The "$+ = Q$" Theorem

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Recall that we have talked about a few notions of K-theory:

- K-theory of rings (via Quillen's $+$-construction);
- K-theory of exact categories (via Quillen's Q-construction);
- and very soon there is a notion of K-theory of symmetric monoidal categories (via the $S^{-1}S$-construction).

The natural question to ask would be how these notions relate. For instance, we know given a ring, its category of projective modules gives rise to an exact category, and we have determined that the ring and the category has the same K_0-group. However, the thing that seemed too good to be true is to confirm that these notions coincide completely, and this is the question we will try to answer.

1 Motivation

Given an exact category $\mathcal{A}$, its maximal groupoid $S = i\mathcal{A}$, i.e., with same objects as $\mathcal{A}$, but morphisms are the isomorphisms of $\mathcal{A}$, has a symmetric monoidal structure as a category. Hence, there are three notions of K-theory on $\mathcal{A}$ we are able to consider:

A. given the exact structure, Quillen defined its K-theory using Q-construction in [Qui06], that is, given the K-theory space $\Omega BQ\mathcal{A}$ which is an infinite loopspace, the K-groups are its homotopy groups; (This is also defined to be the K-theory of a scheme.)

A'. since $\mathcal{A}$ is exact, it is additive and therefore has a symmetric monoidal structure with respect to $\oplus$. The classifying space $|NS|$, known as the geometric realization of its nerve, inherits a group structure by taking group completion;

B. given the symmetric monoidal structure S of $\mathcal{A}$, then the K-theory of a symmetric monoidal category is defined by $K_n(\mathcal{A}) = K_n(S) = \pi_n(B(S^{-1}S))$. I will say more about the construction later in Section 2, but the significance is in Theorem 2.11, that $B(S^{-1}S)$ is a group completion[1] of the H-space BS given some conditions on S.

We will first answer the question of why construction B is the "correct" thing to do for symmetric monoidal categories, instead of the seemingly obvious construction A'.[2]

Lemma 1.1. A category $\mathscr{C}$ with initial (respectively, terminal) object has contractible nerve $|N\mathscr{C}|$.

Proof. Recall that the functor

$$\begin{aligned} \mathbf{Cat} &\to \mathbf{Top} \\ \mathscr{C} &\mapsto |N\mathscr{C}| \end{aligned}$$

turns natural transformations $F_0 \Rightarrow F_1$ into homotopies $BF_0 \simeq BF_1$, therefore sends adjoint functors into inverse homotopy equivalences, as the unit and counit of the adjunction become homotopies to the identity. An initial (respectively, terminal) object is a left (respectively, right) adjoint to the unique functor $\mathscr{C} \to \mathbb{1}$ into the terminal category, therefore gives a homotopy equivalence $|N(\mathscr{C})| \simeq |N(\mathbb{1})|$. Since $|N(\mathbb{1})|$ is an one-point space, then $|N(\mathscr{C})|$ is contractible. $\square$

[1] In general, the group completion Y of X is an extension of H-spaces $X \to Y$ such that the group $\pi_0(Y)$ is the completion of the abelian monoid $\pi_0(X)$, and $H_*(Y;R) \cong \pi_0(X)^{-1}H_*(X;R)$ for all commutative ring R. The functor ΩB, where Ω is the loopspace operator and B is the bar construction, i.e., classifying space, is known as the group completion of a topological monoid, per Segal's method.

[2] As commented by Dustin Clausen in [hm], this construction is mostly motivated as the homotopical analog of the usual Grothendieck approach to direct-sum K_0.

Corollary 1.2. Construction A' gives trivial K-groups.

Corollary 1.3. An adjunction between two categories gives rise to a homotopy equivalence between their nerves.

Proof. Suppose $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ is an adjunction $F \dashv G$, then $d\backslash G$ is isomorphic to the comma category $F(d)\backslash\mathscr{C}$, but this is contractible by Lemma 1.1. We conclude the proof by Quillen's theorem A. □

We will now try to say something interesting about construction A and B. Surprisingly, Grayson's [Gra06] proved Theorem 1.4 initially pointed out by Quillen.

Theorem 1.4. If all short exact sequences of $\mathcal{A}$ splits, then Construction A and Construction B are equivalent.

In particular, over these circumstances the notion of K-groups of a ring (given by $+$-construction) coincides with the notion of K-groups of an exact category (given by Q-construction).

Corollary 1.5 **("$+ = Q$").** For a ring R, the category $\mathcal{P}(R)$ of finitely-generated projective R-modules satisfies $\Omega BQ\mathcal{P}(R) \simeq K_0(R) \times \mathrm{BGL}(R)^+$, the latter being the disjoint union of copies of the connected space $BGL(R)^+$, one for each element of $K_0(R)$. With $K_n(R) := \pi_n(K_0(R) \times BGL(R)^+)$ for all $n \geqslant 0$, we have $K_n(R) \cong K_n(\mathcal{P}(R))$ for all $n \geqslant 0$.

