'] \arrow[dl, "1_{GFC}"] \\
& GFC &
\end{tikzcd}$$

commutes by the triangle identities. Next, the naturality of $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$ implies that for any $C \in \mathscr{C}$, the square

$$\begin{tikzcd}
FGFGFC \arrow[r, "\varepsilon_{FGFC}"] \arrow[d, "FG\varepsilon_{FC}"'] & FGFC \arrow[d, "\varepsilon_{FC}"] \\
FGFC \arrow[r, "\varepsilon_{FC}"'] & FC
\end{tikzcd}$$

and the square after applying functor G

$$\begin{tikzcd}
GFGFGFC \arrow[r, "G\varepsilon_{FGFC}"] \arrow[d, "GFG\varepsilon_{FC}"'] & GFGFC \arrow[d, "G\varepsilon_{FC}"] \\
GFGFC \arrow[r, "G\varepsilon_{FC}"'] & GFC
\end{tikzcd}$$

also commute. □

In general, one loses information when passing from an adjunction $(F, G, \eta, \varepsilon)$ to the associated monad $(GF, \eta, G\varepsilon F)$, but, somewhat surprisingly, there are many cases of interest where it is possible to reconstruct an adjunction from its associated monad. The necessary and sufficient condition are given in Beck's Monadicity Theorem, which we shall consider in time.

For now, however, let us consider some examples.

Example 16.3. Let $M : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U$ be the free-forgetful adjunction between the category of sets and the category of monoids. The right adjoint U sends a monoid to its underlying set and a monoid homomorphism to its underlying function, as ”forgetting” information. The left adjoint M sends a set X to the monoid where the underlying set is $MX = \{$ finite tuples $(x_1, \cdots, x_m)$ of elements of $X\}$ with multiplication given by concatenation and unit given by the empty tuple $(\ )$. The functor M sends a set map $f : X \to Y$ to the monoid homomorphism $Mf : MX \to MY$ that takes $(x_1, \cdots, x_m) \mapsto (f(x_1), \cdots, f(x_m))$.

The unit of $M \dashv U$ is the insertion of generators $\eta_X : X \to UMX$ given by $x \mapsto (x)$ and the counit is multiplication $\varepsilon_N : MUN \to N$ given by $(x_1, \cdots, x_m) \mapsto x_1 \cdots x_m$.

Thus, the monad associated to $(M, U, \eta, \varepsilon)$ has

- $TX = MX$, regarded as a a set,
- $T(f : X \to Y) = Mf$, regarded as a set map,
- $\eta_X : X \to UMX$ as the insertion of generators,
- $\mu_X = U\varepsilon_{MX} : UMUMX \to UMX$ as the set map $MMX \to MX$ that sends a tuple of tuples $((x_{11}, \cdots, x_{1m_1}), \cdots, (x_{n1}, \cdots,$ to the concatenation of the tuples $(x_{11}, \cdots, x_{1m_1}, \cdots, x_{n1}, \cdots, x_{nm_n})$.

Thus, in this concrete example, we see that the monad associated to the adjunction $M \dashv U$ is essentially encoding the properties of free monoids: how their algebra works.

Example 16.4. Consider the adjunction $(-)_+ : \mathbf{Set} \rightleftarrows \mathbf{Set}_* : U$ between the category of sets and the category of pointed sets. The right adjoint U sends a pointed set to its underlying set and a basepoint-preserving function to its underlying function. They left adjoint $(-)_+$ sends a set X to $X \sqcup \{*\}$ and a function $f : X \to Y$ to the function $f_+ : X_+ \to Y_+$ that is f as $X \subseteq X_+$ and sends the new basepoint to the new basepoint. The unit $\eta_X : X \to UX_+$ is the inclusion $X \subseteq X_+$, and the counit $\varepsilon_{(Y,y)} : (U(Y, y))_+ \to (Y, y)$ is the identity on $U \subseteq (U(Y, y))_+$ and sends the new basepoint to $y \in Y$. Thus, the monad associates to this adjunction has

- $TX = X_+$, regarded as a set,

- $T(f : X \to Y) = f_+$, regarded as a set map,
- $\eta : X \to X_+$, the inclusion $X \subseteq X_+$, and
- $\mu : (X_+)_+ \to X_+$ the map that is the identity on $X_{\text{¡}}$ and sends both adjoined points of $(X_+)_+$ to the single adjoined point of X_+.

Example 16.5. Let $\mathbf{Set}$ be the category of sets and $\mathbf{Pos}$ be the category of posets. There is an adjunction

$$(-)^{\text{disc}} : \mathbf{Set} \rightleftarrows \mathbf{Pos} : U.$$

The functor U sends a point ot its underlying set and an order-preserving function to its underlying function. The functor $(-)^{\text{disc}}$ sends a set X to the point $X^{\text{disc}} = (X, x \leqslant y \iff x = y)$ and a function $f : X \to Y$ to itself, regarded as an order-preserving map $X^{\text{disc}} \to Y^{\text{disc}}$. The natural isomorphism here is $\varphi : \mathbf{Pos}(X^{\text{disc}}, (Y, \leqslant)) \cong \mathbf{Set}(X, Y)$ that sends $f : X^{\text{disc}} \to (Y, \leqslant)$ to $Uf : X \to Y$ and has an inverse sending $g : X \to Y$ to $g : X^{\text{disc}} \to (Y, \leqslant)$, same function regarded as an order-preserving map.

Thus, $\eta_X : X \to U(X^{\text{disc}}) = X$ is the identity map for all sets $X \in \mathbf{Set}$, and $\varepsilon_{(Y,\leqslant)} : Y^{\text{disc}} \to (Y, \leqslant)$ is the identity function, but with domain and codomain as indicated. Thus, the monad associated to this adjunction has

- $T = 1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$,
- $\eta_X : X \to X$, the identity function for all sets $X \in \mathbf{Set}$,
- $\mu_X : X \to X$ also the identity function for all sets X.

In other words, we have gotten the trivial monad (from a non-trivial adjunction).

Example 16.6. Let $f : X \to Y$ be a set map, and consider the order adjunction $f : \mathcal{P}(X) \rightleftarrows \mathcal{P}(Y) : f^{-1}$, where f sends a subset $A \subseteq X$ to its image $fA = \{f(a) \mid a \in A\} \subseteq Y$ and $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ sends a subset $B \subseteq Y$ to its inverse image $f^{-1}B = \{x \in X \mid f(x) \in B\}$. Then the unit of the adjunction is the order relation $A \subseteq f^{-1}fA$ and the counit is the order relation $ff^{-1}B \subseteq A$. Whiskering the counit relation gives an inclusion $f^{-1}ff^{-1}fA \subseteq f^{-1}fA$, and applying $f^{-1}f$ to the unit relation gives $f^{-1}fA \subseteq f^{-1}ff^{-1}fA$. Thus, $f^{-1}ff^{-1}fA = f^{-1}fA$, and the monad on $\mathcal{P}(X)$ associated to adjunction has

- $TA = f^{-1}fA$,
- $\eta_A : A \subseteq TA$,
- $\mu_A : T^2A = TA$

We can think of this monad as a closure operation on $\mathcal{P}(X)$. It expands a set and is idempotent. Similar construction apply to monads on any poset category.

17 ADJUNCTIONS FROM MONADS

As we have seen, every adjunction gives rise to a monad via the assignment $(F, G, \eta, \varepsilon) \mapsto (GF, \eta, G\varepsilon F)$, but there are also monads that arise without reference to any obvious adjunction.

Here is an example.

Example 17.1. Consider the covariant power set functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$. There is a natural transformation $\eta_X : X \to \mathcal{P}(X)$ that sends $x \mapsto \{x\}$ and also a natural transformation $\mu_X : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$ that takes $\{A_i \subseteq X \mid i \in I\} \mapsto \bigcup\limits_{i \in I} A_i$, and the tuple $(\mathcal{P}, \eta, \mu)$ is a monad on $\mathbf{Set}$. Naturality of η is the identity $f\{x\} = \{f(x)\}$, naturality of μ is the identity $\bigcup\limits_{i \in I} f(A_i) = f(\bigcup\limits_{i \in I} A_i)$, and verifying that the relevant diagrams for a monad commute is straightforward from here.

Thus, it is natural to ask whether ever monad arises from some adjunction.

It turns out the answer is yes. In fact, every monad generally arises from multiple adjunctions, but there are two canonical ones.

The first is an adjunction in relation to the "Kleisli Category". This is the initial adjunction that constructs a given monad.

The second is an adjunction in relation to the "Eilenberg-Moore Category" or the "Category of Algebras" for the given monad. This is the terminal adjunction that constructs a given monad.

In what follows, we shall focus (exclusively) on the Eilenberg-Moore Category, but for the sake of intuition, let us start with an example.

Example 17.2. Consider the monad associated to the free-forgetful adjunction $M : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U$. It consists of the following data:

- $TX = \{$ finite tuples $(x_1, \cdots, x_m)$ of elements of $X\}$,
- $T(f : X \to Y) : TX \to TY$ sends $(x_1, \cdots, x_m) \mapsto (f(x_1), \cdots, f(x_m))$,
- $\eta_X : X \to TX$ given by $x \mapsto (x)$,
- $\mu_X : TTX \to TX$ sends $((x_{11}, \cdots, x_{1m_1}), \cdots, (x_{n1}, \cdots, x_{nm_n})$ to the concatenation of the tuples $(x_{11}, \cdots, x_{1m_1}, \cdots, x_{n1}, \cdots$

We shall explain how to redefine the notion of a monoid in terms of the monad above.

Recall that a monoid is typically defined as a triple $(X, \cdot, e)$ when X is a set, $\cdot : X \times X \to X$ is a bianry operation, and $e \in X$ is a distinguished element such that $\cdot$ is associative and e serves as a two-sided identity for $\cdot$.

That being said, this is not the only way of presenting a monoid structure on a set X.

Indeed, if $(X, \cdot, e)$ is a monoid, then we can make sense of n-ary products by iterating the binary product. We thus obtain 3-ary products $(xmymz) \mapsto (x \cdot y) \cdot z$ and so on, and since $\cdot$ is associative, we may safely omit parentheses, as all possible parenthezіation of a given n-ary product will be equal.

All told, we obtain a function $\varepsilon : TX \to X$ by sending ( ) to e and $(x_1, \cdots, x_m)$ to $x_1 \cdot \cdots \cdot x_m$, which is the counit of the adjunction $M \dashv U$ but regarded as a set map.

The map $\varepsilon : TX \to X$ is not arbitrary, however. Indeed, it plays nicely with the structure in (T, η, μ) in the sense that the diagram below commutes:

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[dr, "\mathbf{id}"] & TX \arrow[d, "\varepsilon"'] \\
 & X
\end{tikzcd}$$

has elementwise mapping

$$\begin{tikzcd}
a \arrow[r, "\eta"] \arrow[dr, "\mathbf{id}"] & (a) \arrow[d, "\varepsilon"'] \\
 & a
\end{tikzcd}$$

and there is

$$\begin{tikzcd}
T^2X \arrow[r, "T\varepsilon"] \arrow[d, "\mu"'] & TX \arrow[d, "\varepsilon"] \\
TX \arrow[r, "\varepsilon"'] & X
\end{tikzcd}$$

that has elementwise mapping

$$\begin{tikzcd}
((x_{11}, \cdots, x_{1m_1}), \cdots, (x_{n1}, \cdots, x_{nm_n})) \arrow[r, "T\varepsilon"] \arrow[d, "\mu"'] & (x_{11} \cdots x_{1m_1}, \cdots, x_{n1} \cdots x_{nm_n}) \arrow[d, "\varepsilon"] \\
(x_{11}, \cdots, x_{1m_1}, \cdots, x_{n1}, \cdots, x_{nm_n}) \arrow[r, "\varepsilon"'] & x_{11} \cdots x_{m1} \cdots x_{n1} \cdots x_{nm_n}
\end{tikzcd}$$

The first diagram confirms that we are embedding X into TX as the subset of length 1 tuples.

The commutativity of the second diagram is a consequence of the associativity and unitality of $\cdot$ in X.

Thus, if X is a set, then a monoid structure $(\cdot, e)$ on X gives rise to a map $\varepsilon : TX \to X$ making the two diagrams above commute.

Let us turn this around. Suppose that $\varepsilon : TX \to X$ is a set map making the above triangle and square commute. We shall extract a monoid structure $(\varepsilon_2, \varepsilon_0)$ on X.

Indeed, by restricting $\varepsilon : TX \to X$ to length 2 tuples, we obtain a function $\varepsilon_2 : X^2 \subseteq TX \to X$ and evaluating ε at ( ) gives an element $\varepsilon_0 \in X$. We claim that $(X, \varepsilon_2, \varepsilon_0)$ is a monoid.

To see assocativity, we choose $((x, y), (z))$ and $((x), (y, z))$ around the square:

$$\begin{tikzcd}
((x,y),(z)) \arrow[r] \arrow[d] & (\varepsilon_2(x,y),z) \arrow[d] \\
(x,y,z) \arrow[r] & \varepsilon(x,y,z)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(x,(y,z)) \arrow[r] \arrow[d] & (x,\varepsilon_2(y,z)) \arrow[d] \\
(x,y,z) \arrow[r] & \varepsilon(x,y,z)
\end{tikzcd}$$

to get $\varepsilon_2(\varepsilon_2(x, y), z) = \varepsilon(x, y, z) = \varepsilon_2(x, \varepsilon_2(y, z))$.

To get unitality, choose $((\ ), (x))$ and $((x), (\ ))$:

$$\begin{tikzcd}
((\ ),(x)) \arrow[r] \arrow[d] & (\varepsilon_0,x) \arrow[d] \\
(x) \arrow[r] & x
\end{tikzcd}$$

$$\begin{tikzcd}
((x),(\ )) \arrow[r] \arrow[d] & (x,\varepsilon_0) \arrow[d] \\
(x) \arrow[r] & x
\end{tikzcd}$$

With a bit of thought, one can prove that the two constructions we have just described are inverse, i.e. if we start with a monoid structure $(\cdot, e)$ on X, form $\varepsilon : TX \to X$, and then extract a monoid structure $(\varepsilon_2, \varepsilon_0)$, then we recover the original monoid structure, and similarly if we start with $\varepsilon : TX \to X$ and play the same game.

Thus, one can equivalently define a monoid to be a set X, together with a function $\varepsilon : TX \to X$ such that the triangle and square described above commute.

With this example in hand, we now introduces the Eilenberg-Moore Category associated to a monad (T, η, μ). We find the following interpretation of (T, η, μ), following the free monoid monad, to be helpful:

- TC: underlying object of the free algebra on C.
- $\eta : C \to TC$: insertion of generators into the free algebra.
- $\mu : TTC \to TC$: the map that describes how terms in free algebra are ”formally combined”.

Definition 17.3 (Eilenberg-Moore Category/Category of Algebra)**.** Let $\mathscr{C}$ be a category and (T, η, μ) be a monad on $\mathscr{C}$. The Eilenberg-Moore category for T, also called the category of T-algebras, is the category $\mathscr{C}^T$ where:

- objects are pairs $(C \in \mathscr{C}, h : TC \to C \in \mathscr{C})$ such that the diagrams

$$\begin{tikzcd}
C \arrow[r, "\eta_C"] \arrow[dr, "1_C"] & TC \arrow[d, "h"'] \\
& C
\end{tikzcd}$$

and

$$\begin{tikzcd}
T^2 \arrow[r, "Th"] \arrow[d, "\mu_C"'] & TC \arrow[d, "h"] \\
TC \arrow[r, "h"'] & C
\end{tikzcd}$$

commute, and where

- morphisms $\varphi : (C, h) \to (D, k)$ are morphisms $\varphi : C \to D$ in $\mathscr{C}$ such that the square

$$\begin{tikzcd}
TC \arrow[r, "T\varphi"] \arrow[d, "h"'] & TD \arrow[d, "k"] \\
C \arrow[r, "\varphi"'] & D
\end{tikzcd}$$

commutes with composition and identities as in $\mathscr{C}$.

Thus, our previous example shows that monoids are precisely the same thing as algebras over the free monoid monad.

Similar things can be said about morphisms. If $f : (X, \cdot, e) \to (Y, \cdot, e)$ is a monoid homomorphism, then the same set map defines an algebra homomorphism $f : (X, \varepsilon) \to (Y, \varepsilon)$, and if $f : (X, \varepsilon) \to (Y, \varepsilon)$ is an algebra homomorphism, then the same function defines a monoid homomorphism $f : (X, \varepsilon_1, \varepsilon_0(\to (Y, \varepsilon_2, \varepsilon_0)$.

All told, we obtain an isomorphism of categories $K : \mathbf{Mon} \xrightarrow{\cong} \mathbf{Set}^{UM}$ that sends $(X, \cdot, e) \mapsto (X, \varepsilon : MX \to X)$ and $f : (X\cdot, e) \to (Y, \cdot, e)$ to $f : (X, \varepsilon) \to (Y, \varepsilon)$.

We are not done yet. Our original goal was to produce an adjunction that induced a given monad. Thus, it remains to construct an adjunction $F^T : \mathscr{C} \rightleftarrows \mathscr{C}^T : G^T$ such that $(T, \eta, \mu) = (G^T F^T, \eta^T, G^T \varepsilon^T F^T)$.

We shall take the right adjoint G^T to be the forgetful functor $G^T : \mathscr{C}^T \to \mathscr{C}$ that sends $(C, h : TC \to C) \mapsto C$ and $\varphi : (C, h) \to (D, k)$ to $\varphi : C \to D$.

Recalling that we are thinking of TC as the underlying object of the free algebra on C, we define the left adjoint, free T-algebra functor by $F^T : \mathscr{C} \to \mathscr{C}^T$, that sends $C \mapsto (TC, \mu_C : TTC \to TC)$ and $\varphi : C \to D$ to $T\varphi : (TC, \mu_C) \to (TD, \mu_D)$.

One uses the monad axioms to verify that (TC, μ_C) is a T-algebra, and the naturality of μ ensures that $T\varphi : (TC, \mu_C) \to (TD, \mu_D)$ is an algebra homomorphism. For any $C \in \mathscr{C}$ and $(D, k) \in \mathscr{C}^T$, there is a bijection $\phi : \mathscr{C}^T((TC, \mu_C), (D, k) \cong \mathscr{C}(C, D)$ that sends $(f : (TC, \mu_C) \to (D, k)) \mapsto (f \circ \eta_C : C \to TC \to D)$ and sends $(g : C \to D) \mapsto (k \circ Tg : (TC, \mu_C) \to (TD, \mu_D) \to (D, k))$ as an inverse, that is natural in C and (D, k). Note that for T-algebra (D, k), the morphism $k : TD \to D$ defines T-algebra morphism $k : (TD, \mu_D) \to (D, k)$, so the formula for ϕ^{-1} makes sense.

Thus, we obtain an adjunction $F^T \dashv G^T$, where the unit is $\eta^T_C = \phi(1_{(TC, \mu_C)}) = \eta_C : C \to TC$, i.e. the unit of the original monad (T, η, μ), and the counit is $\varepsilon^T_{(D,k)} = \phi^{-1}(1_D) = k : (TD, \mu_D) \to (D, k)$.

From here, one can check that the monad associated to $(F^T, G^T, \eta^T, \varepsilon^T)$ is exactly $(G^T F^T, \eta^T, \mu^T) = (T, \eta, \mu)$, i.e. we have reversed the monad used to define the adjunction $(F^T : \mathscr{C} \rightleftarrows \mathscr{C}^T : G^T, \eta^T, \varepsilon^T)$.

In summary, we have:

Proposition 17.4. Let $\mathscr{C}$ be a category and (T, η, μ) be a monad on $\mathscr{C}$. Then the monad associated to the Eilenberg-Moore adjunction $(F^T, G^T, \eta^T, \varepsilon^T)$ is (T, η, μ).

Thus, every monad comes from an adjunction: if we start with (T, η, μ), form $F^T \dashv G^T$, and then pass to $(G^T F^T, \eta^T, G^T \varepsilon^T F^T)$, we get back to where we started.

Now, another question arises: what if we turn this around? Start with an adjunction $(F, G, \eta, \varepsilon)$, form its monad $(T, \eta, \mu) = (GF, \eta, G\varepsilon F)$, and then construct $(F^T, G^T, \eta^T, \varepsilon^T)$. Do we recover the original adjunction?

In some cases, we do:

Example 17.5. Let $(M, U, \eta, \varepsilon)$ be the free-forgetful adjunction between $\mathbf{Set}$ and $\mathbf{Mon}$. Earlier, we constructed an isomorphism $K : \mathbf{Mon} \xrightarrow{\cong} \mathbf{Set}^{UM}$ sending $X \in \mathbf{Mon}$ to $(EX, \varepsilon_X : MX \to X)$ and $f : X \to Y$ to $f : (X, \varepsilon_X) \to (Y, \varepsilon_Y)$. Strictly speaking, we should write $(UX, U\varepsilon_X : UMUX \to UX)$ and $Uf : (UX, U\varepsilon_X) \to (UY, U\varepsilon_Y)$. Now one can check that both triangles in the diagram

$$\begin{tikzcd}
\mathbf{Mon} \arrow[rr, "K", "\cong"'] \arrow[dr, bend left, "U"'] & & \mathbf{Set}^{UM} \arrow[dl, bend left, "U^T"] \\
& \mathbf{Set} \arrow[ul, bend left, "M"] \arrow[ur, bend left, "M^T"'] &
\end{tikzcd}$$

commutes. In this sense, the $M \dashv U$ adjunction is isomorphic to the $M^T \dashv U^T$ adjunction, so we can recover $M \dashv U$ from $(UM, \eta, U\varepsilon M)$.

However,there are cases where $F \dashv G$ and $F^T \dashv G^T$ are different. We illustrate by example.

Example 17.6. Consider the $(-)^{\text{disc}} : \mathbf{Set} \rightleftarrows \mathbf{Pos} : U$ adjunction, where U is the forgetful functor, and $(-)^{\text{disc}} : \mathbf{Set} \to \mathbf{Pos}$ sends a set X to the discrete poset $(X, x \leqslant y \iff x = y)$ and a set map $f : X \to Y$ to itself. Then, as we saw earlier, the monad associated to this adjunction has

- $T = 1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$.
- $\eta_X : X \to X$ is the identity function for all sets $X \in \mathbf{Set}$.
- $\mu_X : X \to X$ is also the identity function for all sets X.

i.e. is the trivial monad. In this case, one can check that $F^T : \mathbf{Set} \rightleftarrows \mathbf{Set}^T : G^T$ are inverse. Now, if there were a comparison isomorphism $K : \mathbf{Pos} \xrightarrow{\cong} \mathbf{Set}^T$ such that both triangles in

$$\begin{tikzcd}
\mathbf{Pos} \arrow[rr, "K", "\cong"'] \arrow[dr, bend left, "U"'] & & \mathbf{Set}^{T} \arrow[dl, bend left, "G^T"] \\
& \mathbf{Set} \arrow[ul, bend left, "(-)^{\text{disc}}"] \arrow[ur, bend left, "F^T"'] &
\end{tikzcd}$$

commute, then $(-)^{\text{disc}}$ and U would be isomorphisms. But this is false, so we do not recover the $(-)^{\text{disc}} \dashv U$ adjunction up to isomorphism (or even equivalence).

That being said, there is always a canonical comparison functor from $(F, G, \eta, \varepsilon)$ to $(F^T, G^T, \eta^T, \varepsilon^T)$, in the following sense.

Definition 17.7. Let $\mathscr{C}$ be a category and (T, η, μ) be a monad on $\mathscr{C}$. The category $\mathbf{Adj}_T$ has:

- Objects: adjunctions $(F : \mathscr{C} \rightleftarrows \mathscr{D} : G, \eta, \varepsilon)$ such that $(GF, \eta, G\varepsilon F) = (T, \eta, \mu)$, and
- Morphisms: a morphism $K : (F : \mathscr{C} \rightleftarrows \mathscr{D} : G, \eta, \varepsilon) \to (F' : \mathscr{C} \rightleftarrows \mathscr{D} : G', \eta', \varepsilon')$ is a functor $K : \mathscr{D} \to \mathscr{D}'$ such that both triangles below commutes:

$$\begin{tikzcd}
\mathscr{D} \arrow[rr, "K"] \arrow[dr, bend left, "G"] & & \mathscr{D}' \arrow[dl, bend left, "G'"] \\
& \mathscr{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F'"] &
\end{tikzcd}$$

Here is the universal property of $(F^T, G^T, \eta^T, \varepsilon^T)$.

Proposition 17.8. Let $\mathscr{C}$ be a category and (T, η, μ) be a monad on $\mathscr{C}$. Then $(F^T, G^T, \eta^T, \varepsilon^T)$ is terminal in $\mathbf{Adj}_T$.

Before proving this, we need a lemma.

Lemma 17.9. Suppose $\mathscr{C}$ is a category, (T, η, μ) is a monad in $\mathscr{C}$, and $K : (F : \mathscr{C} \rightleftarrows \mathscr{D} : G, \eta, \varepsilon) \to (F' : \mathscr{C} \rightleftarrows \mathscr{D}' : G', \eta', \varepsilon')$ is a morphism in $\mathbf{Adj}_T$:

$$\begin{tikzcd}
\mathscr{D} \arrow[rr, "K"] \arrow[dr, bend left, "G"] & & \mathscr{D}' \arrow[dl, bend left, "G'"] \\
& \mathscr{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F'"] &
\end{tikzcd}$$

Then $\varepsilon' K = K\varepsilon$.

Proof. First of all, note that $(GF, \eta, G\varepsilon F) = (T, \eta, \mu) = (G'F', \eta', \varepsilon')$, so that $\eta = \eta'$. Next, consider

$$\begin{aligned} \phi : \mathscr{D}(FC, D) &\cong \mathscr{C}(C, GD) : \phi^{-1} \\ f &\mapsto Gf \circ \eta_C \\ \varepsilon_D \circ Fg &\mapsfrom g \end{aligned}$$

and

$$\begin{aligned} \psi : \mathscr{D}'(F'C, D') &\cong \mathscr{C}(C, G'D') : \psi^{-1} \\ f &\mapsto G'f \circ \eta'_C \\ \varepsilon'_{D'} \circ F'g &\mapsfrom g \end{aligned}$$

be the natural isomorphisms associated to these two adjunctions. Then, since $KF = F'$, $G'K = G$, and $\eta = \eta'$, the diagram

$$\begin{tikzcd}
\mathscr{D}(FC, D) \arrow[r, "\phi"] \arrow[d, "K"] & \mathscr{C}(C, GD) \arrow[dd, equal] \\
\mathscr{D}'(KFC, KD) \arrow[d, equal] & \\
\mathscr{D}'(F'C, KD) \arrow[r, "\psi"] & \mathscr{C}(C, G'KD)
\end{tikzcd}$$

commutes for all $C \in \mathscr{C}$ and $\mathscr{D}$. Chasing 1_{GD} around the commutative diagram

$$\begin{tikzcd}
\mathscr{D}(FGD, D) \arrow[d, "K"] & \mathscr{C}(GD, GD) \arrow[l, "\phi^{-1}"'] \arrow[dd, equal] \\
\mathscr{D}'(KFGD, KD) \arrow[d, equal] & \\
\mathscr{D}'(F'GD, KD) & \mathscr{C}(GD, G'KD) \arrow[l, "\psi^{-1}"']
\end{tikzcd}$$

shows that $K\varepsilon_D = \varepsilon'_{KD}$ for all $D \in \mathscr{D}$, i.e. $K\varepsilon = \varepsilon' K$. □

We now prove the proposition.