2 Details of Construction B

In this section, we will give details to the said Construction B, and show that it is actually a $+$-construction.

2.1 The $S^{-1}S$-construction

Definition 2.1. Let S be an abelian monoid with an action on X. We say S acts invertibly on X if the translation by $s \in S$, i.e., given by left multiplication

$$\begin{aligned} X &\to X \\ x &\mapsto sx \end{aligned}$$

is a bijection.

It is easy to note that, given S acting invertibly on X as above,

Remark 2.2.

- there is a localization $S^{-1}X := (S \times X)/S$ with componentwise action of S on $S \times X$;
- S acts invertibly on $S^{-1}X$ as well;

  Proof. Note that there is an action of S on $S^{-1}X$ by $t \cdot (s, x) := (s, tx)$. This defines a map

  $$\begin{aligned} X &\to S^{-1}X \\ x &\mapsto (1, x) \end{aligned}$$

  The translation defined by $(s, x) \mapsto (s, tx)$ now has an inverse assignment $(s, x) \mapsto (ts, x)$. □

- the map defined above respects the S-action, and is universal with respect to all arrows from X to a set upon which S acting invertibly.

The said universal property can be generalized to groups: the completion/localization $S^{-1}S$ is a group, and the monoid homomorphism $S \to S^{-1}S$ is universal in the sense of groups. We will try to do something similar for symmetric monoidal categories.

Definition 2.3. A left action of a monoidal category S on a category X is a functor $\otimes : S \times X \to X$ with natural isomorphisms $A \otimes (B \otimes F) \cong (A \otimes B) \otimes F$ and $\mathbb{1} \otimes F \cong F$ for all $A, B \in S$ and $F \in X$, as well as respecting the pentagon diagram and unital diagram from the definition of monoidal category.

We say a functor $g : X \to Y$ of categories with S-action preserves the action if there is a natural isomorphism $A \otimes gF \cong g(A \otimes F)$ such that all suitable diagrams commute.

Definition 2.4. Let S be a monoidal category with action on category X, then we say S acts invertibly on X if the translation

$$\begin{aligned} X &\to X \\ F &\mapsto A \otimes F \end{aligned}$$

is a homotopy equivalence (on classifying space) for each $A \in S$.

Definition 2.5. The category $\langle S, X \rangle$ has the same objects as X, and a morphism is represented by an isomorphism class of tuples $(F, G, A, A \otimes F \to G)/\sim$ where $A \in S$ and $F, G \in X$,[3] and morphisms $f : A \otimes F \to G$ and $f' : A \otimes F \to G$ land in the same class if and only if we have an isomorphism $A \cong A'$ such that the diagram

$$\begin{array}{ccc} A \otimes F & \xrightarrow{\cong} & A' \otimes F \\ & \searrow \quad \swarrow & \\ & G & \end{array}$$

commutes.

The localization of X at S is the category $S^{-1}X := \langle S, S \times X \rangle$, where S acts on $S \times X$ diagonally. There is an induced action of S on $S^{-1}X$ given by $A \otimes (B, F) = (B, A \otimes F)$ if S is commutative up to natural isomorphism.

Remark 2.6. Now S acts invertibly on $S^{-1}X$, since the translation $(B, F) \mapsto (B, A \otimes F)$ has a homotopy inverse $(B, F) \mapsto (A \otimes B, F)$: given these two functors, there is a natural transformation $\mathrm{id}_{S^{-1}X}$ to the composition $(B, F) \mapsto (A \otimes B, A \otimes F)$ of two functors, which induces a homotopy equivalence.

Remark 2.7.
- The proof strategy is to apply our knowledge of S to categories like $F(R)$, the category of based free modules over a ring R, or $i\mathcal{P}(R)$, the maximal groupoid in a category of projective R-modules, which are both symmetric monoidal.
- In particular, $\langle S, S \rangle$ has initial object $\mathbb{1}$ and is therefore contractible by Lemma 1.1.

Definition 2.8. Let $\rho : S^{-1}X = \langle S, S \times X \rangle \to \langle S, S \rangle$ to be the projection onto the first factor, that is, mapping $(A, B \times F) \mapsto (A, B)$ on objects and $(A, A \otimes B \to B', A \otimes F \to F') \mapsto (A, A \otimes B \to B')$ on morphisms.

Let $B \to B'$ be a morphism in $\langle S, S \rangle$, then we can write it as $(A, A \otimes B \to B')$ with respect to some unique up to (not necessarily unique) isomorphism A. In this sense, an automorphism is given by an automorphism $a : A \cong A$ such that the isomorphism $a \otimes \ell : A \otimes B \cong A \otimes B$ for some translation ℓ gives a commutative diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{a \otimes \ell} & A \otimes B \\ & \searrow \quad \swarrow & \\ & B' & \end{array}$$

In particular, suppose $A \otimes B \to B'$ is monic and $\mathrm{Hom}(A, A) \to \mathrm{Hom}(A \otimes B, A \otimes B)$ is injective, then a is uniquely determined and therefore must be the identity map. More generally,

Lemma 2.9. If every morphism of S is monic and every translation $S \to S$ is faithful, then every arrow in $\langle S, S \rangle$ determines the choice A up to unique isomorphism and ρ is cofibred.