Proof. Let $\mathscr{C}$ be a category, (T, η, μ) be a monad on $\mathscr{C}$, and $(F, \mathscr{C} \rightleftarrows \mathscr{D} : G, \eta, \mu) \in \mathbf{Adj}_T$. Then there is a morphism

$$\begin{tikzcd}
\mathscr{D} \arrow[rr, "K"] \arrow[dr, bend left, "G"] & & \mathscr{C}^T \arrow[dl, bend left, "G^T"] \\
& \mathscr{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] &
\end{tikzcd}$$

in $\mathbf{Adj}_T$ defined by $KD = (GD, G\varepsilon_D : GFGD \to GD)$ with $K(f : D \to D') = Gf : (GD, G\varepsilon_D) \to (GD', G\varepsilon_{D'})$.

We must show that this is the only such morphism. So suppose $L : \mathscr{D} \to \mathscr{C}^T$ is another morphism. Since $G^T L = G$, it follows that for any object $D \in \mathscr{D}$, if $LD = (C, h)$, then $C = G^T(C, h) = G^T LD = GD$. Thus, $LD = (GD, h_D : GFGD \to GD)$ for some h_D for all $D \in \mathscr{D}$. We must show that $h_D = G\varepsilon_D$. Note that $\varepsilon^T_{LD} = h_D : (GFGD, G\varepsilon_{FGD}) \to (GD, h_D)$, and by the previous lemma, $\varepsilon^T L = L\varepsilon$, so $\varepsilon^T_{LD} = L\varepsilon_D$. Applying G^T, we find that

$G\varepsilon_D = G^T L\varepsilon_D = G^T \varepsilon^T_{LD} = h_D : GFGD \to GD$, and thus $LD = (GD, h_D) = (GD, G\varepsilon_D) = KD)$. Thus, $K = L$ on objects.

For morphisms, suppose $f : D \to D'$ in $\mathscr{D}$, and write $L(f : D \to D') = \varphi : (GD, G\varepsilon_D) \to (GD', G\varepsilon_{D'})$. Applying G^T, we see that $\varphi = G^T(\varphi : (GD, G\varepsilon_D) \to (GD', G\varepsilon_{D'})) = G^T L(f : D \to D') = G(f : D \to D')$. Thus, $L(f : D \to D') = Gf : (GD, G\varepsilon_D) \to (GD', G\varepsilon_{D'}) = K(f : D \to D')$, so that $K = L$ on morphisms, too. Thus, $K = L$. ☐

So, to summarize, suppose we have an adjunction $(F, G, \eta, \varepsilon)$, we pass to the associated monad $(T, \eta, \mu) = (GF, \eta, G\varepsilon F)$, and then we form the adjunction $(F^T, G^T, \eta^T, \varepsilon^T)$ relative to the Eilenberg-Moore category of (T, η, μ). Then $(F^T, G^T, \eta^T, \varepsilon^T)$ need not be $(F, G, \eta, \varepsilon)$, but there is a canonical comparison map $K : (F, G, \eta, \varepsilon) \to (F^T, G^T, \eta^T, \varepsilon^T)$ in the category of adjunctions that induce the monad (T, η, μ).

A natural question is: when is K an isomorphism? In other words, when do we recover the original adjunction from the Eilenberg-Moore adjunction?

Beck's Monadicity Theorem gives us a complete answer, and we now turn our attention to it.

18 CANONICAL PRESENTATIONS

Let (T, η, μ) be a monad. Before we can state and prove the Beck's Monadicity Theorem, we shall need to know more about the structure of T-algebras.

The key observation is that every T-algebra has a canonical presentation as a quotient of a free T-algebra. We illustrate how this works for monoids, before turning to the general theory.

To start with, let us see how quotients of monoids work.

Example 18.1. Suppose that $(X, \cdot, e)$ is a monoid. A congruence relation on X is a binary relation $\sim$ such that

1. $\sim$ is an equivalence relation, and
2. if $x \sim x'$ and $y \sim y'$, then $x \cdot y \sim x' \cdot y'$.

If $\sim$ is a congruence relation on X, then we write $[x] = \{y \in X \mid y \sim x\}$ for the congruence class of $x \in X$ and $X/\sim = \{[x] \mid x \in X\}$ for the set of all congruence classes of elements of X.

The quotient of X by $\sim$ is the monoid where

1. underlying set if $X/\sim$.
2. multiplication is defined by $[x] \cdot [y] = [x \cdot y]$: this is well-defined by the second axiom of a congruence relation, and
3. unit is $[e]$.

We think of the monoid $X/\sim$ as obtained by setting $\sim$-equivalent elements equal to each other.

Now suppose that R is a binary relation on X. If R is not a congruence relation, then we won't necessarily be able to form a quotient X/R, but we can first expand R to a congruence relation $\sim_R$, and then form $X/\sim_R$.

The congruence relation generated by R is the smallest congruence relation that contains R. We shall denote it $\sim_R$. Explicitly, $x \sim_R y$ if and only if there is an integer $n \geqslant 0$ and elements $x_0, x_1, \cdots, x_n \in X$ such that

1. $x = x_0$,
2. $y = x_n$, and
3. for each $0 \leqslant k < n$, there are elements $a, b, b', c \in X$ such that
    - $x_k = abc$,
    - $x_{k+1} = ab'c$, and
    - either bRb' or $b'Rb$.

The relation $\sim_R$ has the following properties:

1. $\sim_R$ is a congruence relation.
2. if xRy, then $x \sim_R y$.
3. if $\approx$ is a congruence relation such that xRy implies $x \approx y$, then for all $x, y \in X$: if $x \sim_R y$, then $x \approx y$.

This is the sense in which $\sim_R$ is the smallest congruence relation that contains R.

Now, for any monoid X and binary relation R on X, there is a projection homomorphism $\pi : X \to X/\sim_R$ that sends $x \mapsto [x]$, which has the following universal property:

1. for all $x, y \in X$: if xRy, then $\pi(x) = \pi(y)$, and
2. if $\varphi : X \to Y$ is a monoid homomorphism such that xRy implies $\varphi(x) = \varphi(y)$, then there is a unique homomorphism $\bar{\varphi} : X/\sim_R \to Y$ such that $\varphi = \bar{\varphi} \circ \pi$:

$$\begin{tikzcd}
X \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & X/\sim_R \arrow[d, dashed, "\exists!\bar{\varphi}"'] \\
& Y
\end{tikzcd}$$

namely $\bar{\varphi}[x] = \varphi(x)$.

This can be seen by noting that $x \sim_\varphi y$ if and only if $\varphi(x) = \varphi(y)$ is a congruence relation. Then, if xRy implies $\varphi(x) = \varphi(y)$ (i.e. $x \sim_\varphi y$), then $x \sim_R y$ implies $x \sim_\varphi y$ and $\varphi(x) = \varphi(y)$, so that $\bar{\varphi}[x] = \varphi(x)$ is well-defined.

Thus, we think of $X/\sim_R$ as the monoid obtained from X by setting R-related elements equal.

With these preliminaries on quotient monoids done, we now consider presentation of monoids.

Suppose X is a monoid, and $G \subseteq X$ is a subset. We say that G generates X or that G is a set of generators of X if every element of X can be expressed as a product of elements in G.

Equivalently, the relation $i : G \hookrightarrow UX$ induced a monoid homomophism $\varphi : MG \to X$ from the free monoid MG on teh set G, and G is a set of generators if and only if $\varphi : MG \to X$ is surjective.

Now suppose $G \subseteq X$ generates X. Then every element of X can be built from the elements of G, but we have ignored the algebraic properties of these elements. For example, there may be $a, b, c \in G$ such that $ab = c$, but we have not kept track of this information.

A relation between the generators $G \subseteq X$ is a pair $(r, s) \in MG \times MG$ such that $\varphi(r) = \varphi(s)$, i.e. a relation is a pair of words in G, which are equal when regarded as elements of X.

If R is a set of relations, then we can regard R as a binary relation on MG with xRy if and only if $(x, y) \in R$, and we can form the quotient $MG/\sim_R$. By the universal property of quotient, these is a unique homomorphism $\bar{\varphi} : MG/\sim_R \to X$ such that $\bar{\varphi} \circ \pi = \varphi$.

We say that R is a complete set of relations for X relative to the generator G if the map $\bar{\varphi} : MG/\sim_R \to X$ is an isomorphism. In such a case, we refer to the pair (G, R) as a **presentation** of X by generators and relations.

Thus, a presentation of a monoid X is a specification of ”building blocks” of X, together with a description of the algebraic equations relating these building blocks.

If (G, R) is a presentation of X, then $X \cong MG/\sim_R$ is obtained by forming the set of all words in G and then identifying the pairs of words specified by R.

Now, a given monoid X will typically have many different presentations, but one is canonical:

1. Take X itself as a set of generators, so that $\varphi = \varepsilon : MX \to X$ is the counit of the free-forgetful adjunction $M : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U$, and then
2. take $R = \{((x_1, \cdots, x_m), (x_1 \cdots x_m)) \mid x_1, \cdots, x_m \in X\}$ as a set of relations (i.e. identify every ”formal product” in MX with its ”actual product” in X). For $m = 0$, we understand $((\ ), (e)) \in R$.

Then $X \cong MX/\sim_R$ as can be seen by noting that $\varepsilon : MX \to X$ also has the universal property of the quotient projection $\pi : MX \to MX/\sim_R$.

Now, presentations of monoids can be recast as categorical coequalizers. let X be a monoid, $G \subseteq X$ bea. set of generators, and $R \subseteq MG \times MG$ be a complete set of relations. Observe that a monoid homomorphism $\psi : MG \to Y$ has the property $(*)$ $\forall \alpha, \beta \in MG$, if $\alpha R \beta$, then $\psi(\alpha) = \psi(\beta)$ if and only if the diagram $(**)$

$$\begin{tikzcd} R \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & UMG \arrow[r, "U\psi"] & UY \end{tikzcd}$$

is a fork, i.e. $U\psi \circ p_1 = U\psi \circ p_2$. Here, p_1 and p_2 are the first and second coordinate projections, respectively. Transposing along the free-forgetful adjunction $M \dashv U$, we see that $(**)$ is, in turn, equivalent to the diagram $(***)$

$$\begin{tikzcd} MR \arrow[r, shift left, "\bar{p_1}"] \arrow[r, shift right, "\bar{p_2}"'] & MG \arrow[r, "\psi"] & Y \end{tikzcd}$$

being a fork, i.e. having $\psi \circ \bar{p_1} = \psi \circ \bar{p_2}$, where $\bar{p_1}$ and $\bar{p_2}$ are the transposes of p_1 and p_2 respectively.

Since (G, R) is a presentation of X_1, we have

$$\begin{tikzcd} MG \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & MG/\sim_R \arrow[d, dashed, "\cong"'] \\ & X \end{tikzcd}$$

It follows that $\varphi : MG \to X$ is an initial morphisms with property $(*)$, and hence $(***)$, i.e.

$$\begin{tikzcd} MR \arrow[r, shift left, "\bar{p_1}"] \arrow[r, shift right, "\bar{p_2}"'] & MG \arrow[r, "\varphi"] & X \end{tikzcd}$$

is a coequalizer. Conversely, if this diagram is a coequalizer, then $\varphi : MG \to X$ is initial with property $(***)$, and hence with property $(*)$, which implies that the induced morphisms in

$$\begin{tikzcd} MG \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & MG/\sim_R \arrow[d, dashed] \\ & X \end{tikzcd}$$

is an isomorphism, so (G, R) is a presentation of X.

Thus, we can encode presentations using coequalizers.

Specializing to the canonical presentations of X, it follows that

$$\begin{tikzcd} MMX \arrow[r, shift left, "\varepsilon_{MX}"] \arrow[r, shift right, "M\varepsilon_X"'] & MX \arrow[r, "\varepsilon_X"] & X \end{tikzcd}$$

is a coequalizer of monoids.

Note that this coequalizer has the following special property: if we consider its underlying set maps, then there are unit maps

$$\begin{tikzcd} MMX \arrow[r, shift left, "\varepsilon_{MX}"] \arrow[r, shift right, "M\varepsilon_X"'] & MX \arrow[r, "\varepsilon_X"] \arrow[l, bend left=60, "\eta_{MX}"] & X \arrow[l, bend left=60, "\eta_X"] \end{tikzcd}$$

that split the coequalizer, in the following sense:

Definition 18.2 (Split Coequalizer)**.** A split coequalizer diagram is a collection

$$\begin{tikzcd} A \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & B \arrow[r, "e"] \arrow[l, bend left=60, "t"] & C \arrow[l, bend left=60, "s"] \end{tikzcd}$$

Figure 18.1: Split Coequalizer

of objects and morphisms such that the diagram

$$\begin{tikzcd}
B \arrow[r, "t"] \arrow[d, "e"] \arrow[rr, bend left, "1_B"] & A \arrow[r, "d_0"] \arrow[d, "d_1"] & B \arrow[d, "e"] \\
C \arrow[r, "s"] \arrow[rr, bend right, "1_C"'] & B \arrow[r, "e"] & C
\end{tikzcd}$$

commutes. If there is such a diagram, we say that e is a split coequalizer. This makes sense because of the following proposition.

Proposition 18.3. If

$$\begin{tikzcd}
A \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & B \arrow[r, "e"] \arrow[l, bend left=50, "t"] & C \arrow[l, bend left=50, "s"]
\end{tikzcd}$$

is a split coequalizer diagram in a category $\mathscr{C}$, then

$$\begin{tikzcd}
A \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & B \arrow[r, "e"] & C
\end{tikzcd}$$

is an **absolute coequalizer**, i,e, it is a coequalizer that is preserved by every functor out of $\mathscr{C}$.

Proof. Since functors preserve commutative diagrams, the functorial image of any split coequalizer is a split coequalizer. Thus, it will be enough to show that every split coequalizer diagram is a coequalizer diagram.

Given a morphism $f : B \to D$ such that $f \circ d_0 = f \circ d_1$, let $\bar{f} : C \to D$ be the composite $f \circ s$. Then $\bar{f} \circ e = f \circ s \circ e = f \circ d_1 \circ t = f \circ d_0 \circ t = f \circ 1_B = f$, so that $\bar{f}$ factors f through e. On the other hand, if $f = g \circ e$, then applying s gives $f \circ s = g \circ e \circ s = g$, so that $\bar{f} = f \circ s$ gives the unique factorization. ☐

To summarize, every monoid $X \in \mathbf{Mon}$ has a canonical presentation encoded by the coequalizer diagram

$$\begin{tikzcd}
MMX \arrow[r, shift left, "\varepsilon_{MX}"] \arrow[r, shift right, "M\varepsilon_X"'] & MX \arrow[r, "\varepsilon_X"] & X
\end{tikzcd}$$

in $\mathbf{Mon}$. Moreover, its underlying diagram extends to a split coequalizer diagram

$$\begin{tikzcd}
MMX \arrow[r, shift left, "\varepsilon_{MX}"] \arrow[r, shift right, "M\varepsilon_X"'] & MX \arrow[r, "\varepsilon_X"] \arrow[l, bend left=50, "\eta_{MX}"] & X \arrow[l, bend left=50, "\eta_X"]
\end{tikzcd}$$

in $\mathbf{Set}$.

Something like this is true in every Eilenberg-Moore category $\mathscr{C}^T$, as we briefly explain.

Let (T, η, μ) be a monad on a category $\mathscr{C}$, let $\mathscr{C}^T$ be the category of T-algebra, and let $(F^T : \mathscr{C} \rightleftarrows \mathscr{C}^T : G^T, \eta^T, \varepsilon^T)$ be the free-forgetful adjunction.

Given any T-algebra $(C, h) \in \mathscr{C}^T$, we have

$$\begin{tikzcd}
(TTC, \mu_{TC}) \arrow[r, shift left, "\varepsilon^T_{(TC,\mu_C)}"] \arrow[r, shift right, "T\varepsilon^T_{(C,h)}"'] & (TC, \mu_C) \arrow[r, "\varepsilon^T_{(C,h)}"] \arrow[d, equal] & (C, h) \\
(TTC, \mu_{TC}) \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Th"'] & (TC, \mu_C) \arrow[r, "h"] & (C, h)
\end{tikzcd}$$

Proposition 18.4. For any T-algebra $(C, h) \in \mathscr{C}^T$, the diagram

$$\begin{tikzcd} TTC \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Th"'] & (TC, \mu_C) \arrow[r, "h"] & (C, h) \end{tikzcd}$$

is a coequalizer in $\mathscr{C}^T$.

Proof. Suppose that $\varphi : (TC, \mu_C) \to (A, \alpha)$ is a T-algebra homomorphism such that $\varphi \circ \mu_C = \varphi \circ Th$. Consider the morphism $\varphi \circ \eta_C : C \to A$ in $\mathscr{C}$. Then $\varphi \circ \eta_C$ is a T-algebra homomorphism $(C, h) \to (A, \alpha)$ because

1. $\alpha \circ T(\varphi \circ \eta_C) = \alpha \circ T\varphi \circ T\eta_C = \varphi \circ \mu_C \circ T\eta_C = \varphi$, and also
2. $\varphi \circ \eta_C \circ h = \varphi \circ Th \circ \eta_{TC} = \varphi \circ \mu_C \circ \eta_{TC} = \varphi$.

Moreover, the second property shows that the triangle

$$\begin{tikzcd} (TC, \mu_C) \arrow[r, "h"] \arrow[dr, "\varphi"'] & (C, h) \arrow[d, "\varphi\circ\eta_C"'] \\ & (A, \alpha) \end{tikzcd}$$

commutes. Thus, there is a factorization of φ through h, namely $\varphi \circ \eta_C$.

To see that this factorization is unique, suppose that $\psi : (C, h) \to (A, \alpha)$ is also such that $\varphi = \psi \circ h$ as algebra homomorphisms. Then $\varphi = \psi \circ h$ in $\mathscr{C}$ as well, so that $\varphi \circ \eta_C = \psi \circ h \circ \eta_C = \psi \circ 1_C = \psi$. Thus, $\varphi \circ \eta_C$ is the unique factorization of φ through h. ☐

Thus, every T-algebra has a canonical presentation

$$\begin{tikzcd} TTC \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Th"'] & (TC, \mu_C) \arrow[r, "h"] & (C, h) \end{tikzcd}$$

Moreover, its underlying diagram in $\mathscr{C}$ extends to a split coequalizer diagram.

Proposition 18.5. For any T-algebra $(C, h) \in \mathscr{C}^T$,

$$\begin{tikzcd} TTC \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Th"'] & TC \arrow[r, "h"] \arrow[l, bend left=60, "\eta_{TC}"] & C \arrow[l, bend left=60, "\eta_C"] \end{tikzcd}$$

is a split coequalizer diagram in $\mathscr{C}$.

Proof. Observe that the diagram

$$\begin{tikzcd} TTC \arrow[r, "\eta_{TC}"] \arrow[d, "h"] \arrow[rr, bend left=50, "1_{TC}"] & TTC \arrow[r, "\mu_C"] \arrow[d, "Th"] & TC \arrow[d, "h"] \\ C \arrow[r, "\eta_C"] \arrow[rr, bend right=50, "1_C"'] & B \arrow[r, "h"] & C \end{tikzcd}$$

commutes. ☐

19 BECK'S MONADICITY THEOREM

We first review some basic knowledge.

Suppose that $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, together with $\eta : 1_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$ is an adjunction, then there is an associated monad $(GF, \eta, G\varepsilon F)$ on the category $\mathscr{C}$.

Conversely, if (T, η, μ) is a monad on $\mathscr{C}$, then we can construct the Eilenberg-Moore category $\mathscr{C}^T$ of T-algebra and T-algebra homomorphisms, together with a free-forgetful adjunction $F^T : \mathscr{C} \rightleftarrows \mathscr{C}^T : G^T$.

This free-forgetful adjunction has the property that $(G^T F^T, \eta^T, G^T \varepsilon^T F^T) = (T, \eta, \mu)$.

Thus, if we start with a monad, form the Eilenberg-Moore adjunction, and then extend the assocaited monad, we return to where we started.

On the other hand, if we start with an adjunction $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, pass to the associated monad, and then form the corresponding Eilenberg-Moore adjunction, we do not necesarily recover the original adjunction.

However, the Eilenberg-Moore adjucntion $F^T : \mathscr{C} \rightleftarrows \mathscr{C}^T : G^T$ is terminal in the category $\mathbf{Adj}_T$ of adjunctions that induce the monad (T, η, μ). Thus, there is a unique comparison functor $F : \mathscr{D} \to \mathscr{C}^T$ that maps $F \dashv G$ to $F^T \dashv G^T$.

The (strict) Beck's Monadicity Theorem gives necessary and sufficient conditions for K to be an isomorphism. In such a case, we say that $F \dashv G$ is a strictly monadic adjunction and that G is a strictly monadic functor.

There is also a version for when K is an equivalence, but for ease of exposition, we shall not consider it.

The condition in Beck's Theorem is somewhat technical, so we shall try to motivate it before starting and proving the theorem.

Consider the problem of constructing an inverse to K:

$$\begin{tikzcd}
\mathscr{D} \arrow[rr, bend left, "K"] \arrow[dr, shift left, "G"] & & \mathscr{C}^T \arrow[ll, dashed, bend left, "K^{-1}"] \arrow[dl, shift left, "G^T"] \\
& \mathscr{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^T"] &
\end{tikzcd}$$

The functor $K^{-1} : \mathscr{C}^T \to \mathscr{D}$ must have a number of properties. In particular:

1. it must be an isomorphism,
2. we must have $K^{-1}F^T = F$ and $GK^{-1} = G^T$, and
3. by our earlier work on maps in $\mathbf{Adj}_T$, we must have $K^{-1}\varepsilon^T = \varepsilon K^{-1}$.

Now let $(C, h) \in \mathscr{C}^T$. Then (C, h) has a canonical presentation

$$\begin{tikzcd}
F^T G^T F^T C = F^T TC \arrow[r, shift left, "\varepsilon^T_{(TC,\mu_C)}"] \arrow[r, shift right, "T\varepsilon^T_{(C,h)}"'] & (TC, \mu_C) \arrow[r, "\varepsilon^T_{(C,h)}"] \arrow[d, equal] & (C, h) \\
(TTC, \mu_{TC}) \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Th"'] & (TC, \mu_C) \arrow[r, "h"] & (C, h)
\end{tikzcd}$$

Thus, if K^{-1} exists, then applying it would give a coequalizer $(**)$

$$\begin{tikzcd}
FGFC \arrow[r, shift left, "\varepsilon_{FC}"] \arrow[r, shift right, "Fh"'] & FC \arrow[r, "K^{-1}h"] & K^{-1}(C, h)
\end{tikzcd}$$

It follows that we should define $K^{-1}(C, h)$ as the coequalizer of $\varepsilon_{FC}, Fh : FGFC \to FC$, but there is a problem: coequalizers are not generally unique, but the inverse to a morphism is unique if it exists. Which coequalizer do we choose?

To answer this, let us go back to $(**)$. The coequalizer that describes $K^{-1}(C, h)$ is not arbitrary. If we apply $G : \mathscr{D} \to \mathscr{C}$ to it, we get $(* * *)$

$$\begin{tikzcd}
TTC \arrow[r, shift left, "\mu_C"] \arrow[r, shift right, "Th"'] & (TC, \mu_C) \arrow[r, "h"] & (C, h)
\end{tikzcd}$$

so that $K^{-1}h : FC \to K^{-1}(C, h)$ lifts the (split) coequalizer $h : TC \to C$.

Thus, in order to pin down $K^{-1}(C, h)$, it will be enough to require such lifts to be unique. This motivates the next definition.

Definition 19.1 (Strictly Creates Coequalizers)**.** A functor $G : \mathscr{D} \to \mathscr{C}$ strictly creates coequalizers for a parallel pair $f, g : D \to E$ in $\mathscr{D}$ when to each coequalizer $q : GE \to Q$ of $Gf, Gg : GD \to GD$ in $\mathscr{C}$, there is a unique $\tilde{Q}$ and a unique arrow $\tilde{q} : E \to \tilde{Q}$ with $G\tilde{Q} = Q$ and $G\tilde{q} = q$ and when moreover this unique arrow is a coequalizer of f and g.

Thus, in our construction of K^{-1}, we would like $G : \mathscr{D} \to \mathscr{C}$ to strictly create coequalizers for certain parallel pairs, which brings us to Beck's Theorem.

Theorem 19.2 (Beck's Monadicity Theorem, Strict Version)**.** Let $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$, together with η and ε be an adjunction. Then $(F, G, \eta, \varepsilon)$ is strictly monadic if and only if (†) G strictly creates coequalizers for these parallel pairs $f, g : D \to E$ in $\mathscr{D}$ for which Gf and Gg have a split coequalizer in $\mathscr{C}$.

Before starting the proof, we prove the following lemma.

Lemma 19.3. Suppose $(F : \mathscr{C} \rightleftarrows \mathscr{D} : G, \eta, \varepsilon) \in \mathbf{Adj}_T$ satisfies condition (†). Then $(F, G, \eta, \varepsilon)$ is terminal in $\mathbf{Adj}_T$.