Proof. The cobase-change map for morphism $(A, A \otimes B \to B')$ is defined by

$$\begin{aligned} \rho^{-1}B &\to \rho^{-1}B' \\ (B, F) &\mapsto (B', A \otimes F). \end{aligned}$$

That is, there is a pushout diagram given by

$$\begin{array}{ccc} (A, A \otimes B) & \longrightarrow & (B, F) \\ \downarrow & & \downarrow \\ B' & \longrightarrow & (B', A \otimes F) \end{array}$$

where the left column represents $B \to B'$ in $\langle S, S \rangle$, and the right column represents $\rho^{-1}(B) \to \rho^{-1}(B')$ in $S^{-1}X$. □

[3] We made some choices here. Fix $F, G \in X$, then there is a particular (but not necessarily unique) choice of A; see below.

By projection on second coordinate, locally the fibers in the right column can be identified with X. Therefore the cobase-change map above is just translation by A on X, i.e., $F \mapsto A \otimes F$.

Theorem 2.10. The defined localization map $X \to S^{-1}X$ is a homotopy equivalence if and only if S acts invertibly on X.

Proof. If S acts invertibly on X, then every translation on X in particular is a homotopy equivalence, so all cobase-change maps are by identification of the form above. In particular, the square

$$\begin{array}{ccc} X & \longrightarrow & S^{-1}X \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & \langle S, S\rangle \end{array}$$

is homotopy Cartesian. By Remark 2.7, then the map $X \to S^{-1}X$ is a homotopy equivalence.

If $X \to S^{-1}X$ is a homotopy equivalence, then since S acts invertibly on $S^{-1}X$ and we know the functor preserves the action, then the pullback action of S on X is invertible as well. □

This motivates us to study invertible actions, and allows the following calculation. Based on a well-known fact about homotopy commutative, homotopy associative H-spaces, $\pi_0(S)$ is a multiplicatively closed subset of the ring $H_0(S) = \mathbb{Z}[\pi_0(S)]$, it has an action on $H_*(X)$ and therefore acts invertibly on $H_*(S^{-1}X)$. Therefore the functor $X \to S^{-1}X$ defined by $F \mapsto (\mathbb{1}, F)$ induces a map

$$(\pi_0(S))^{-1}H_*(X) \to H_*(S^{-1}(X)).$$

Theorem 2.11. This map is an isomorphism, under the given assumption that every morphism in S is an isomorphism, and translations in S are faithful.[4] Checking the definition, this means $B(S^{-1}S)$ is the group completion of BS.

2.2 Proof of Theorem 2.11

Definition 2.12. For each functor $F : \mathscr{C} \to \mathbf{Ab}$, we define $H_i(\mathscr{C}; F)$ to be the ith homology of the telescope

$$\cdots \longrightarrow \coprod_{c_0 \to \cdots \to c_n} F(c_0) \longrightarrow \cdots \longrightarrow \coprod_{c_0 \to c_1} F(c_0) \longrightarrow \coprod_{c_0} F(c_0).$$

Remark 2.13. For instance, the last boundary map sends the copy of $F(c_0)$ indexed by $f : c_0 \to c_1$ to $F(c_0) \oplus F(c_1)$ by $x \mapsto (-x, fx)$. The cokernel of this map is the usual description for the colimit of the functor F, so $H_0(\mathscr{C}; F) = \operatorname{colim}_{c \in \mathscr{C}} F(c)$.

Definition 2.14. A functor $\mathscr{C} \to \mathbf{Set}$ is morphism-inverting if it sends morphisms to isomorphisms. By Lemma 2.15, we know the morphism-inverting functors $F : \mathscr{C} \to \mathbf{Ab}$ are in one-to-one correspondence with local coefficient systems on the topological space $B\mathscr{C}$, i.e., $H_i(\mathscr{C}; F) \cong H_i(B\mathscr{C}; F)$ canonically.

Lemma 2.15. Morphism-inverting functors $\mathscr{C} \to \mathbf{Set}$ are in one-to-one correspondence with covering spaces of $B\mathscr{C}$.

Proof Sketch. Let $F : \mathscr{C} \to \mathbf{Set}$ be morphism-inverting, then the forgetful functor on the category of elements $\mathscr{C} \int F \to \mathscr{C}$ makes $B(\mathscr{C} \int F)$ into a covering space of $B\mathscr{C}$ with fiber $F(c)$ over each vertex c of $B\mathscr{C}$.

Let $\pi : E \to B\mathscr{C}$ be a covering space, then $F(c) = \pi^{-1}(c)$ defines a morphism-inverting functor on $\mathscr{C}$, where c is considered as a 0-cell of $B\mathscr{C}$. □

Definition 2.16. Let M be a $\pi_0(S)$-module, then there is a functor $\bar{M} : \langle S, S\rangle \to \mathbf{Ab}$ that sends object B to abelian group M and morphism $(A, A \otimes B \xrightarrow{\sim} B')$ to $M \xrightarrow{\cdot [A]} M$.

Remark 2.17. If $\pi_0(S)$ acts invertibly on M, then $\bar{M}$ is morphism-inverting, and the homology group $H_*(\langle S, S\rangle, \bar{M})$ reduces to singular homology on the classifying space $B\langle S, S\rangle$ with coefficients in the local coefficient system determined by $\bar{M}$. Since $\langle S, S\rangle$ is contractible, then the homology is just $M_{(0)}$ concentrated at degree 0.

[4] The examples we care about satisfy this condition. For instance, take S to be $i\mathcal{P}(R)$ or $F(R) \cong \coprod_n \mathrm{GL}_n(R)$, the category of based free modules over a ring R. Note that only the former has the symmetric monoidal structure.

First note that each fiber of the cofibred functor $\rho : S^{-1}X \to \langle S, S\rangle$ is identified with X, and by our previous argument, the cobase-change maps are given by the action of S on X.

Lemma 2.18. This functor induces a spectral sequence

$$E^2_{p,q} = H_p(\langle S, S\rangle, \overline{H_q(X)}) \Rightarrow H_{p+q}(S^{-1}X).$$

Here $\overline{H_q(X)}$ is interpreted as $H_q\rho^{-1}$, the functor mapping $A \mapsto H_q(\rho^{-1}A; \mathbb{Z})$.

Proof Sketch. This is the construction of a Serre's spectral sequence, with filtering with respect to the columns. ☐

Recall that localization at multiplicatively closed subset $\pi_0(S)$ of $H_0(S)$ is exact, then we end up with another spectral sequence with E^2-page

$$E^2_{p,q} = H_p(\langle S, S\rangle, \overline{(\pi_0(S))^{-1}H_q(X)}) \Rightarrow H_{p+q}(S^{-1}X).$$

since $\pi_0(S)$ acts invertibly on $H_*(S^{-1}(X))$. Because of Remark 2.17, we know $E^2 = E^\infty$, therefore it only has one column, so that implies edge morphisms are isomorphisms, thus

$$(\pi_0(S)^{-1}H_q(X))_{(0)} \cong H_p(\langle S, S\rangle, \overline{(\pi_0(S))^{-1}H_q(X)}) \cong H_{p+q}(S^{-1}X) \cong H_p(\langle S, S\rangle, \overline{H_q(X)}) \cong H_q(X)_{(0)}.$$

2.3 Identification with $+$-construction

Definition 2.19. Let $f : X \to Y$ be a functor between categories with S-actions, and that f is compatible with the actions.

- If S acts trivially on Y, then we say the S-action on X is fiberwise with respect to f, as S does act on the fibers $f^{-1}(Y)$.
- If, in addition, f is fibered and the base-change maps respect the action on the fibers, then the action is said to be Cartesian. This gives $S^{-1}X$ is fibered over Y, and its fibers are of the form $S^{-1}f^{-1}(Y)$, and the base-change maps are induced by those of f.

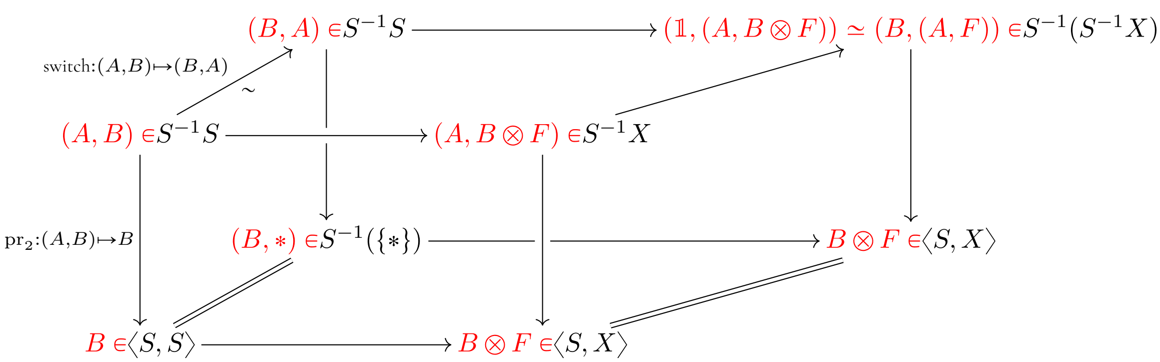
We now consider the projections $q : S^{-1}X \to \langle S, X\rangle$ on the second factor. If we assume all morphisms in X are monic and that for each $F \in X$, the map $S \to X$ given by $B \mapsto B \otimes F$ is a faithful functor, then using the same argument as in the proof of Theorem 2.10, we conclude that q is cofibred where each fiber can be identified as S, and the cobase-change maps are translations.