Proof. Suppose $(F', G', \eta', \varepsilon') \in \mathbf{Adj}_T$ as well. We define a morphism

$$\begin{tikzcd}
\mathscr{D}' \arrow[rr, "M"] \arrow[dr, bend left, "G'"] & & \mathscr{D} \arrow[dl, bend left, "G"] \\
& \mathscr{C} \arrow[ul, bend left, "F'"] \arrow[ur, bend left, "F"] &
\end{tikzcd}$$

in $\mathbf{Adj}_T$ as follows. Given any $D' \in \mathscr{D}'$ consider the parallel pair $\varepsilon_{FG'D'}, FG'\varepsilon'_{D'} : GFGG'D' = FG'F'G'D' \to FG'D'$ in $\mathscr{D}$. Apply $\mathscr{D}$, we obtain $\mu_{G'D'}, TG'\varepsilon'_{D'} : T^2G'D' \to TG'D'$ which has a split coequalizer $G'\varepsilon'_{D'} : TG'D' \to G'D'$. Thus, there are unique MD' and $q_{D'} : FG'D' \to MD'$ in $\mathscr{D}$ such that $G(q_{D'} : FG'D' \to MD') = G'\varepsilon'_{D'} : TG'D' \to G'D'$, and moreover, $q_{D'}$ is a coequalizer of $\varepsilon_{FG'D'}$ and $FG'\varepsilon'_{D'}$. This defines M on objects. For morphisms, if $\varphi : D'_1 \to D'_2$ in $\mathscr{D}$, then both left hadn squares in

$$\begin{tikzcd}
FG'F'G'D'_1 \arrow[r, shift left, bend left] \arrow[r, shift right, bend right] \arrow[d, "FG'F'G'\varphi"'] & FG'D'_1 \arrow[r, "q_{D'_1}"] \arrow[d, "FG'\varphi"] & MD'_1 \arrow[d, dashed, "\exists! M\varphi"] \\
FG'F'G'D'_2 \arrow[r, shift left, bend left] \arrow[r, shift right, bend right] & FG'D'_2 \arrow[r, "q_{D'_2}"'] & MD'_2
\end{tikzcd}$$

commute, so there is a unique $M\varphi : MD'_1 \to MD'_2$ such that the right hand square commutes. This defines M as morphisms. Functoriality of M follows from the uniqueness of $M\varphi$. To see that $GM = G'$, note this is true on objects by definition. For morphisms, note that applying G to the diagram above shows that $GM\varphi : G'D'_1 \to G'D'_2$ is the unique morphism such that the square

$$\begin{tikzcd}
TG'D'_1 \arrow[r, "G'\varepsilon'_{D'_1}"] \arrow[d, "TG'\varphi"'] & G'D'_1 \arrow[d, "GM\varphi"] \\
TG'D'_2 \arrow[r, "G'\varepsilon'_{D'_2}"'] & G'D'_2
\end{tikzcd}$$

commutes. Since $G'\varphi$ also has this property, it follows that $GM\varphi = G'\varphi$. Thus, $GM = G'$. Next, we explain why $MF' = F$. On objects, note that for any $C \in \mathscr{C}$, $q_{F'C} : FG'F'C \to MF'C$ is, by definition, the unique lift of $\mu_C : T^C \to TC$. Since the morphism $\varepsilon_{FC} : FGFC \to FC$ has this property, $MF'C = FC$ and $q_{F'C} = \varepsilon_{FC}$. For morphisms, if $\varphi : C_1, C_2$ in $\mathscr{C}$, then $MF'\varphi : FC_1 \to FC_2$ is, by definition, the unique morphism such that the square

$$\begin{tikzcd}
FGFC_1 \arrow[r, "\varepsilon_{FC_1}"] \arrow[d, "FGF\varphi"'] & FC_1 \arrow[d, "MF'\varphi"] \\
FGFC_2 \arrow[r, "\varepsilon_{FC_2}"'] & FC_2
\end{tikzcd}$$

commutes, and $F\varphi$ also has this property. Thus, $MF'\varphi = F\varphi$, so that $MF' = F$. This proves that $M : \mathscr{D}' \to \mathscr{D}$ is a morphism in $\mathbf{Adj}_T$.

Finally, we establish the uniqueness of M. Suppose that $L : \mathscr{D}' \to \mathscr{D}$ is a morphism in $\mathbf{Adj}_T$. For any $D' \in \mathscr{D}'$, consider the morphisms

$$\begin{tikzcd}
F'G'F'G'D' \arrow[r, bend left, "\varepsilon'_{F'G'D'}"] \arrow[r, bend right, "F'G'\varepsilon'_{D'}"'] & F'G'D' \arrow[r, "\varepsilon'_{D'}"] & D'
\end{tikzcd}$$

Applying L gives

$$\begin{tikzcd}
FG'F'G'D' \arrow[r, bend left, "\varepsilon'_{FG'D'}"] \arrow[r, bend right, "FG'\varepsilon'_{D'}"'] & FG'D' \arrow[r, "L\varepsilon'_{D'}"] & LD'
\end{tikzcd}$$

because $LF' = F$ and $L\varepsilon' = \varepsilon L$. Applying G to these new morphisms gives

$$\begin{tikzcd}
T^2G'D' \arrow[r, shift left, "\mu_{G'D'}"] \arrow[r, shift right, "TG'\varepsilon'_{D'}"'] & TG'D' \arrow[r, "G'\varepsilon'_{D'}"] \arrow[l, dashed, bend left=60, "\eta'_{TG'D'}"] & G'D' \arrow[l, dashed, bend left=60, "\eta'_{G'D'}"]
\end{tikzcd}$$

Since G satisfies (†), it follows that LD' and $L\varepsilon'_{D'} : FG'D' \to LD'$ are the unique object and morphism lifting $G'D'$ and $G'\varepsilon'_{D'}$, respectively. Thus, $LD' = MD'$, and $L\varepsilon'_{D'} = q_{D'}$ is a coequalizer. Now suppose $\varphi : D'_1 \to D'_2$ in $\mathscr{D}'$. Then the square

$$\begin{tikzcd}
F'G'D'_1 \arrow[r, "\varepsilon'_{D'_1}"] \arrow[d, "F'G'\varphi"'] & D'_1 \arrow[d, "\varphi"] \\
F'G'D_2 \arrow[r, "\varepsilon'_{D'_2}"'] & D'_2
\end{tikzcd}$$

commutes. Applying L, we obtain a commutative square

$$\begin{tikzcd}
FG'D'_1 \arrow[r, "q_{D'_1}"] \arrow[d, "FG'\varphi"'] & MD'_1 \arrow[d, "L\varphi"] \\
FG'D_2 \arrow[r, "q_{D'_2}"'] & MD'_2
\end{tikzcd}$$

so that $L\varphi \circ q_{D'_1} = q_{D'_2} \circ FG'\varphi = M\varphi \circ q_{D'_1}$. Since $q_{D'_1}$ is a coequalizer, it is an epimorphism, and hence $L\varphi = M\varphi$. This proves that $L = M$, so M is unique. ☐

We now prove the "if" direction in Beck's Monadicity Theorem.

Proof. Suppose that $(F : \mathscr{C} \rightleftarrows \mathscr{D} : G, \eta, \varepsilon)$ is an adjunction and $(T, \eta, \mu) = (GF, \eta, G\varepsilon F)$ is its associated monad. Suppose further that G satisfies (†). Then $(F, G, \eta, \varepsilon)$ is terminal in $\mathbf{Adj}_T$ by the lemma. However, we also know that the Eilenberg-Moore adjunction $(F^T, G^T, \eta^T, \varepsilon^T)$ is terminal. Thus the unique comparison $K : (F, G, \eta, \varepsilon) \to (F^T, G^T, \eta^T, \varepsilon^T)$ is an isomorphism, i.e. $(F, G, \eta, \varepsilon)$ is strictly monadic. ☐

Now we consider the "only if" direction.

Proof. Suppose that $(F : \mathscr{C} \rightleftarrows \mathscr{D} : G, eta, \varepsilon)$ is a strictly monadic adjunction. We wish to show that G satsifies (†). Since the comparison $K : (F, G, \eta, \varepsilon) \to (F^T, G^T, \eta^T, \varepsilon^T)$ to the Eilenberg-Moore adjunction is an isomorphism, it will be enough to show that $G^T : \mathscr{C}^T \to \mathscr{C}$ strictly creates coequalizers for these pairs $f, g : (C, h) \to (D, k)$ in $\mathscr{C}^T$ such that $f, g : \mathscr{C} \to \mathscr{D}$ has a split coequalizer in $\mathscr{C}$.

So, suppose $f, g : (C, h) \to (D, k)$ in $\mathscr{C}^T$ are such that the underlying map $f, g : C \to D$ has a split coequalizer in $\mathscr{C}$. Then every coequalizer of f and g is split, and hence absolute. Now let $q : D \to Q$ be a coequalizer of f and g. We start by lifting Q and q back to $\mathscr{C}^T$. Since $C \overset{f}{\underset{g}{\rightrightarrows}} D \xrightarrow{q} Q$ is an absolute coequalizer, $TC \overset{Tf}{\underset{g}{\rightrightarrows}} TD \xrightarrow{Tq} TQ$ is a coequalizer as well. Consider the diagram below:

$$\begin{tikzcd}
TC \arrow[r, shift left, "Tf"] \arrow[r, shift right, "g"'] \arrow[d, "h"'] & TD \arrow[r, "Tq"] \arrow[d, "k"] & TQ \arrow[d, dashed, "\exists! l"] \\
C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D \arrow[r, "q"] & Q
\end{tikzcd}$$

Since the upper and lower hand square commute, there is a unique $l : TQ \to Q$ such that the right hand square commutes. Assuming that (Q, l) is a T-algebra, we would have (Q, l), $q : (D, k) \to (Q, l)$ in $\mathscr{C}^T$ lifting Q and q. So we need to prove (Q, l) is a T-algebra. Consider the diagram

$$\begin{tikzcd}
Q \arrow[rrr, "\eta Q"] \arrow[dddrrr, bend right, "1_Q"'] & & & TQ \arrow[ddd, "l"] \\
& D \arrow[ul, "q"] \arrow[r, "\eta_D"] \arrow[dr, "1_D"'] & TD \arrow[ur, "Tq"] \arrow[d, "k"] & \\
& & D \arrow[dr, "q"] & \\
& & & Q
\end{tikzcd}$$

Diagram chasing shows that $l \circ \eta_Q \circ q = 1_Q \circ q$, and since q is a coequalizer, it is an epimorphism. Thus, $l \circ \eta_Q = 1_Q$. Next, consider the diagram

$$\begin{tikzcd}
T^2Q \arrow[rrr, "Tl"] \arrow[ddd, "\mu_Q"'] & & & TQ \arrow[ddd, "l"] \\
& T^2D \arrow[ul, "T^2q"] \arrow[r, "Tk"] \arrow[d, "\mu_D"] & TD \arrow[ur, "Tq"'] \arrow[d, "k"] & \\
& TD \arrow[r, "k"] \arrow[dl, "Tq"] & D \arrow[dr, "q"] & \\
TQ \arrow[rrr, "l"'] & & & Q
\end{tikzcd}$$

Then diagram chasing shows $l \circ Tl \circ T^2q = l \circ \mu_Q \circ T^2q$, but q is an absolute coequalizer, so T^2q is a coequalizer, and hence an epimorphism. Thus, $l \circ Tl = l \circ \mu_Q$. These two diagrams prove that (Q, l) is a T-algebra. Thus, (Q, l) and $q : (D, k) \to (Q, l)$ lift Q and $q : D \to Q$ to $\mathscr{C}^T$.

Next, we explain why the lift of Q and q is unique. If (R, m) and $r : (D, k) \to (R, m)$ are any other lift of Q and $q : D \to Q$, then applying G^T shows that $R = Q$ and $r = q$. It remains to show that $m = l$. Since $r = q$ is a T-algebra homomorphism, the square

$$\begin{tikzcd}
TD \arrow[r, "Tq"] \arrow[d, "k"'] & TQ \arrow[d, "m"] \\
D \arrow[r, "q"'] & Q
\end{tikzcd}$$

commutes. Thus, $m \circ Tq = q \circ k = l \circ Tq$, but q is an absolute coequalizer, so Tq is coequalizer and hence an epimorphism. Therefore, $m = l$. Thus, the lift is unique.

Finally, we shall show that $q : (D,k) \to (Q,l)$ is a coequalizer in $\mathscr{C}^T$. First of all, $q \circ f = q \circ g$ in $\mathscr{C}$ because it is a coequalizer of $f, g : C \to D$. Thus, $q \circ f = q \circ g$ in $\mathscr{C}^T$ as well. Now, let $r : (D,k) \to (S,s)$ be any T-algebra homomorphism such that $r \circ f = r \circ g$:

$$\begin{tikzcd}
(C,h) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (D,k) \arrow[r, "q"] \arrow[dr, "r"'] & (Q,l) \\
 & & (S,s)
\end{tikzcd}$$

Then $r \circ f = r \circ g$ in $\mathscr{C}$, so there is a unique morphism $\bar{r} : Q \to S$ such that $\bar{r} \circ q = r$ in $\mathscr{C}$. We claim that $\bar{f}$ is a T-algebra homomorphism $\bar{r} : (Q,l) \to (S,s)$. Indeed, consider the diagram below

$$\begin{tikzcd}
TQ \arrow[rr, "T\bar{r}"] \arrow[dddd, "l"'] & & TS \arrow[dddd, "s"] \\
 & TD \arrow[ul, "Tq"] \arrow[ur, "Tr"'] \arrow[d, "k"] & \\
 & D \arrow[ddl, "q"] \arrow[ddr, "r"] & \\
 & & \\
Q \arrow[rr, "\bar{r}"'] & & S
\end{tikzcd}$$

Diagram chasing shows that $s \circ T\bar{r} \circ Tq = \bar{r} \circ l \circ Tq$, but Tq is epimorphism by the same argument as before. Thus, $s \circ T\bar{r} = \bar{r} \circ l$, i.e. $\bar{r} : (Q,l) \to (S,s)$ is a T-algebra homomorphism that factors $r : (D,k) \to (S,s)$ through $q : (D,k) \to (Q,l)$. The morphism $\bar{r}$ is unique with this property because if $\bar{r}' : (Q,l) \to (S,s)$ and $\bar{r}' \circ q = r$ in $\mathscr{C}^T$, then applying G^T gives $F' \circ q = r = \bar{r} \circ q$ in $\mathscr{C}$, and hence $\bar{r} = \bar{r}'$ because q is an epimorphism. This shows that the map $q : (D,k) \to (Q,l)$ is a coequalizer of $f, g : (C,h) \to (D,k)$.