Let S act on $S^{-1}X = \langle S, S \times X\rangle$ via the first factor, then the action is Cartesian with respect to q, therefore applying localization of S on q yields a cofibred map $S^{-1}(S^{-1}(X)) \to \langle S, X\rangle$ where each fiber of which may be identified with $S^{-1}S$. Since S acts invertibly on $S^{-1}S$, then the cobase-change maps are homotopy equivalences, therefore

$$\begin{array}{ccc} S^{-1}S & \longrightarrow & S^{-1}(S^{-1}X) \\ \downarrow & & \downarrow \\ S^{-1}(\{*\}) & \longrightarrow & \langle S, X\rangle \end{array}$$

is a homotopy Cartesian square by mimicking the argument in Theorem 2.10.

The map $S^{-1}S \to S^{-1}(S^{-1}X)$ is given by $(A, B) \mapsto (A, (B, F))$ for some fixed F in X. This can be extended to



where every square but the top commutes; the top square is homotopy commutative, given by the natural transformations of functors $S^{-1}S \to S^{-1}(S^{-1}X)$:

$$(\mathbb{1}, (A, B \otimes F)) \xrightarrow{\sim} (B, (B \otimes A, B \otimes F)) \xleftarrow{\sim} (B, (A, F)).$$

By Theorem 2.10, $S^{-1}X \to S^{-1}(S^{-1}X)$ is a homotopy equivalence, therefore the front square is homotopy Cartesian. Since $\langle S, S\rangle$ is contractible, then we conclude

Theorem 2.20. If $\langle S, X\rangle$ is contractible, then the map $S^{-1}S \to S^{-1}X$ defined by $(A, B) \mapsto (A, B \otimes F)$ for some fixed F in X is a homotopy equivalence.

Now we should think of everything we have constructed over an exact category where every exact sequence splits, then the isomorphism subcategory with the direct sum inherits a monoidal category structure. With that, $S^{-1}S$ is an H-space with multiplication

$$\begin{aligned} S^{-1}S \times S^{-1}S &\to S^{-1}S \\ ((A, B), (C, D)) &\mapsto (A \oplus C, B \oplus D). \end{aligned}$$

To set it up for Corollary 1.5, we should fix a ring R and let $\mathcal{P}(R)$ be the category finitely-generated projective R-modules, then $\pi_0(S^{-1}S) = K_0(R)$ for $S = i(\mathcal{P}(R))$.[5] Note that we can interpret $S = F(R) \cong \coprod_n \mathrm{GL}_n(R) = \operatorname{colim}_n \mathrm{GL}_n(R)$, the category of based free R-modules. Indeed, the objects are based free R-modules $\{0, R, R^2, \ldots\}$, and there are no maps in $F(R)$ between R^m and R^n whenever $m \neq n$, and note that $\mathrm{Aut}(R^n) \cong \mathrm{GL}_n(R)$. The symmetric monoidal operation is now the concatenation of bases, i.e., for $R^m \otimes R^n = R^{m+n}$ and if a, b are morphisms on A and B respectively, then $a \otimes b$ is the matrix $\operatorname{diag}(A, B)$. With this, the classifying space is isomorphic to the disjoint union of classifying spaces $\mathrm{BGL}_n(R)$.

For now, let $S = F(R) \cong \operatorname{colim}_n \mathrm{GL}_n(R)$ be the category of based free modules, then for any $n \geqslant 1$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{g \mapsto (g,1)} & \mathrm{Aut}(R^n) \\ {\scriptstyle -\oplus R}\downarrow & & \downarrow{\scriptstyle -\oplus R} \\ \mathrm{GL}_{n+1}(R) & \xrightarrow[g \mapsto (g,1)]{} & \mathrm{Aut}(R^{n+1}) \end{array}$$

By the natural transformation $(A, B) \to (A \oplus R, B \oplus R)$ on $S^{-1}S$, we know the diagram

$$\begin{array}{ccc} \mathrm{Aut}(A) & \longrightarrow & \mathrm{Aut}(A \oplus R) \\ & \searrow \quad \swarrow & \\ & S^{-1}S & \end{array}$$

commutes up to homotopy. Taking the colimit, we extend to a map $\mathrm{BGL}(R) = \varinjlim \mathrm{Aut}(R^n) \to BS^{-1}S$ using the telescope construction, and the image lands in the connected component of the identity $(S^{-1}S)_0$, which is also an H-space.[6]

Theorem 2.21. For $S = \operatorname{colim}_n \mathrm{GL}_n(R)$, $B(S^{-1}S) \simeq \mathbb{Z} \times \mathrm{BGL}(R)^+$.

Proof. Define

$$f : BL \to BS^{-1}S$$

[5] Indeed, we know $\pi_0(S^{-1}S)$ is the group completion of $\pi_0(S) = i(S)$. Let $i(S)^\dagger$ be its completion, then there is a natural homomorphism $\alpha(m, n) = [m] - [n]$ from $S^{-1}S$ to $i(S)^\dagger$. One can check that this extends to a map $\pi_0(S^{-1}S) \to i(S)^\dagger$ which is the inverse to the universal homomorphism $i(S)^\dagger \to S^{-1}S$.

[6] To construct this, let $\mathbf{n}$ be the simplicial set of n objects, then $\mathcal{N}$, the ordered set of positive integers, is the colimit of such simplicial sets, then a functor $C : \mathcal{N} \to \mathbf{Cat}$ is a sequence $C_0 \to C_1 \to C_2 \to \cdots$ of categories. (Alternatively, let S_n be the component of S which contains R^n, then S_n is a groupoid equivalent to $\mathrm{Aut}(R^n) = \mathrm{GL}_n(R)$. Define $S_n \to S_{n+1}$ by $B \mapsto R \oplus B$ inductively.) Since $C_n \simeq \mathbf{n} \int C$, then the geometric realization of $L = \mathcal{N} \int C$ is homotopy equivalent to BC, where C is the colimit of C_n. With this construction, $BL \simeq \mathrm{BGL}(R)$. If we interpret L to be the colimit of S_n's, then its objects are pairs (n, B) for $B \in S_n$ and a morphism $(n, B) \to (n + m, C)$ is just an isomorphism $R^m \oplus B \cong C$.

$$(n, B) \mapsto (R^n, B),$$

which restricts to $f : BL \to B(S^{-1}S)_0$, the connected component at identity. It suffices to show that $H_*((BS^{-1}S)_0; \mathbb{Z}) \cong H_*(BL; \mathbb{Z})$ is an isomorphism. First, since BL and $(BS^{-1}S)_0$ are H-spaces with homotopy type of a CW complex, then one can show that $\pi_1((BS^{-1}S)_0)$ acts trivially on the homotopy fiber F of $f : \mathrm{BGL}(R) \to (S^{-1}S)_0$, therefore $\pi_*(F) = 0$ by the relative Hurewicz theorem. By definition, the map $f : L \to (S^{-1}S)_0$ is acyclic. Taking the long exact sequence of homotopy groups give $\pi_1(BL) \cong \pi_1((BS^{-1}S)_0)$, then the fundamental group is the abelianization, therefore we have a perfect normal subgroup of $\pi_1(\mathrm{BGL}(R))$. Therefore $\mathrm{BGL}(R) \to (S^{-1}S)_0$ is a $+$-construction. In particular, since $K_0(S) \cong \mathbb{Z}$, this gives the isomorphism we want: since the multiplication on the H-space $S^{-1}S$ has a homotopy inverse given by the switch map, then all components are homotopy equivalent.

Let $e = [R] \in \pi_0(BS)$ be the class of R, then by Theorem 2.11 we know $H_*(B(S^{-1}S))$ is the localization of the ring $H_*(BS)$ at $\pi_0(BS)$. But $\pi_0(BS)$ is exactly generated by R, so this is $H_*(B(S^{-1}S)) \cong H_*(BS)[\pi_0^{-1}(BS)] \cong H_*(BS)\left[\frac{1}{e}\right]$. But note that this is just the homology given by the colimit of

$$H_*(BS) \xrightarrow{e} H_*(BS) \xrightarrow{e} \cdots$$

induced by $\oplus R : S \to S$. Therefore $H_*(B(S^{-1}S)) \cong H_*((BS^{-1}S)_0) \otimes \mathbb{Z}[e, e^{-1}]$ where $\mathbb{Z}[e, e^{-1}] \cong \mathbb{Z}[\pi_0(S)] = H_0(S)$. In particular $H_*((BS^{-1}S)_0) \cong H_*(BS) \cong \coprod H_*(\mathrm{BGL}_n(R))$. ☐

Definition 2.22. Here the notion of cofinality can be defined similarly on categories: let $M \subseteq P$ be split exact categories where M is a full subcategory, then we say M is cofinal in P if given $A \in P$, there exists $B \in P$ and $C \in M$ such that $A \oplus B \cong C$.

A monoidal functor $F : S \to T$ is cofinal if given $A \in T$, there exists $B \in T$ and $C \in S$ such that $A \otimes B \cong FC$.

Lemma 2.23. If $f : S \to T$ is cofinal, and suppose T acts on X, then $S^{-1}X = T^{-1}X$.

Proof. Note that S acts on X via the pullback along f. Now S acts invertibly on X if and only if T acts invertibly on X, so $S^{-1}X \cong T^{-1}(S^{-1}X) \cong S^{-1}(T^{-1}X) \cong T^{-1}X$, c.f., the fact we proved before Theorem 2.20. ☐

Theorem 2.24 (Gersten/K-book Cofinality Theorem IV.4.11)**.** If M is cofinal in P, then $QM \to QP$ is a covering space, and $K_*(M) \to K_*(P)$ is an isomorphism for $* > 0$ and is injective for $q = 0$.