In total, we have shown that $G^T : \mathscr{C}^T \to \mathscr{C}$ strictly creates coequalizers for those pairs $f, g : (C,h) \to (D,k)$ in $\mathscr{C}^T$ such that $f, g : C \to D$ has a split coequalizer in $\mathscr{C}$. ☐

This complete our proof of Beck's Monadicity Theorem. We conclude by showing that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ from the category ofmonoids to the category of sets is strictly monadic.

We proved this directly in a previous note, but the point is to see how Beck's Monadicity Theorem works in practice.

Example 19.4. Let $U : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U$ be the free-forgetful adjunction, and suppose $f, g : (X, \cdot, e) \to (Y, \cdot, et)$ are monoid homomorphisms such that $f, g : X \to Y$ has a split coequalizer in $\mathbf{Set}$. We must show that U strictly creates coequalizers for $f, g : (X, \cdot, e) \to (Y, \cdot, e)$.

So suppose $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] & Q \end{tikzcd}$ is a coequalizer in $\mathbf{Set}$. Since $f, g : X \to Y$ have a split coequalizer, it follows this coequalizer is also split, and hence absolute. We must show that there is a unique lift of the map q to a monoid homomorphism, and that this lift is a coequalizer. Define $e_Q \in Q$ by $q(e_Y)$. Now consider the diagram below:

$$\begin{tikzcd}
X \times X \arrow[r, shift left, "f \times f"] \arrow[r, shift right, "g \times g"'] \arrow[d, "\cdot"'] & Y \times Y \arrow[r, "q \times q"] \arrow[d, "\cdot"] & Q \times Q \arrow[d, dashed, "\exists! \cdot"'] \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"'] & Q
\end{tikzcd}$$

Since q is an sbsolute coequalizer, the top row is also a coequalizer, and then since both the upper and lower left hand squares commute, it follows that there is a unique $\cdot : Q \times Q \to Q$ such that the right hand square commutes. Once we show that $(Q, \cdot, e)$ is a monoid, we will have produced a lift of $q : Y \to Q$ to $\mathbf{Mon}$. To that send, we check associativity and unitality. For associativity, consider the commutative diagram below:

$$\begin{tikzcd}
Q \times Q \times Q \arrow[rrr, "\cdot_Q \times 1"] \arrow[ddd, "1 \times \cdot_Q"'] & & & Q \times Q \arrow[ddd, "\cdot_Q"] \\
& Y \times Y \times Y \arrow[ul, "q \times q \times q"] \arrow[r, "\cdot_Y \times 1"] \arrow[d, "1 \times \cdot_Y"] & Y \times Y \arrow[ur, "q \times q"] \arrow[d, "\cdot_Y"] & \\
& Y \times Y \arrow[r, "\cdot_Y"'] \arrow[dl, "q \times q"] & Y \arrow[dr, "q"] & \\
TQ \arrow[rrr, "\cdot_Q"'] & & & Q
\end{tikzcd}$$

Diagram chasing shows that $\cdot_Q \circ (\cdot_Q \times 1) \circ q \times q \times q = \cdot_Q \circ (1 \times \cdot_Q) \cdot q \times q \times q$, and then since q is an absolute coequalizer, $q \times q \times q$ is a coequalizer, and hence an epimorphism. Thus, $\cdot_Q \circ (\cdot_Q \times 1) = \cdot_Q \circ (1 \times \cdot_Q)$, i.e. $\cdot_Q$ is associative. The verifications of the left and right unit axioms are similar. Thus, $(Q, \cdot, e)$ is a monad, and we have found a lift of $q : Y \to Q$.

To see that the lift is unique, note that if $r : (Y, \cdot, e) \to (R, \cdot, e)$ is another lift, then applying U shows that $r : Y \to R = q : Y \to Q$, so that $R = Q$ and $r = q$. It remains to check that the rest of the structure on $(R, \cdot_R, e_R) = (Q, \cdot_R, e_R)$. Given that $r = q$ is a monoid homomorphism, we must have $e_R = q(e_N) = e_Q$, and the diagram

$$\begin{tikzcd}
Y \times Y \arrow[r, "q \times q"] \arrow[d, "\cdot_Y"'] & Q \times Q \arrow[d, "\cdot_R"] \\
Y \arrow[r, "q"'] & Q
\end{tikzcd}$$

must commute. Thus, $\cdot_R \circ q \times q = q \circ \cdot_Y = \cdot_Q \circ q \times q$, and since $q \times q$ is an epimorphism, $\cdot_R = \cdot_Q$. Thus, $(R, \cdot, e) = (Q, \cdot, e)$, so that the lift of $q : Y \to Q$ to **Mon** is unique.

Finally, we must check that $\begin{tikzcd} (X, \cdot, e) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & (Y, \cdot, e) \arrow[r, "q"] & (Q, \cdot, e) \end{tikzcd}$ is a coequalizer in **Mon**. We have $q \circ f = q \circ g$ because q is a coequalizer of f and g in **Set**. Next, suppose that $t : (Y, \cdot, e) \to (T, \cdot, e)$ is a monoid homomorphism such that $t \circ f = t \circ g$. Then there is a unique set msp $\bar{t} : Q \to T$ such that $t = \bar{t} \circ q$:

$$\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "t"'] & Q \arrow[d, dashed, "\exists ! \bar{t}"'] \\
& & T
\end{tikzcd}$$

We claim that $\bar{t} : (Q, \cdot, e) \to (T, \cdot e)$ is a monoid homomorphism. To see that $\bar{t}$ preserves $\cdot$, consider the diagram below:

$$\begin{tikzcd}
Q \times Q \arrow[rr, "\bar{t} \times \bar{t}"] \arrow[ddd, "\cdot"'] & & T \times T \arrow[ddd, "\cdot"] \\
& Y \times Y \arrow[ul, "q \times q"] \arrow[ur, "t \times t"'] \arrow[d, "\cdot"] & \\
& Y \arrow[dl, "q"] \arrow[dr, "t"] & \\
Q \arrow[rr, "\bar{t}"'] & & T
\end{tikzcd}$$

We have that $\cdot_T \circ \bar{t} \times \bar{t} \circ q \times q = \bar{t} \circ \cdot_Q \circ q \times q$, and since $q \times q$ is an epimorphism, it follows $\cdot_T \circ \bar{t} \times b\bar{t} = \bar{t} \circ \cdot_Q$. Thus, $\bar{t}$ preserves multiplication. For the unit, note that $\bar{t}(e_Q) = \bar{t}(q(e_Y)) = t(e_Y) = e_T$. Thus, $\bar{t} : (Q, \cdot, e) \to (T, \cdot, e)$ is a monoid homomorphism that factors $t : (Y, \cdot, e) \to (T, \cdot, e)$ through $q : (Y, \cdot, e) \to (Q, \cdot, e)$. To see that $\bar{t}$ is unique, note that if $\bar{t}'$ is another factorization, then $\bar{t} \circ q = t = t \circ q$ in **Set**, so that $\bar{t} = t$ as functions. Thus, $\bar{t}$ is the unique factorization, and

$$\begin{tikzcd}
(X,\cdot,e) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & (Y,\cdot,e) \arrow[r, "q"] & (Q,\cdot,e)
\end{tikzcd}$$

is a coequalizer. This shows that $U : \mathbf{Mon} \to \mathbf{Set}$ strictly creates coequalizers for $f, g : (X,\cdot,e) \to (Y,\cdot,e)$, so by Beck's Monadicity Theorem, the free-forgetful adjunction $M : \mathbf{Set} \rightleftarrows \mathbf{Mon} : U$ is strictly monadic.

20 Kan Extensions

In this note, we shall introduce Kan extensions, but by way of motivation, let us consider the following problem.

Suppose $\mathscr{C}$ is a locally small category, and let $y : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}^{\text{op}}}$ be the Yoneda Embedding.

By the Yoneda Lemma, y is full and faithful. Thus, $\mathscr{C}$ is equivalent to the full subcategory of $\mathbf{Set}^{\mathscr{C}^{\text{op}}}$ whose objects are the hom functors $\mathscr{C}(-, C)$. Accordingly, we shall identify $\mathscr{C}$ with this subcategory and $C \in \mathscr{C}$ with $\mathscr{C}(-, C) \in \mathbf{Set}^{\mathscr{C}^{\text{op}}}$.

Now suppose that $F : \mathscr{C} \to \mathscr{E}$ is a functor. A question that comes up is: is there a natural way of extending F to $\mathbf{Set}^{\mathscr{C}^{\text{op}}}$?

To answer this question, let us examine F to $\mathbf{Set}^{\mathscr{C}^{\text{op}}}$ more closely.

Let $P \in \mathbf{Set}^{\mathscr{C}^{\text{op}}}$. By the Yoneda Lemma, there are natural bijections $\mathbf{Set}^{\mathscr{C}^{\text{op}}}(\mathscr{C}(-, C), P) \cong PC$ given by evaluating a natural transformation $\eta : \mathscr{C}(-, C) \Rightarrow P$ at $1_C \in \mathscr{C}(C, C)$. It follows that the structure of $P : \mathscr{C}^{\text{op}} \to \mathbf{Set}$ is completely determined by how hom functors $\mathscr{C}(-, C)$ map into P. We can pick out individual elements $X \in PC$ using natural transformations $\eta : \mathscr{C}(-, C) \Rightarrow P$, and we can encode the action of $P(f : D \to C)$ by looking at how f_* precomposes with such η.

Thus, it makes sense to ask whether we can somehow recover P from the hom functor $\mathscr{C}(-, C)$, thought of as representing elements of P, and morphisms between the $\mathscr{C}(-, C)$, encoding the action of P as its elements.

The answer is yes, in the sense that P is canonically a colimit of the hom functors $\mathscr{C}(-, C)$. We make this precise.

Consider all of the morphisms $\eta : \mathscr{C}(-, C) \Rightarrow P \in \mathbf{Set}^{\mathscr{C}^{\text{op}}}$. These are parenthesized by objects of the comma category $Y \downarrow P$, i.e. pairs $(C \in \mathscr{C}, \eta : (\mathscr{C}(-, C) \Rightarrow P)$, and moreover, for any $\varphi : (C, \eta) \to (D, \theta)$ in $Y \downarrow P$, the diagram

$$\begin{tikzcd}
\mathscr{C}(-, C) \arrow[rr, Rightarrow, "\varphi_*"] \arrow[dr, Rightarrow, "\eta"'] & & \mathscr{C}(-, D) \arrow[dl, Rightarrow, "\theta"] \\
& P &
\end{tikzcd}$$

commutes (by definition).

Thus, if D is the composite diagram

$$\begin{gathered}
Y \downarrow P \xrightarrow{\pi^P} \mathscr{C} \xrightarrow{y} \mathbf{Set}^{\mathscr{C}^{\text{op}}} \\
(C, \eta) \mapsto C \mapsto \mathscr{C}(-, C) \\
\varphi : (C, \eta) \to (D, \theta) \mapsto \varphi : C \to D \mapsto \varphi_* : \mathscr{C}(-, C) \Rightarrow \mathscr{D}(-, D)
\end{gathered}$$

then the tuple $(\lambda_{(C,\eta)} = \eta : \mathscr{C}(-, C) \Rightarrow P)_{(C,\eta) \in Y \downarrow P}$ is a cocone under D with vertex P. The interesting thing is that it is a colimit.

Theorem 20.1 (Density Theorem)**.** Let $\mathscr{C}$ be a locally small category, $Y : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}^{\text{op}}}$ be the covariant Yoneda Embedding, and $P \in \mathbf{Set}^{\mathscr{C}^{\text{op}}}$. Then $P \cong \mathbf{colim}_{(C,\eta) \in Y \downarrow P} \mathscr{C}(-, C) = \mathbf{colim}(Y \downarrow P \xrightarrow{\pi^P} \mathscr{C} \xrightarrow{y} \mathbf{Set}^{\mathscr{C}^{\text{op}}}$, with $\lambda = (\lambda_{(C,\eta)} = \eta : \mathscr{C}(-, C) \Rightarrow P)_{(C,\eta) \in Y \downarrow P}$ being a colimiting cocone.

Proof. As explained above, $\lambda : Y \circ \pi^P \Rightarrow P$ is a cocone, by the definition of $Y \downarrow P$. Then, given any $T \in \mathbf{Set}^{\mathscr{C}^{\text{op}}}$ and $\tau : y \circ \pi^P \Rightarrow T$ a cocone, there is a unique natural transformation $t : P \Rightarrow T$ such that the triangle

$$\begin{tikzcd}
& \mathscr{C}(-, C) \arrow[dl, Rightarrow, "\eta = \lambda_{(C,\eta)}"'] \arrow[dr, Rightarrow, "\tau_{(C,\eta)}"] & \\
P \arrow[rr, Rightarrow, "t"'] & & T
\end{tikzcd}$$

commutes for all $(C, \eta) \in Y \downarrow P$, namely $t_C : PC \to TC$ by sending $x \mapsto (\tau_{(C,\eta(x))})_C(1_C)$, where $\eta(x) : \mathscr{C}(-, C) \Rightarrow P$ is the unique natural transformation that sends $1_C \in \mathscr{C}(C, C)$ to $x \in PC$. □

Now let us return to the problem of extending $F : \mathscr{C} \to \mathscr{E}$ to a functor on all of $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$.

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, hook] & \mathscr{E} \\
\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}} \arrow[ru, dashed, "E"'] &
\end{tikzcd}$$

For E to be an extension, i.e. for $E \circ y = F$ to be true, we must have $E(\mathscr{C}(-, C)) = E(yC) = FC$.