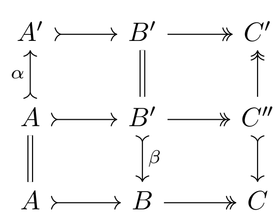
The argument we gave in Theorem 2.21 really only proved the case for $S = \mathrm{colim}_n \mathrm{GL}_n(R)$.[7] Note that $S = \mathrm{colim}_n \mathrm{GL}_n(R) \to \mathcal{P}(R)$ is cofinal, since every projective module is a summand of a free module. By Theorem 2.24, the K-groups of S agree with the K-groups of $\mathcal{P}(R)$ for $n \geqslant 1$. Therefore,

Theorem 2.25. Let $S = i(\mathcal{P}(R))$ be the isomorphism category of finitely-generated projective R-modules, then $B(S^{-1}S) \simeq K_0(R) \times \mathrm{BGL}(R)^+$.

3 Proof of Theorem 1.4 and Corollary 1.5

Let $\mathcal{A}$ be an exact category where every exact sequence splits, and set $S = i(\mathcal{A})$.

Definition 3.1. A fibred category $\mathcal{E}$ over $\mathcal{A}$ has objects the admissible exact sequences in $\mathcal{A}$, and a morphism from $E' : (A' \rightarrowtail B' \twoheadrightarrow C')$ to $E : (A \rightarrowtail B \twoheadrightarrow C)$ is an equivalence class of diagrams of the form



where the rows are exact sequences in $\mathcal{A}$.

[7]Note that if R satisfies the invariant basis property, then this category is equivalent to a full subcategory of $i(\mathcal{P}(R))$.

Remark 3.2. For instance, given injective map $C'' \rightarrowtail C$, the induced map on exact sequences $A \to B \to C$ is the pullback exact sequence $A \to B' \to C''$. Similarly, given surjective map $C'' \to C'$, we have a surjection $B' \to C'$ and the kernel A' extends it to a short exact sequence $A' \to B' \to C'$.

Definition 3.3. Fix $C \in \mathcal{A}$, let E_C be the category with objects are all exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

from $\mathcal{A}$ and morphisms are all isomorphisms that are identity on C, i.e.,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

In particular, every morphism of E_C is an isomorphism.

Remark 3.4. Note that the right column from is a morphism $\varphi : C'' \to C$ in the Q-construction $Q\mathcal{A}$. Therefore, C can be thought of as a fibred functor $t : \mathcal{E} \to Q\mathcal{A}$ by sending $(A \rightarrowtail B \twoheadrightarrow C)$ to C. Using this notation, $E_C = t^{-1}(C)$ is just the fibre of the category $\mathcal{E}$.

Remark 3.5. The category E_0 is homotopy equivalent to $S = i(\mathcal{A})$ via the full embedding

$$\begin{aligned} i(\mathcal{A}) &\to E_0 \\ A &\mapsto (A = A \twoheadrightarrow 0). \end{aligned}$$

Remark 3.6. E_C obtains the structure of a symmetric monoidal category as follows: given $E_i = (A_i \rightarrowtail B_i \twoheadrightarrow C)$, we have $E_1 * E_2 = (A_1 \oplus A_2 \rightarrowtail (B_1 \times_C B_2) \twoheadrightarrow C)$ with identity $e : (0 \rightarrowtail C \twoheadrightarrow C)$, which gives a faithful monoidal functor

$$\begin{aligned} \eta_C : S &\to E_C \\ A &\mapsto (A \rightarrowtail A \oplus C \twoheadrightarrow C). \end{aligned}$$

Remark 3.7. There is an S-action on $\mathcal{E}$ given by $A' \otimes (0 \to A \to B \to C \to 0) = (0 \to A' \oplus A \to A' \oplus B \to C \to 0)$. Then $\mathcal{E} \to Q\mathcal{A}$ is fibrewise and Cartesian with respect to this action.

Theorem 3.8. Fix $C \in \mathcal{A}$ in a split exact category. One can show that $M = \langle S, E_C \rangle$ is contractible by showing

i. M is connected;

ii. M is an H-space;

iii. the multiplication on M has a homotopy inverse;

iv. the endomorphism $x \mapsto x^2$ on M is homotopic to the identity.

Proof. i. By the symmetric monoidal structure defined in Remark 3.6, consider the projection

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 \oplus A_2 & \longrightarrow & B_1 \times_C B_2 & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \| & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

we choose a splitting for the surjections an obtain an isomorphism $A_2 \otimes E_1 = E_1 * E_2$, therefore this determines a morphism $E_1 \to E_1 * E_2$ in M. Similarly we obtain a morphism $E_2 \to E_1 * E_2$. Therefore, this connects E_1 and E_2.

ii. The operation of symmetric monoidal structure defines the H-space.

iii. As a connected H-space, the category M has a homotopy inverse. Consider

$$\begin{array}{ccccc} M & \xrightarrow{g} & M \times M & \xrightarrow{\mathrm{pr}_2} & M \\ \| & & \downarrow f & & \| \\ M & \xrightarrow[g]{} & M \times M & \xrightarrow[\mathrm{pr}_2]{} & M \end{array}$$

where $f : (x, y) \mapsto (xy, y)$ and $g : x \mapsto (x, e)$. Since M is connected, the rows are fibrations, then the five lemma of fibrations says that f has to be a homotopy equivalence as well. Therefore, let h be its inverse, then the inverse of multiplication on M is given by $x \mapsto \mathrm{pr}_1(h(e, x))$.

iv. Take $E = E_1 = E_2$, then the projection diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \longrightarrow & B \times_C B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

gives a canonical splitting of surjections and we obtain a natural morphism $E \to E * E$ as in i. This is the homotopy we want.

Consider the homotopy classes of maps $[M, M]$. By ii. and iii. we know this is a group, and iv. says that every element x of $[M, M]$ is such that $x^2 = x$. In particular, $[M, M] = \{e\}$ is the trivial group, hence contractible. □

Theorem 3.9. The square

$$\begin{array}{ccc} S^{-1}S & \longrightarrow & S^{-1}\mathcal{E} \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & Q\mathcal{A} \end{array}$$

is homotopy Cartesian. In particular,

$$S^{-1}S \longrightarrow S^{-1}\mathcal{E} \longrightarrow Q\mathcal{A}$$

is a homotopy fibration.

Proof. Since $\mathcal{E} \to Q\mathcal{A}$ is fibred, and by Remark 3.7 we conclude that $S^{-1}\mathcal{E} \to Q\mathcal{A}$ is also fibred. By a corollary of Quillen's Theorem B, it suffices to show that the base-change maps $\varphi^* : E_C \to E_{C'}$ of $\varphi : C' \to C$ in $Q\mathcal{A}$ for the fibred map $S^{-1}\mathcal{E} \to Q\mathcal{A}$ are homotopy equivalences. Note that if suffices to prove it for injective morphisms $0 \rightarrowtail C$ and surjective morphisms $C \twoheadrightarrow 0$ of $Q\mathcal{A}$.

Consider the surjective map $C \twoheadrightarrow 0$, and the injective case follows in a similar fashion. Recall that we can identify E_0 and S, now the base-change $\varphi^* : E_C \to E_0$ is just the map sending $(0 \to A \to B \to C \to 0)$ to B. Let $f : E_0 \to E_C$ be the map $A \mapsto (0 \to A \to A \oplus C \to C \to 0)$. Since $\langle S, E_C \rangle$ is contractible, then $S^{-1}f : S^{-1}S \cong S^{-1}E_0 \to S^{-1}E_C$ is a homotopy equivalence by Theorem 2.20. Now $\varphi^* \circ S^{-1}f : S^{-1}E_0 \to S^{-1}E_0$ is the composition defined by $(A', A) \mapsto (A', A \oplus C)$ in $S^{-1}S$, therefore we know this is a homotopy equivalence already. Hence, we conclude that φ^* has to be a homotopy equivalence. □

Theorem 3.10. $S^{-1}\mathcal{E}$ is contractible.

Proof.

Definition 3.11. The subdivision $\mathrm{Sub}(X)$ of a category of X is the category with objects given by the arrows $\mathrm{Mor}(X)$ and a morphism $f \to g$ is a pair of arrows $h, k : X \to X$ such that $kfh = g$.

First note that the codomain map $\mathrm{Sub}(X) \to X$ is a homotopy equivalence. Now let X be the subcategory of $Q\mathcal{A}$ of injective morphisms, then $\mathcal{E}$ is equivalent to $\mathrm{Sub}(X)$, therefore equivalent to X. Since X has an initial object $\mathbb{1}$, then X is contractible, hence $\mathcal{E}$ is contractible. Now S acts invertibly on $\mathcal{E}$, therefore $\mathcal{E}$ and $S^{-1}\mathcal{E}$ are homotopy equivalent by Theorem 2.10, thus $S^{-1}\mathcal{E}$ is contractible. □

Proof of Theorem 1.4. By Theorem 3.9 and Theorem 3.10, we have a fibration with contractible total space. Taking the long exact sequence of homotopy groups, we know $S^{-1}S$ and the infinite loopspace $\Omega BQ\mathcal{A}$ have the same homotopy groups: $\pi_n(BS^{-1}S) \cong \pi_{n+1}(BQ\mathcal{A}) \cong \pi_n(\Omega BQ\mathcal{A})$. Since they are both of CW complex structure, then by Whitehead's theorem, we conclude that they are homotopy equivalent. □

Proof of Corollary 1.5. By Theorem 1.4, $B(S^{-1}S) \cong \Omega BQ\mathcal{A}$, but we know $B(S^{-1}S) \cong K_0(R) \times \mathrm{BGL}(R)^+$ by Theorem 2.25 already, so we are done. □

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