Next, given an arbitrary $P \in \mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$, we know that $P = \mathbf{colim}_{(C,\eta)\in Y\downarrow P}\mathscr{C}(-, C)$, so one sensible thing to do is to extend E to be cocontinuously, i.e. to send the colimit to a colimit $E(P) = E(\mathbf{colim}_{(C,\eta)\in Y\downarrow P}(\mathscr{C}(-, C)) := \mathbf{colim}_{(C,\eta)\in Y\downarrow P}FC$, where the right-hand term is $\mathbf{colim}(Y \downarrow P] \xrightarrow{\pi^P} \mathscr{C} \xrightarrow{F} \mathscr{E}$.

This is a natural candidate for an extension of $F : \mathscr{C} \to \mathscr{E}$ along $y : \mathscr{C} \to \mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$. We shall soon prove that E is an extension of F (up to natural isomorphism, but first we generalize.

Suppose that

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathscr{E} \\
\mathscr{D} &
\end{tikzcd}$$

are any functors whatsoever. Then given any $D \in \mathscr{D}$, we can approximate D "from the left" relative to K by considering all maps $f : KC \to D$. They are parametrized by objects $(C, f) \in K \downarrow D$, and just as before, the object D is the vertex of a canonical cocone under the diagram $K \downarrow D \xrightarrow{\pi^D} \mathscr{C} \xrightarrow{K} \mathscr{D}$, given by $\lambda_{(C,f)} = f : KC \to D$.

Now, regardless of whether $D = \mathbf{colim}_{(C,f)\in K\downarrow D}KC$ (or even whether this colimit exists), we can still "try to extend by cocontinuity" as above, and define $L(D) := \mathbf{colim}_{(C,f)\in K\downarrow D}FC = \mathbf{colim}(K \downarrow D \xrightarrow{\pi^D} \mathscr{C} \xrightarrow{F} \mathscr{E}$, provided that these colimits exist. If they do, then we can make L into a functor as follows:

1. for each $D \in \mathscr{D}$, choose a colimit $(LD, \lambda^D : F\pi^D \Rightarrow LD)$ of the diagram $F \circ \pi^D$, and then

2. for each $g : D \to D'$ in $\mathscr{D}$, note that there is a functor $g_* : K \downarrow D \to K \downarrow D'$, given by post-composition with g, such that

$$\begin{tikzcd}
K \downarrow D \arrow[rr, "g_*"] \arrow[rd, "\pi^D"'] & & K \downarrow D' \arrow[ld, "\pi^{D'}"] \\
& \mathscr{C} &
\end{tikzcd}$$

commutes. Thus, any cocone under a diagram indexed by $K \downarrow D'$ can be whiskered to a cocone under a diagram indexed by $K \downarrow D$. In particular, the colimit cocone $\lambda^{D'} : F\pi^{D'} \Rightarrow LD'$ whiskers to a cocone $\lambda^{D'}_{g_*} : F\pi^D \Rightarrow LD'$, so by the universal property of $\lambda^D : F\pi^D \Rightarrow LD$, there is a unique morphism $Lg : LD \to LD'$ such that

$$\begin{tikzcd}
& FC \arrow[ld, "\lambda_{(C,f)^D}"'] \arrow[rd, "\lambda^{D'}_{(C,gf)}"] & \\
LD \arrow[rr, "Lg"'] & & LD'
\end{tikzcd}$$

commutes for all $(C, f) \in K \downarrow D$.

Furthermore, there is a natural transformation $\eta : F \Rightarrow LK : \mathscr{C} \to \mathscr{E}$ whose components are

3. $\eta_C = \lambda^{KC}_{(C,1_{KC})} : FC \to LKC$.

As alluded to above, one can ask whether the functor L is an extension of F along K, and we have the following proposition.

Proposition 20.2. Suppose that

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathscr{E} \\
\mathscr{D} &
\end{tikzcd}$$

are functors, that K is fully faithful, and that for all $D \in \mathscr{D}$, the colimits $\mathbf{colim}(K \downarrow D \xrightarrow{\pi^D} \mathscr{C} \xrightarrow{F} \mathscr{E})$ exist. Let L and η be as described above. Then η is a natural isomorphism.

Proof. For any $C \in \mathscr{C}$, the category $K \downarrow KC$ has a terminal object, namely $(C, 1_{KC})$, because K is fully faithful. Now consider the diagram $K \downarrow KC \xrightarrow{\pi^{KC}} \mathscr{C} \xrightarrow{F} \mathscr{E}$. Then, since $(C, 1_{KC}) \in K \downarrow KC$ is terminal, the tuple $(F\pi^{KC}(C, 1_{KC}), (F\pi^{KC}(! : (D,g) \to (C, 1_{KC})))_{(D,g) \in K \downarrow KC}$ is a colimit cocone under $F\pi^{KC}$. However, we also have a colimit $(LKC, \lambda^{KC} : F\pi^{KC} \Rightarrow LKC)$, so there is a unique isomorphism $\varphi : FC \to LKC$ such that

$$\begin{tikzcd}
 & FD \arrow[dl, "F\pi^{KC}(!:(D,g)\to(C,1_{KC}))"'] \arrow[dr, "\lambda^{KC}_{(D,g)}"] & \\
FC \arrow[rr, "\varphi"'] & & LKC
\end{tikzcd}$$

commutes for all $(D,g) \in K \downarrow KC$. Taking $(D,g) = (C, 1_{KC})$ shows that $\varphi = \varphi \circ 1_{FC} = \lambda^{KC}_{(C,1_{KC})} = \eta C$, so that ηC is an isomorphism for all $C \in \mathscr{C}$. ☐

Thus, our ”L construction” is an extension (up to natural isomorphism) when we are extending along a fully faithful functor. In general, however, this is not the case: indeed there are examples when no true extension can exist.

Nonetheless, if the colimits $\mathbf{colim}(K \downarrow D \xrightarrow{\pi^D} \mathscr{C} \xrightarrow{F} \mathscr{E})$ all exist, then we can still construct (L, η), and happily, this pair always has a universal property.

Definition 20.3 (Left Kan Extension)**.** Suppose that

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathscr{E} \\
\mathscr{D} &
\end{tikzcd}$$

are functors. A left Kan extension of F along K is a functor $\mathbf{Lan}_K F : \mathscr{D} \to \mathscr{E}$, together with a natural transformation $\eta : F \Rightarrow \mathbf{Lan}_K F \circ K$ such that for any functor $G : \mathscr{D} \to \mathscr{E}$, together with a natural transformation $\gamma : F \Rightarrow GK$, there is a unique natural transformation $\bar{\gamma} : \mathbf{Lan}_K F \Rightarrow G$ such that $\gamma = \bar{\gamma} K \circ \eta$:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathscr{E} \\
\mathscr{D} \arrow[ur, "G"'] &
\end{tikzcd}$$

being equivalent to

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathscr{E} \\
\mathscr{D} \arrow[ur, "G"] \arrow[ur, bend right=60, "\mathbf{Lan}_K F"'] &
\end{tikzcd}$$

Theorem 20.4. Suppose that $F : \mathscr{C} \to \mathscr{E}$ and $K : \mathscr{C} \to \mathscr{D}$ are functors, and that for any $D \in \mathscr{D}$, the colimit $\mathbf{colim}(K \downarrow D \xrightarrow{\pi^D} \mathscr{C} \xrightarrow{F} \mathscr{E})$ exists. Then the functor L and natural transforamtion $\eta : F \Rightarrow LK$ defined above are a left Kan extension of F along K.

Proof. One must check that (L, η) has the necessary universal property. Suppose that $G : \mathscr{D} \to \mathscr{E}$ is a functor and $\gamma : F \Rightarrow GK$ is a natural transformation. Given any $D \in \mathscr{D}$, we need a morphism $\bar{\gamma}_D : LD \Rightarrow GD$ that is natural in D. Equivalently, if $(LD, \lambda^D : F\pi^D \Rightarrow LD)$ is the colimit used to define LD, then we need a cocone $F\pi^D \Rightarrow GD$. For each object $(C \in \mathscr{C}, f : KC \to D) \in K \downarrow D$, consider the morphism $FC \xrightarrow{\gamma_C} GKC \xrightarrow{Gf} GD$. The morphism $(Gf \circ \gamma_C)_{(C,f) \in K \downarrow D}$ define a cocone $F\pi^D \Rightarrow GD$, so by the universal property of λ^D, there is a unique morphism $\bar{\gamma}_D : LD \to GD$, such that

$$\begin{tikzcd}
FC \arrow[r, "\gamma_C"] \arrow[d, "\lambda^D_{(C,f)}"'] & GKC \arrow[d, "Gf"] \\
LD \arrow[r, dashed, "\bar{\gamma}_D"'] & GD
\end{tikzcd}$$

commutes for all $(C, f) \in K \downarrow D$. One can check that $\bar{\gamma}$ is the unique natural transformation $L \Rightarrow G$ such that $\gamma = \bar{\gamma}K \circ \eta$. ☐

Corollary 20.5. Suppose $\mathscr{C}$ is small, $\mathscr{D}$ is locally small, and $\mathscr{E}$ is cocomplete. Then, given any functors

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathscr{E} \\
\mathscr{D} \arrow[ur, "G"'] &
\end{tikzcd}$$

there is a left Kan extension $(\mathbb{L}\text{⅁}\ltimes_K F, \eta)$ of along K, and it can constructed using the colimit formula above.

Proof. For any $D \in \mathscr{D}$, the category $K \downarrow D$ is small, and therefore the colimits $\mathbf{colim}(K \downarrow D \xrightarrow{\pi^D} \mathscr{C} \xrightarrow{F} \mathscr{E})$ all exist. ☐

So far, we have only focused on left Kan extension, but as with thing categorized, there is a dual story.

We briefly indicate how this works.

Definition 20.6 (Right Kan Extension)**.** Suppose $F : \mathscr{C} \to \mathscr{D}$ and $K : \mathscr{D} \to \mathscr{E}$ are functors. A right Kan extension of F along K is a functor $\mathbf{Ran}_K F : \mathscr{D} \to \mathscr{E}$, together with a natural transformation $\varepsilon : \mathbf{Ran}_K F \circ K \Rightarrow F$ such that for any functor $G : \mathscr{D} \to \mathscr{E}$ together with a natural transformation $\delta : GK \Rightarrow F$, there is a unique natural transformation $\bar{\delta} : G \Rightarrow \mathbf{Ran}_K F$ such that $\delta = \varepsilon \circ \bar{\delta}K$:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathscr{E} \\
\mathscr{D} \arrow[ur, "G"'] &
\end{tikzcd}$$

being equivalent to

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathscr{E} \\
\mathscr{D} \arrow[ur, bend left=40, "G"] \arrow[ur, "\mathbf{Ran}_K F"'] &
\end{tikzcd}$$

Given that left Kan extensions can be constructed using colimits, right Kan extensions can be constructed using limits. We spell this out.

Suppose $F : \mathscr{C} \to \mathscr{E}$ and $K : \mathscr{C} \to \mathscr{D}$ are functors, and for each $D \in \mathscr{D}$, let $\pi_D : D \downarrow K \to \mathscr{C}$ be the canonical projection functor. Suppose further that for each $D \in \mathscr{D}$, the diagram $F\pi_D : D \downarrow K \to \mathscr{C} \to \mathscr{E}$ has a limit. We construct a functor $R : \mathscr{D} \to \mathscr{E}$ and a natural transformation $\varepsilon : RK \Rightarrow F$ as follows:

1. For each $D \in \mathscr{D}$, choose a limit $(RD, \lambda^D : RD \Rightarrow F\pi_D)$,
2. For each $g : D \to D'$ in $\mathscr{D}$, note that the tuple $(\lambda^D_{(C,f'g)} : RD \to FC)_{(C,f')\in D'\downarrow K}$ is a cone over the diagram $F\pi_{D'} : D' \downarrow K \to \varepsilon$. Thus, there exists a unique morphism $Rq : RD \to RD'$ such that the triangle

$$\begin{tikzcd}
RD \arrow[rr, "Rg"] \arrow[dr, "\lambda^D_{(C,f'g)}"'] & & RD' \arrow[dl, "\lambda^{D'}_{(C,f')}"] \\
& FC &
\end{tikzcd}$$

commutes for all $(C, f') \in D' \downarrow K$.
3. For each $C \in \mathscr{C}$, we define $\varepsilon_C = \lambda^{KC}_{(C,1_{KC})} : RKC \to FC$.

Theorem 20.7. Suppose that $F : \mathscr{C} \to \mathscr{E}$ and $K : \mathscr{C} \to \mathscr{E}$ are functors and that for any $D \in \mathscr{D}$, the limit $\lim(K \downarrow D \xrightarrow{\pi_D} \mathscr{C} \xrightarrow{F} \varepsilon)$ exists. Then the functor $R : \mathscr{D} \to \mathscr{E}$ and and natural transformation $\varepsilon : RK \Rightarrow F$ defined above are a right Kan extension of F along K. Moreover, if K is fully faithful, then ε is a natural isomorphism.

Corollary 20.8. Suppose $\mathscr{C}$ is small, $\mathscr{D}$ is locally small, and $\mathscr{E}$ is complete. Then, given any functors

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathscr{E} \\
\mathscr{D} \arrow[ur, "G"'] &
\end{tikzcd}$$

there is a right Kan extension $(\mathbf{Ran}_K F, \eta)$ of along K, and it can constructed using the limit formula above.

21 ALL CONCEPTS ARE KAN EXTENSIONS

All of the fundamental categorical concepts that we have considered in these notes can be formulated in terms of Kan extensions.

In what follows, we shall illustrate.

Proposition 21.1 (Yoneda Lemma as Kan Extension)**.** Suppose that $\mathscr{C}$ is a locally small category, $\mathbb{1}$ is the terminal category, and $* : \mathbb{1} \to \mathbf{Set}$ is the functor that sends the objects of $\mathbb{1}$ to the singleton $* \in \mathbf{Set}$. For any $C \in \mathscr{C}$, consider the diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "*"] \arrow[d, "C"'] & \mathbf{Set} \\
\mathscr{C} \arrow[ur, "\mathscr{C}(C,-)"'] &
\end{tikzcd}$$

where $C : \mathbb{1} \to \mathscr{C}$ is the functor the object of $\mathbb{1}$ to $C \in \mathscr{C}$, and $1_C : * \Rightarrow \mathscr{C}(C,-) \circ C$ is the natural transformation where the only component is the function $* \to \mathscr{C}(C,-)$ that picks out 1_C. Then the assertion that the diagram above is a left Kan extension is equivalent to the assertion that $\mathbf{ev}_1 : \mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C,-), F) \to FC$ is a bijection.

Proof. ($\Rightarrow$) Suppose the diagram is a left Kan extension. Then, for any $F : \mathscr{C} \to \mathbf{Set}$, there is a natural bijection $\mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C,-), F) \cong \mathbf{Set}^{\mathbb{1}}(*, F \circ C)$ that sends $(\eta : \mathscr{C}(C,-), F)$ to $\eta C \circ 1_C$.

However, there are also bijections $\mathbf{Set}^{\mathbb{1}}(*, F \circ C) \cong \mathbf{Set}(*, FC) \cong FC$ that sends $\lambda : * \Rightarrow F \circ C$ to $\lambda_\circ : * : FC \mapsto \lambda_\circ(*) \in FC$, where $\circ$ is the single object in $\mathbb{1}$. The composite bijection $\mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C,-), F) \cong FC$ is the evaluation at 1_C.

($\Leftarrow$) Suppose that $\mathbf{ev}_1 : \mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C,-), F) \to FC$ is a bijection. Then the composite $\mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C,-), F) \to \mathbf{Set}^{\mathbb{1}}(*, F \circ C) \cong \mathbf{Set}(*, FC) \cong FC$ that sends $\eta \mapsto \eta C \circ 1_C \mapsto (\eta C \circ 1_C)_\circ \mapsto (\eta C \circ 1_C)_\circ(*) = \mathbf{ev}_1(\eta)$, is a bijection, and hence $\mathbf{Set}^{\mathscr{C}}(\mathscr{C}(C,-), F) \to \mathbf{Set}^{\mathbb{1}}(*, F \circ C)$ that sends $\eta \mapsto \eta C \circ 1_C$ is bijective. This says that the diagram above is a left Kan extension. □

Proposition 21.2 ((Co)Limits as Kan Extensions)**.** Suppose that $F : J \to \mathscr{D}$ is a functor, $\mathbb{1}$ is the terminal category, $! : J \to \mathbb{1}$ is the unique factor, and $D : \mathbb{1} \to \mathscr{D}$ is the functor that sends the object of $\mathbb{1}$ to $D \in \mathscr{D}$. Then:

1. A diagram of the form

$$\begin{tikzcd} J \arrow[r, "F"] \arrow[d, "!"'] & \mathscr{D} \\ \mathbb{1} \arrow[ur, "D"'] & \end{tikzcd}$$

   is a left Kan extension if and only if $(D, \eta : F \Rightarrow D \circ !)$ is a colimit cocone.

2. A diagram of the form

$$\begin{tikzcd} J \arrow[r, "F"] \arrow[d, "!"'] & \mathscr{D} \\ \mathbb{1} \arrow[ur, "D"'] & \end{tikzcd}$$

   is a right Kan extension if and only if $(D, \varepsilon : D \circ ! \Rightarrow F)$ is a limit cone.

Proof. We prove the first statement, and the second statement follows from a dual argument.

($\Rightarrow$) Suppose that the diagram is a left Kan extension. Then for any $T \in \mathscr{D}$, we have a bijection $\mathscr{D}(D, T) \cong \mathscr{D}^{\mathbb{1}}(D, T) \cong \mathscr{D}^{J}(F, T \circ !) = \mathbf{Cocone}(F, T)$ that sends $f \mapsto (f) \mapsto (f \circ \eta_j)_{j \in J}$. This says precisely that $\eta : F \Rightarrow D$ is a colimit cocone.

($\Leftarrow$) Suppose that $\eta : F \Rightarrow D$ is a colimit cocone. Then the composite $\mathscr{D}(D, T) \cong \mathscr{D}^{\mathbb{1}}(D, T) \to \mathscr{D}^{J}(F, T \circ !) = \mathbf{Cocone}(F, T)$ sends $f \mapsto (f) \mapsto (f)! \circ \eta = (f \circ \eta_j)_{j \in J}$ is a bijection. Therefore, $\mathscr{D}^{\mathbb{1}}(D, T) \to \mathscr{D}^{J}(F, T \circ !)$ is a bijection, which says precisely that the diagram is a left Kan extension. ☐

We shall momentarily express adjunctions in terms of Kan extensions, but first, a bit of terminology.

Definition 21.3 (Preserves Kan Extension)**.** Suppose that

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathscr{E} \\ \mathscr{D} \arrow[ur, "L"'] & \end{tikzcd}$$

is a left Kan extension of F along K with $\eta : F \Rightarrow L \circ K$, and that G is a functor with domain in $\mathscr{E}$. We say that G preserves the left Kan extension (L, η) if $(GL, G\eta)$ is a left Kan extension of GF along K.

We say that (L, η) is an absolute left Kan extension if every function with domain in G preserves (L, η).

Proposition 21.4 (Adjunctions as Kan Extensions)**.** Suppose that $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ as functors and that $\eta : 1_{\mathscr{C}} \Rightarrow GF$ is a natural transformation. Then the following are equivalent:

1. $F \dashv G$ with unit η.
2. (G, η) is an absolute left Kan extension of $1_{\mathscr{C}}$ along $F : \mathscr{C} \to \mathscr{D}$.
3. (G, η) is a left Kan extension of $1_{\mathscr{C}}$ along $F : \mathscr{C} \to \mathscr{D}$ that is preserved by F.

Proof. (1) $\Rightarrow$ (2): Suppose that $F \dashv G$ with unit η. Then for any category $\mathscr{E}$, there is a "precomposite adjunction" $G^* : \mathscr{E}^{\mathscr{C}} \rightleftarrows \mathscr{E}^{\mathscr{D}} : F^*$ and $\mathscr{E}^{\mathscr{D}}(HG, L) \cong \mathscr{E}^{\mathscr{C}}(H, LF)$ that sends $(\theta : HG \Rightarrow L) \mapsto (\theta F \circ H\eta : H \Rightarrow HGF \Rightarrow LF)$. Specializing to the case $\mathscr{E} = \mathscr{C}$ and $H = 1_{\mathscr{C}}$, we see that $\mathscr{C}^{\mathscr{D}}(G, L) \cong \mathscr{C}^{\mathscr{C}}(1_{\mathscr{C}}, LF)$ that sends $(\theta : G \Rightarrow L) \mapsto (\theta F \circ \eta : 1_{\mathscr{C}} \Rightarrow GF \Rightarrow LF)$, which says precisely that the pair (G, η) below

$$\begin{tikzcd} \mathscr{C} \arrow[r, "1_{\mathscr{C}}"] \arrow[d, "F"'] & \mathscr{C} \\ \mathscr{D} \arrow[ur, "G"'] & \end{tikzcd}$$

is a left Kan extension of $1_{\mathscr{C}}$ along F, with $\eta : 1_{\mathscr{C}} \Rightarrow G \circ F$. Now let $\mathscr{E}$ and $H : \mathscr{C} \to \mathscr{E}$ be fixed, but arbitrary. Then the bijection $\mathscr{E}^{\mathscr{D}}(HG, L) \cong \mathscr{E}^{\mathscr{C}}(H, LF)$ that sends $(\theta : HG \Rightarrow L) \mapsto (\theta F \circ H\eta : H \Rightarrow HGF \Rightarrow LF)$ says precisely that the pair $(HG, H\eta)$

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "1_{\mathscr{C}}"] \arrow[d, "F"'] & \mathscr{C} \arrow[r, "H"] & \mathscr{E} \\
\mathscr{D} \arrow[ru, "G"'] & &
\end{tikzcd}$$

is the left Kan extension of $H1_{\mathscr{C}}$ along F. Thus, (G, η) is an absolute left Kan extension of $1_{\mathscr{C}}$ along F.

$(2) \Rightarrow (3)$ is obvious.

$(3) \Rightarrow (1)$: Suppose that (G, η) is a left Kan extension of $1_{\mathscr{C}}$ along F that is preserved by F. Then $(FG, F\eta)$ is a left Kan extension of F along F:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "1_{\mathscr{C}}"] \arrow[d, "F"'] & \mathscr{C} \arrow[r, "H"] & \mathscr{D} \\
\mathscr{D} \arrow[ru, "G"'] & &
\end{tikzcd}$$

Consider the pair $(1_{\mathscr{D}}, 1_F : F \circ 1_{\mathscr{C}} \Rightarrow 1_{\mathscr{D}} \circ F)$. By the universal property of $(FG, F\eta)$, there is a unique $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$ such that $\varepsilon F \circ F\eta = 1_F$. This is one of the triangle identities, and our proof will be complete once we know $G\varepsilon \circ \eta G = 1_G$. To do this, recall that (G, η) is a left Kan extension of $1_{\mathscr{C}}$ along F. Thus, there is a bijection $\phi : \mathscr{C}^{\mathscr{D}}(G, G) \cong \mathscr{C}^{\mathscr{C}}(1_{\mathscr{C}}, GF)$ that sends $(\theta : G \Rightarrow G) \mapsto \theta F \circ \eta$.

Now, $\theta = G\varepsilon \circ \eta G : G \Rightarrow GFG \Rightarrow G$, and by the naturality of η, $\theta F \circ \eta = G\varepsilon F \circ \eta GF \circ \eta = G\varepsilon F \circ GF\eta \circ \eta = G(\varepsilon F \circ F\eta) \circ \eta$. We already established that $\varepsilon F \circ F\eta = 1_F$, and therefore this simplifies to η. Thus, $\phi(G\varepsilon \circ \eta G) = (G\varepsilon \circ \eta G)F \circ \eta = \eta = (1_G)F \circ \eta = \phi(1_G)$, and since ϕ is injective, $G\varepsilon \circ \eta G = 1_G$. Thus, $(F, G, \eta, \varepsilon)$ is an adjunction, so $F \dashv G$ with unit η. □

We conclude by explaining the construction of a monad $(GF, \eta, G\varepsilon F)$ from the adjunction $(F, G, \eta, \varepsilon)$ in terms of Kan extensions.

So, suppose $(F : \mathscr{C} \rightleftarrows \mathscr{D} : G, \eta, \varepsilon)$ is an adjunction. By the dual of the previous proposition,

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "1_{\mathscr{D}}"] \arrow[d, "G"'] & \mathscr{D} \\
\mathscr{D} \arrow[ru, "F"'] &
\end{tikzcd}$$

is an absolute right Kan extension for $\varepsilon : FG \Rightarrow 1_{\mathscr{D}}$, so in particular,

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "G"] \arrow[d, "G"'] & \mathscr{C} \\
\mathscr{C} \arrow[ru, "GF"'] &
\end{tikzcd}$$

is a right Kan extension where $G\varepsilon : GF \circ G \Rightarrow G$, i.e. $GF \cong \mathbf{Lan}_G G$.

Now, the identity transformation $1_G : 1_{\mathscr{C}} G \Rightarrow G$ factors uniquely through GF, i.e. there is a unique $\theta : 1_{\mathscr{C}} \Rightarrow GF$ such that $G\varepsilon \circ \theta G = 1_G$, and thus $\theta = \eta$, the unit of the adjunction, by the triangular identity:

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "G"] \arrow[d, "G"'] & \mathscr{C} \\
\mathscr{C} \arrow[ru, "1_{\mathscr{C}}"'] &
\end{tikzcd}$$

with respect to $1_G : 1_{\mathscr{C}} \circ G \Rightarrow G$ is equivalent to

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "G"] \arrow[d, "G"'] & \mathscr{C} \\
\mathscr{C} \arrow[ru, "GF"'] \arrow[ru, bend right=60, "1_{\mathscr{C}}"'] &
\end{tikzcd}$$

with respect to $G\varepsilon \circ \eta : 1_{\mathscr{C}} \Rightarrow GF \Rightarrow G$.

Finally, for $\mu = G\varepsilon F : GFGF \Rightarrow GF$, note that the natural transformation $G\varepsilon \circ GFG\varepsilon : GFGFG \Rightarrow GFG \Rightarrow G$ factors uniquely through GF, i.e. there is a unique $\theta : GFGF \Rightarrow GF$ such that $G\varepsilon \circ \theta G = G\varepsilon \circ GFG\varepsilon$. By the naturality of ε, we have $G\varepsilon \circ G\varepsilon FG = G\varepsilon GFG\varepsilon$, so that $\theta = G\varepsilon F$:

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "G"] \arrow[d, "G"'] \arrow[dr, "G"] & \mathscr{C} \\
\mathscr{C} \arrow[r, "GF"'] & \mathscr{C} \arrow[u, "GF"']
\end{tikzcd}$$

with respect to $G\varepsilon : GF \circ G \Rightarrow G$ is equivalent to

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "G"] \arrow[d, "G"'] & \mathscr{C} \\
\mathscr{C} \arrow[ur, "GF"'] \arrow[ur, bend right, "GFGF"'] &
\end{tikzcd}$$

with respect to $G\varepsilon F : GFGF \to GF$ and $G\varepsilon : GF \circ G \Rightarrow G$.